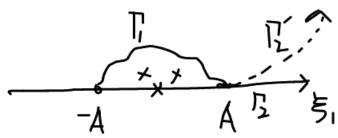
0.1 Continuation of Proof

We have derived

$$\langle E, \varphi\rangle = (2\pi)^{-n} \int\limits_{\Gamma_1+\Gamma_2} \frac{\widetilde{\widehat{\varphi}}(\zeta)}{P(\zeta)}\, d\zeta, \tag{1}$$

and proved that $E \in \mathscr{D}'$ and $P(\partial)E = \delta$. The contribution from Γ_1 is C^∞. In order to show that the



contribution from Γ_2 is also smooth, we consider a deformation Γ_2' of Γ_2 such that

$$\Gamma_2' \;:\; \xi + ig(\xi)\eta_0, \qquad \eta_0 \cdot x_0 > 0$$

and consider the integral

$$I(x) = (2\pi)^{-n} \int\limits_{\Gamma_1+\Gamma_2'} \frac{e^{ix\cdot\zeta}}{P(\zeta)}\, d\zeta. \tag{2}$$

We define g such that $g \geq 0$, $g \in C$ and $g = 0$ except for large ξ, where $g(\xi) = |\xi|^\gamma$ otherwise. Take $\gamma > 0$ sufficiently small so that

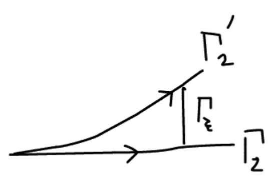
$$|P(\zeta)| \geq |\xi|^b \text{ on } \Gamma_2',$$

where we recall that $\xi = \text{Re }\zeta$. The integrand of I, which is defined to be

$$\frac{e^{-|\text{Re }\zeta|^\gamma}}{|\text{Re }\zeta|^b}$$

belongs to Schwartz class functions. In particular, this implies that $I \in C^\infty$ in a neighbourhood of x_0. It is left to prove that the modifying/deforming Γ_2 into Γ_2' leaves the value of the integral unchanged. With out loss of generality, we will consider the one dimensional case.

We need to show



$$\int I(x)\varphi\, dx = (2\pi)^{-n} \int\limits_{\Gamma_1+\Gamma_2'} \left(\int\limits_{\mathbb{R}^n} \frac{e^{ix\cdot\zeta}}{P(\zeta)}\, d\zeta \right) \cdot \varphi(x)\, dx = (2\pi)^{-n} \int\limits_{\Gamma_1+\Gamma_2'} \frac{\widetilde{\widehat{\varphi}}(\zeta)}{P(\zeta)}\, d\zeta.$$

Let $x_0 \neq 0,\ \varphi \in \mathscr{D}(\mathbb{R}^n \backslash \{0\})$,

$$|\widetilde{\widehat{\varphi}}(\zeta)| \leq C_N(1+|\text{Re } \zeta|)^{-N} \cdot e^{A|\text{Im } \zeta|}$$

by Paley-Weiner. On Γ_ξ:

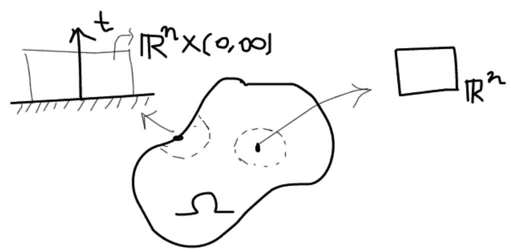
$$|\widetilde{\widehat{\varphi}}(\zeta)| \leq C_N(1-|\xi|)^{-N} e^{A|\xi|^\gamma}$$

we have

$$|P(\zeta)| \geq |\zeta|^b$$

Replace Γ_2' by Γ_2 in $I(x)$. [QED]

0.2 Half Space Problem



Studying Lu with constant coefficients in $\mathbb{R}^n$ is essentially studying locally the behaviour of $L'u$, where L' is the variable coefficient differential operator whose coefficients satisfy sufficient regularity. We will consider the so called *half space problem.*

It is always possible via some coordinate transformation to achieve

$$Lu \mapsto \partial_t^m u = \sum_{k=0}^{m-1} P_k(D_x)\partial_t^k u,$$

which can then always be reduced to a system of first order equations

$$\partial_t u_j = \sum_{k=0}^{m-1} P_{ij}(D_x)u_k, \qquad j = 0..m-1.$$

We denote the previous in matrix form by $\partial_t u = P(D_x)u$. Carrying on by taking the Fourier transform in x,

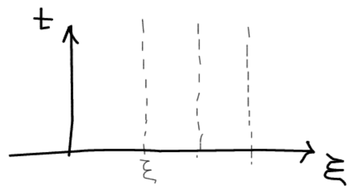
$$\partial_t \widehat{u}(\xi,t) = P(\xi)\widehat{u}(\xi,t)$$

which yields a system of ODE's.

