Lecture[1] 11

Definition 1. *Let $u \in \mathscr{D}'(\Omega)$. We define its* singular support *by*

$$\operatorname{sing\,supp} u = \Omega \backslash \{\omega \subset \Omega \text{ open s.t } u\big|_\omega \in C^\infty(\omega)\}$$

and analytic singular support *by*

$$\operatorname{sing\,supp}_a u = \Omega \backslash \bigcup \{\omega \subset \Omega \text{ open s.t } u\big|_\omega \in C^\omega(\omega\}$$

The sets defined above are relatively closed in Ω. Moreover

$$\operatorname{supp} u \supset \operatorname{sing\,supp}_a u \supset \operatorname{sing\,supp} u.$$

FIG

Definition 2. *An operator $L : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is called* hypoelliptic *if*

$$\operatorname{sing\,supp} u \subset \operatorname{sing\,supp} Lu.$$

and analytic-hypoelliptic *if*

$$\operatorname{sing\,supp}_a u \subset \operatorname{sing\,supp}_a Lu.$$

Consider a constant coefficient operator $P(\partial)$. Suppose $P(\partial)E = \delta$.

- If $P(\partial)$ is hypoelliptic $\implies \operatorname{sing\,supp} E \subset \{0\}$.
- If $P(\partial)$ analytic hypoelliptic $\implies \operatorname{sing\,supp}_a E \subset \{0\}$.

The converse statements are also true.

Theorem 1 (Schwartz)**.** *Let $P(\partial)E = \delta$.*

a) $\operatorname{sing\,supp} E \subset \{0\} \implies P(\partial)$ *is hypoelliptic.*

b) $\operatorname{sing\,supp}_a E \subset \{0\} \implies P(\partial)$ *is analytic hypoelliptic.*

Proof. 1. FIG $V \in \mathcal{N}(0)$. Choose $\chi \in \mathscr{D}(\Omega)$ such that $\chi \equiv 1$ on V. Then,

$$P(\partial)(\chi u) = \chi P(\partial)u + v$$

where v is some tail sum of derivative who vanishes in a neighbourhood of y.

$$E * P(\partial)(\chi u) = [P(\partial)E] * \chi u = \chi u.$$

$$\chi u = \underbrace{E * (\underbrace{\chi f}_{\in C^\infty})}_{\in C^\infty} + E * v$$

Consider $\zeta \in \mathscr{D}(B_\epsilon)$ such that $\zeta \equiv 1$ on $B_{\epsilon/2}$.

$$E = \zeta E + (1-\zeta)E$$

$$E * v = \underbrace{(\zeta E) * v}_{=0 \ nbhd \ of \ y} + \underbrace{\underbrace{[(1-\zeta)E]}_{\in C^\infty} * v}_{\in C^\infty}$$

$$\operatorname{supp} (\zeta E) * v \subset \operatorname{supp} \zeta E + \operatorname{supp} v.$$

[1] Notes by Ibrahim Al Balushi

2. By Cauchy- Kovalevskaya, there exists $h \in C^\omega(N),\ N \in \mathscr{N}(y)$, i.e $P(\partial)h = f$,

$$\implies P(\partial)(u-h) = 0$$

in a neighbourhood of y. Without loss of generality, we can assume $f = 0$. Define

$$w = [(1-\zeta)E] * v.$$

We want to show analyticity.

$$\partial^\alpha w = [(1-\zeta)\partial^\alpha E] * v + e * v$$

where e is a residual from $\partial^\alpha(1-\zeta)$. For $K \subset \Omega$ compact

$$\partial^\alpha w(z) = \int_K [1-\zeta(x)]\partial^\alpha E(x) v(z-x)\ dx + \int e(z)v(x-z)\ dx$$

If $z \in \mathcal{N}(y),\ e(z) = 0.\ z \in B_\delta(y)$.

$$K \subset \{|x| > \epsilon/2\} \cap \underbrace{(\{z\} - \text{supp } v)}_{\subset B_\delta(y) - \text{supp } v}$$

$$|\partial^\alpha w(z)| \leq C \sup_K |\partial^\alpha E| \cdot \|v\|_{L^1(\tilde{K})}$$

where $\tilde{K} = \{|x| > \epsilon/2\} \cap (B_\delta(y) - \text{supp } v)$. Analyticity is equivalent to

$$E \in C^\omega(\Omega) \iff \forall K \subset \Omega \textit{ compact } \exists r > 0,\ c \textit{ s.t } \sup_K |\partial^\alpha E| \leq C\alpha!/r^{|\alpha|}, \forall \alpha$$

Thus E analytic and so is w.

