Lecture[1] 12

Schwartz Class $\mathcal{S}(\mathbb{R}^n)$

Definition 1. *We define the* ***Schwartz class*** *functions $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ by the set*

$$\{\varphi \in C^\infty(\mathbb{R}^n) : P_{\alpha,\beta}(\varphi) < \infty, \ \forall \alpha, \beta\} \tag{1}$$

where

$$P_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi(x) \right| \tag{2}$$

defines a family of separating seminorms.

Another way to view the definition above is to consider the space of C^∞ functions $\varphi(x)$ satisfing

$$\left| \frac{\partial^k \varphi(x)}{\partial x^k} \right| \leq C_{m,k}(1 + |x|)^{-m} \tag{3}$$

for any k and any positive integer m. As a direct consequence of this definition, Schwartz class functions are C^∞ functions whose derivatives decay faster than any polynomial. The topology on $\mathcal{S}$ generated by the family of seminorms $\{P_{\alpha,\beta}\}$ is a Frechet topology. Moreover, the following topological embedding holds

$$\mathscr{D} \subset \mathcal{S} \subset L^1.$$

In particular, any sequence $\varphi_n \in \mathscr{D}$ convergent in the topology of $\mathscr{D}$ is also convergent in the topology of $\mathcal{S}$. Also, $\mathscr{D}$ is dense in $\mathcal{S}$. This can be easily shown by considering a cut off function $\chi(x/n)$ to construct a sequence of compactly supported C^∞ functions converging to a target C_o^∞ function which lies in $\mathcal{S}$.

The Fourier Transform

Definition 2. *Let $u \in L^1(\mathbb{R}^n)$. The Fourier transform is defined by*

$$\widehat{u}(\xi) = (\mathcal{F}u)(\xi) = \int e^{-i\xi \cdot x} u(x) \ dx. \tag{4}$$

- If u is continuous then its transform $\widehat{u} \in C_o(\mathbb{R})^n$, due to the Riemann-Lebesgue Lemma.
- If $\widehat{u} \in L^1$ then,

$$u(x) = (2\pi)^{-n} \int e^{i\xi \cdot x} \widehat{u}(\xi) \ d\xi. \tag{5}$$

$$\widehat{\widehat{u}} = (-2\pi)^n \widehat{u}. \tag{6}$$

Theorem 1. *The map $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism with $\mathcal{F}^{-1}$ given by*

$$\mathcal{F}^{-1}\psi = (2\pi)^{-n} \widetilde{\widehat{\psi}}. \tag{7}$$

moreover, we have

$$\widehat{\partial^\alpha \varphi}(\xi) = (i\xi)^\alpha \widehat{\varphi}(\xi) and \tag{8}$$

$$\widehat{x^\alpha \varphi} = (i\partial)^\alpha \widehat{\varphi} \tag{9}$$

[1] Notes by Ibrahim Al Balushi

Proof. The proof of this theorem is strictly computational.

$$\partial^\alpha \widehat{\varphi}(\xi) = \int e^{-i\xi\cdot x} \underbrace{(-ix)^\alpha \varphi(x)}_{=:\psi\in\mathcal{S}} \, dx \implies \widehat{\varphi} \in C^\infty \tag{10}$$

$$\implies \widehat{(-ix)^\alpha}\varphi = \partial^\alpha \widehat{\varphi} \tag{11}$$

$$\int e^{i\xi\cdot x}\psi(\xi)\widehat{\varphi}(\xi)\, d\xi = \int \varphi(y)\, dy \int e^{-i\xi(y-x)}\psi(\xi)\, d\xi \tag{12}$$

$$= \int \varphi(y)\widehat{\psi}(y-x)\, dy = \int \widehat{\psi}(y)\varphi(x+y)\, dy. \tag{13}$$

we notice if $x = 0$:

$$\int \psi\widehat{\varphi} = \int \widehat{\psi}\varphi. \tag{14}$$

Now consider the transformation $\psi(\xi) \mapsto \psi(\epsilon x)$ so that $\widehat{\psi}(y) \mapsto \epsilon^{-n}\widehat{\psi}(y/\epsilon)$,

$$\int e^{-\xi\cdot x}\psi(\epsilon x)\widehat{\varphi}(\xi)\, d\xi = \int \epsilon^{-n}\widehat{\psi}(y/\epsilon)\varphi(x+y)\, dy = \int \widehat{\psi}(y)\varphi(x+\epsilon)\, dy. \tag{15}$$

Sending $\epsilon \to 0$ we obtain

$$\psi(0)\int e^{-\xi\cdot x}\widehat{\varphi}(\xi)\, d\xi = \varphi(x)\int \widehat{\psi}(y)\, dy. \tag{16}$$

Take $\psi(x) = e^{-|x|^2}$ we obtain the constant $(2\pi)^{-n}$. This rises from the Gaussian integral. □

Facts:

- Parseval's Formula.
- $\int u\overline{v} = (2\pi)^{-n}\int \widehat{u}\overline{\widehat{v}}$.
- $\widehat{u * v} = \widehat{u}\cdot\widehat{v}$.
- $\widehat{u\cdot v} = (2\pi)^{-n}\widehat{u} * \widehat{v}$.