We denote the system by $\dot{y} = Ay$. The eigenvalues $\{\lambda_j\}_{j=0}^{m-1}$ of A, are ordered so that

$$\text{Re } \lambda_0 \leq \cdots \leq \text{Re } \lambda_{r-1} \leq \alpha < \text{Re } \lambda_r \leq \cdots$$

where α is a given real number. Let $\{e_j\}_{j=0}^{m-1}$ be corresponding eigenvectors (including root vectors / generalized eigenvectors should the eigenvalue multiplicities demand that).



Definition 1. *A root vector (or a generalized eigenvector) of height k corresponding to λ_j is a vector v satisfying*

$$(A - \lambda_j I)^k v = 0.$$

From finite dimensional linear algebra and ordinary differential equation theory, the solution to problem $\dot{y} = Ay$ is

$$y(t) = \sum_{k=0}^{m-1} C_k t^{\beta_k} e^{\lambda_k t} \widehat{e}_k, \qquad |\beta_k| \leq m-1. \tag{3}$$

Theorem 1. *Let* (3) *be a solution. Then,*

$$y(t) = \mathcal{O}\left(t^{\beta} e^{\alpha t}\right) \iff y(0) \in Q_- := span\{e_0, ..., e_{r-1}\}$$

for some β.

Recalling that u can be obtained by solving a system of 1st order ODE's as described above, a similar argument on the ODE system will be made to treat the PDE case directly. Let $\{\lambda_j(\xi)\}_{j=0}^{m-1}$ be the eigenvalues of $P(\xi)$. We also order them so that

$$\text{Re } \lambda_0(\xi) \leq \cdots \leq \text{Re } \lambda_{r-1}(\xi) \leq \alpha < \text{Re } \lambda_r(\xi) \leq \cdots$$

and define

$$Q_-(\xi) = \text{span}\{e_0(\xi), ..., e_{r-1}(\xi)\}.$$

We use the facts

- For any matrix norm

$$\|e^{tA}\| \leq e^{\alpha t}\left(1 + 2t\|A\| + \cdots + \frac{(2t)^{m-1}}{(m-1)!}\|A\|^{m-1}\right),$$

  on Q_-.

- The spectral norm is bounded by the Frobenius norm, i.e.,

$$\|P(\xi)\|^2 \leq \sum_{jk} |P_{jk}(\xi)|^2 \leq C^2(1+|\xi|^2)^p.$$

So we infer

$$|\widehat{u}(x,t)| \leq Ce^{\alpha t}[1 + t^{m-1}(1+|\xi|)^{p(m-1)}]|\widehat{u}(\xi,0)|, \tag{$*$}$$

for $\widehat{u}(\xi, 0) \in Q_-(\xi)$.

Def: *Motivated by this, we define the **Sobolev norm**:*

$$\|u\|_{H^s}^2 = \int (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi. \tag{4}$$

Using this definition we may write ($*$) into the following theorem:

Theorem 2. *Let $\widehat{u}(\xi, 0) \in Q_-(\xi)$ for almost every ξ, then*

$$\|u(\cdot, t)\|_{H^s} \leq C(1 + t^{m-1})e^{\alpha t}\|u(\cdot, 0)\|_{H^{s+p(m-1)}}.$$

If $\widehat{u}(\xi, 0) \notin Q_-(\xi)$ on a set of ξ with positive measure, then there exists $c > 0$ such that

$$\|u(\cdot, t)\|_{L^2} \geq ce^{(\alpha+c)t}.$$

Corollary 1. *The Cauchy problem*

$$\partial_t u = P(D_x)u,$$

with $u(x, 0)$ specified, is well posed (to H_s from some H_r), if all eigenvalues of $P(\xi)$ have real part which are bounded above independently of ξ.

Similarly, the Cauchy problem $\partial_t^m u = \sum_{k=0}^{m-1} P_k(D_x)\partial_t^k u$ with $u(x, 0), ..., \partial_t^{m-1}u(x, 0)$ specified, is well posed if all roots $\tau^m = \sum_{k=0}^{m-1} P_k(\xi)\tau^k$ have real part which are bounded above independently of ξ.

Examples

- The heat equation $\partial_t u = \partial_x^2 u$

  $$\tau = -\xi^2$$

  and thus well posed.

- The wave equation $\partial_t^2 u = \partial_x^2 u$

  $$\tau^2 = -\xi^2 \implies \tau = \pm i\xi$$

  and thus well posed.

- The Laplace equation $\partial_t^2 u = -\partial_x^2 u$

  $$\tau^2 = \xi^2 \implies \tau = \pm\xi$$

  take $\alpha = 0$, $Q_-(\xi)$ =span(e_0). Specify $u(x, 0)$ (Dirichlet problem).

- Backward heat equation $\partial_t u = -\partial_x^2 u$

  $$\tau = \xi^2,$$

  hence not well posed.