□

Clarification

$$P(\partial)(\chi_u) = \chi P(\partial)u + v$$

$v \in C_c^\infty$ due to (a).

$$w = [(1-\xi)E] * v \implies \partial^\alpha w = [(1-\xi)\partial^\alpha E] * v + e * v$$

where e is zero except on a small neighbourhood of the origin.
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Laurent Expansion

$u \in C(\Omega \backslash \{0\})$, with $0 \in \Omega$. We want to define a distribution $\hat{u} \in \mathscr{D}'(\Omega)$ that extends u. I.e such that

$$\langle \hat{u}, \varphi \rangle = \int u\varphi, \qquad \forall \varphi \in \mathscr{D}(\Omega \backslash \{0\}). \tag{1}$$

For example $u = \frac{1}{x^{1-\alpha}}$ and suppose $|u(x)| = \mathcal{O}(|x|^{-N})$ as $x \to 0$.

We define regularization

$$\langle \hat{u}, \varphi \rangle = \int_{|x|\leq\epsilon} u(x) \left[\underbrace{\varphi(x) - \sum_{|\alpha|\leq N} \frac{\varphi^{(\alpha)}(0)}{\alpha!} x^\alpha}_{=:\Psi(x)} \right] dx + \int_{|x|>\epsilon} u(x)\varphi(x)\ dx \tag{2}$$

Fig

$\varphi \in \mathscr{D}(\Omega\backslash\{0\})$ then all derivatives $\partial^\alpha \varphi$. In particular the sum $\sum_{|\alpha|\leq N} \frac{\varphi^{(\alpha)}(0)}{\alpha!} x^\alpha$ vanishes. We obtain

$$\langle \hat{u}, \varphi \rangle = \int_{|x|\leq\epsilon} u(x)\varphi(x)\ dx + \int_{|x|>\epsilon} u(x)\varphi(x)\ dx = \int_{\Omega\backslash\{0\}} u\varphi \tag{3}$$

It is left to show that continuity of $\hat{u}(\varphi)$. Derivatives of Ψ at zero, $\partial^\alpha \Psi(0) = 0$, for all $|\alpha| \leq N$,

$$\implies \Psi(x) = \mathcal{O}\left(|x|^{N+1} \cdot \|\varphi\|_{C^{N+1}B_\epsilon}\right)$$

and thus $|\langle \hat{u}\varphi \rangle| \leq C \cdot \|\phi\|_{C^{N+1}(\Omega)}$.

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Application

Suppose $P(\partial)$ is hypoelliptic. $P(\partial)u = 0$ in $\Omega\backslash\{0\}$ with $\text{supp}P(\partial)\hat{u} = \{0\}$,

$$\implies P(\partial)\hat{u} = \sum_\alpha a_\alpha \delta^{(\alpha),} \qquad |\alpha| < \infty.$$

Let $v = \sum_\alpha a_\alpha E^{(\alpha)}$. Then

$$P(\partial)v = \sum_\alpha a_\alpha P(\partial) E^{(\alpha)} = \sum_\alpha a_\alpha \delta^{(\alpha)} = P(\partial)\hat{u}, \tag{4}$$

$$\begin{aligned} &\implies P(\partial)(\hat{u} - v) = 0 \\ &\implies \hat{u} - v = w \in C^\infty(\Omega) \qquad P(\partial) \text{ hypoelliptic} \\ &\implies \hat{u} = \sum_\alpha E^{(\alpha)} + w \\ &\implies u(x) = w(x) + \sum_\alpha a_\alpha E^{(\alpha)}(x), \qquad x \neq 0 \end{aligned}$$

while noting that $P(\partial)w = 0$.

Example 1. *Let $P(\partial)$ denote the Cauchy-Riemann operator and take $E(z) = \frac{1}{\pi z}$.*

$$E^{(\alpha)}(z) = \frac{1}{z^k}$$

Example 2. *Let $P(\partial)$ denote the Laplacian operator for $n = 2$. Take $E(x) = \log|x|$ then*

$$E^{(\alpha)}(x) = \frac{1}{|x|^{|\alpha|}} P(x).$$