## 0.1 Sobolev Space $H^s(\mathbb{R}^n)$

Last lecture have talked about the solvability of the Cauchy problem for the system $u_t = P(D_x)u$, or the single equation $\partial_t^m u = \sum_{k=0}^{m-1} P_k(D_x)\partial_t^k u$. In this lecture we will formally present the idea more precisely. Note that the PDE may also be written in the following forms:

- $P(D_x, \partial_t) = \sum_{k=0}^{m} P_k(D_x)\partial_t^m u = 0$ — "hybrid" form.
- $\widehat{P}(D_x, D_t) = \sum_{k=0}^{m} P_k(D_x)(iD_t)^m u = 0$ — the full symbol.

  The polynomials are related through

- $P(\xi, \tau) = \widehat{P}(\xi, -i\tau)$ and $\widehat{P}(\xi, \tau) = P(\xi, i\tau)$.

So for instance, that the real part of $\lambda$ satisfying $P(\xi, \lambda) = 0$ is bounded from above is equivalent to saying that the imaginary part of $\tau$ satisfying $\widehat{P}(\xi, \tau) = 0$ is bounded from below.

**Definition 1.** *The **Sobolev space** (or the **Bessel potential space**) $H^s = H^s(\mathbb{R}^n)$ is defined to be the space of tempered distributions whose Sobolev norm defined to be*

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi, \qquad s \in \mathbb{R} \tag{1}$$

*is finite.*

The space $H^s$ is a Hilbert space with respect to the inner product:

$$\langle u, v \rangle_{H^s} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \, d\xi. \tag{2}$$

It is useful to use the notation $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ for convenience.

**Definition 2.** *Let us define $C_\alpha^k H^s \equiv C_\alpha^k([0, \infty), H^s(\mathbb{R}^n))$ by*

$$C_\alpha^k H^s = \left\{ u : \overline{\mathbb{R}}_+ \to H^s(\mathbb{R}^n) : u \in C^k, \ \sup_{t \geq 0} (1 + t)^m e^{-\alpha t} \|u(t)\|_{H^s} < \infty \right\}, \tag{3}$$

*for some $m \in \mathbb{R}$.*

Note that in order to have the Banach space property the norm must contain $t$-derivatives of $u$, however for the purposes of this study we will not concern ourselves with it. The spaces $C_\alpha^k H^s$ will be used merely to abbreviate statements that would otherwise be unnecessarily long.

## 0.2 Petrovsky Well-Posedness Condition

Suppose $u \in C_\alpha^k H^s$ satisfies the system $\partial_t u = P(D_x)u$ for $k \geq 1$. Then by taking the Fourier transform in $x$

$$\partial_t \widehat{u} = P(\xi)\widehat{u}, \qquad \left[P(\xi, \partial_t)\widehat{u} = 0 \text{ for single equations}\right]$$

in a weighted $L^2$ sense, i.e "$C_\alpha^k L_s^2$". So

$$\partial_t \widehat{u}(\xi, t) = P(\xi)\widehat{u}(\xi, t), \qquad \text{for } a.e\ \xi \in \mathbb{R}^n.$$

Suppose $\lambda_0(\xi), ..., \lambda_{m-1}(\xi)$ be the eigenvalues of $P(\xi)$, ie. the roots of $P(\xi, \lambda) = 0$, ordered so that

$$\text{Re } \lambda_0(\xi) \leq \cdots \leq \text{Re } \lambda_{r-1}(\xi) \leq \alpha < \cdots$$

corresponding to eigenvectors $e_0(\xi), ..., e_{m-1}(\xi)$. Define

$$Q_\alpha(\xi) = \text{span}\{e_0(\xi), ..., e_{r-1}(\xi)\}.$$

Since $u \in C_\alpha^k H^s$, we must have $\widehat{u}(\xi, 0) \in Q_\alpha(\xi)$ for almost every $\xi$. In particular, in order for the Cauchy problem to have a solution in $C_\alpha^k H^s$ with some $\alpha$ and $s$, for any initial data in

$$H^{-\infty} = \bigcup_s H^s,$$

we must satisfy the following criterion:

$$\exists \ c \in \mathbb{R} \text{ such that Re } \lambda_{m-1}(\xi) \leq c, \text{ for } a.e \ \xi \in \mathbb{R}^n,$$

otherwise known as the **Petrovsky Well-Posedness Condition** abbreviated PWP.

**Definition 3.** *If PWP is satisfied, P is called Petrovsky well-posed.*

We have proved for $PWP$ problems, a unique solution exists in some $C_\alpha^k H^s$ for any given initial data in $H^{-\infty}$.