Tempered Distributions $\mathcal{S}'$

Definition 3. *Linear continuous functional on Schwartz class $\mathcal{S}$ is called a **tempered distribution**. The linear space of tempered distributions is denoted by $\mathcal{S}'$.*

We have seen the embedding relation between test functions $\mathscr{D}$ and Schwartz class function $\mathcal{S}$. Thus any tempered distribution is also a linear continuous distribution on $\mathscr{D}$. Particularly, since any $\varphi_n \in \mathscr{D}$ and $\varphi_n \to \varphi$ in $\mathscr{D}$ implies that $\varphi_n \to \varphi$ in $\mathcal{S}$, then if $f_n(\varphi) \to f(\varphi)$ for all $\varphi \in \mathcal{S}$ implies $f_n(\varphi) \to f(\varphi)$ for all $\varphi \in \mathscr{D}$,

$$\mathcal{S}' \subset \mathscr{D}'.$$

The definitions for differentiating tempered distributions and test functions coincide. Moreover, multiplication of element of $\mathcal{S}'$ with smooth functions in the performed similarly as in $\mathscr{D}$, with the one exception that $a \in C^\infty$ must also satisfy

$$\left|\frac{\partial^k a(x)}{dx^k}\right| \le C_k(1+|x|)^{n_k}, \qquad \forall k. \tag{17}$$

Recall that distributions may be represented locally as some derivative of a bounded function. A similar theorem holds for tempered distributions.

Theorem 2. *Any $f \in \mathcal{S}'$ can be represented in the following form:*

$$f = \sum_{|\alpha|=0}^{m_1} \partial_\alpha f_\alpha \tag{18}$$

where f_α are regular functionals in $\mathcal{S}'$ corresponding to continuous functions $f_\alpha(x)$ satisfying the estimates

$$|f_\alpha(x)| \leq C_\alpha (1 + |x|)^{m_2}$$

where m_1 and m_2 are integers.

We define the Fourier transform for tempered distributions:

Definition 4. *If $u \in \mathcal{S}'$ then define*

$$\langle \widehat{u}, \varphi \rangle = \langle u, \widehat{\varphi} \rangle, \qquad \forall \varphi \in \mathcal{S}. \tag{19}$$

Theorem 3. *The map $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is an isomorphism. Moreover,*

$$\widehat{\widehat{u}} = (2\pi)^{-n} \widetilde{u}, \qquad u \in \mathcal{S}' \tag{20}$$

Proof. Let $\varphi \in \mathcal{S}$. Then,

$$\left\langle \widehat{\widehat{u}}, \varphi \right\rangle = \left\langle u, \widehat{\widehat{\varphi}} \right\rangle = (2\pi)^{-n} \langle u, \widehat{\varphi} \rangle = (2\pi)^{-n} \langle \widehat{u}, \varphi \rangle. \tag{21}$$

□

Example 1. *Consider the following computation: For any $\varphi \in \mathcal{S}$,*

$$\left\langle \widehat{\delta}, \varphi \right\rangle = \langle \delta, \widehat{\varphi} \rangle = \widehat{\varphi(0)} = \int \varphi(x)\ dx = \langle 1, \varphi \rangle \tag{22}$$

$$\implies \widehat{\delta} = 1.$$

Example 2. $\widehat{\partial^\alpha \delta}(\xi) = (i\xi)^\alpha$.

Simple application: Generalized Liouville's theorem

Suppose $P(\xi) \neq 0$ if $\xi \neq 0$. For example this holds for the Laplacian or the heat operator. Let $u \in \mathcal{S}'$ that satisfies $P(\partial)u = 0$. Then

$$P(i\xi)\widehat{u}(\xi) = 0$$
$$\text{supp } \widehat{u} \subset \{0\}$$
$$\implies \widehat{u} = \sum_\alpha \partial^\alpha \delta, \quad |\alpha| < \infty$$

u is a polynomial.

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