**Example**: Consider the Laplace equation. It is known that the Cauchy problem for the Laplace equation generally has no solution. This was established by verifying that the Green's function which solves the Cauchy problem. From the Sobolev theory we may draw similar conclusion. The Laplace equation $\partial_t^2 - D_x^2 u = 0$, where the negative sign rises from using this notation. We have

$$P(\xi, \lambda) = -\xi^2 + \lambda^2 \implies \lambda = \pm\xi$$

where we note that $\lambda$ is not bounded independently of $\xi$. It follows that the Laplace equation is not PWP.

## 0.3 $\alpha$-Regularity

Let us focus on the single equation case $P(D_x, \partial_t)u = 0$.

**Definition 4.** *We say that $P$ is $\alpha$-**regular of order** $r$ if $r(\xi) = r$ almost everywhere, i.e., $\lambda_{r-1}(\xi) \leq \alpha < \lambda_r(\xi)$ for almost every $\xi \in \mathbb{R}^n$.*

**Example**: The Laplacian is 0-regular of order 1.

Consider the boundary condition:

$$B_j(D_x, \partial_t)u = \sum_{k=0}^{m_j} B_{jk}(D_x)\partial_t^k u = g_j, \qquad j = 0, ..., l-1 \qquad \text{at } \{t = 0\}.$$

Suppose for any $g_j \in H^{-\infty}$, the problem has a unique solution, $u \in \bigcup_s C_\alpha^k H^s$. This means $\widehat{u}(\xi, 0) \in Q_\alpha(\xi)$ for all $g_j$. From the Fourier perspective the boundary condition gives

$$B_j(\xi, \partial_t)\widehat{u}(\xi, t) = \widehat{g}_j(\xi), \qquad \text{at } \{t = 0\}.$$

Combining this with

$$\widehat{u}(\xi, t) = \sum_{\text{Re } \lambda_k \leq \alpha} C_k t^{\beta_k} e^{\lambda_k t} \vec{e_k}$$

gives

$$\sum_{k=0}^{r(\xi)-1} B'_{ik}(\xi) C_k(\xi) = \widehat{g_j}(\xi),$$

where $B'_{ik}(\xi)$ is a matrix uniquely determined by $B_{jk}$ and $\{\lambda_k\}$. From uniqueness the system may not be underdetermined and hence may have at least $l$ boundary conditions satisfying $l \geq r$. Meanwhile we require sufficiently many eigen-basis $r$ to span the solution space and thus $r \geq l$ which implies $r = l$. It follows that $P$ is $\alpha$-regular of order $l$. In addition to the previous requirement, we need $\det[B'_{jk}(\xi)] \neq 0$ for almost every $\xi$. This is an instance of the so-called **Lopatinsky-Shapiro conditions**.

Conversely, if $P$ is $\alpha$-regular of order $l$ and $\det[B'_{jk}(\xi)] \neq 0$ for all $\xi \in \mathbb{R}^n$, then for any $g_j \in H^{-\infty}$, there exists a unique solution in $\cup_s C^k_\alpha H^s$.

*Proof sketch.* By Seidenberg-Tarski, $\|B'(\xi)\|^{-1} \leq (1 + |\xi|)^a$. From that we may conclude $|\widehat{u}(\xi, 0)| \leq (1 + |\xi|)^a \sum_j |\widehat{g}_j(\xi)|$. $\square$

**Example**: Referring to the Laplace equation with

- Dirichlet boundary problem $B(D_x, \partial_t)u = u$.
- Neumann : $B(D_x, \partial_t)u = \partial_t u \implies B(\xi, \lambda) = \lambda$.

$$\widehat{u}(\xi, t) = C(\xi) e^{-\xi t}$$

$$\begin{cases} \partial_t = g & \text{at } \{t = 0\} \\ \partial_t \widehat{u}(\xi, 0) = \widehat{g}(\xi) & \end{cases}$$

$$-\xi C(\xi) = \widehat{g}(\xi)$$

  where in this case the factor $-\xi$ defines the matrix $B'$. It follows that if $c = 0$ then we would not have invertibility. If we consider the perturbed Neumann problem boundary condition $\partial_t u + \epsilon u = g$ for some $\epsilon \neq 0$ then we have solvability.

- The backward heat equation is not regular.
- Schrödinger equation $\partial_t u = i\Delta u$. The roots are purely imaginary and thus $\alpha$-regular for all $\alpha \geq 0$.
- The wave equation with strong friction $\partial_t^2 u = \Delta u + \partial_t \Delta u$.

$$\implies P(\xi, \lambda) = \lambda^2 + \xi^2 + \xi^2 \lambda = \lambda^2 + \xi^2 \lambda + \xi^2$$

$$\implies \lambda_{1,2} = -\frac{\xi^2}{2} \pm \sqrt{\frac{\xi^4}{4} - \xi^2} \implies \text{Re } \lambda_{1,2} \leq 0$$

  and thus Petrovsky well posed.