Lecture[1] 10

Some preliminaries

- Define translate operator τ and the reflection of a function v respectively:

$$(\tau_x v)(y) = v(y-x) \qquad \tilde{v}(y) = v(-y).$$

- We previously defined for $v \in L^1_c$ and $\phi \in \mathscr{D}$:

$$< v * u, \phi > = < u, \tilde{v} * \phi >,$$

- By direct computation we have the relation:

$$\tilde{v}(x-y) = v(y-x) = \tau_x v(y) = \tau_y \tilde{v}(x).$$

Theorem 1. *Let $u \in \mathscr{D}$. The operator $C_u : \mathscr{D} \to \mathscr{E}$ defined by*

$$C_u v = v \mapsto v * u$$

is continuous. Moreover,

a) $(v * u)(x) = < u, \tau_x \tilde{v} >$.

b) $supp\ (u * v) \subset supp\ v + supp\ u$.

c) $\partial^\alpha (v * u) = \partial^\alpha v * u = v * \partial^\alpha u$.

Proof. By definition

$$< u * v, \phi > = < u, \tilde{v} * \phi > = < u, \int \tau_y \tilde{v} \cdot \phi(y)\ dy >$$

We will show the last term on the right hand side

$$< u, \int \tau_y \tilde{v} \cdot \phi(y)\ dy > = \int \underbrace{< u, \tau_y \tilde{v} >}_{=:w(y)} \phi(y)\ dy$$

To show this, we use Riemann's theory of integration; that is, we will inspect their Riemann sums. This is possible since $\int \tau_y \tilde{v} \cdot \phi(y) \in \mathscr{D}$ in the argument of the LHS, and $< u, \tau_y \tilde{v} >$ in the RHS is continuous; $\tau_{y+\epsilon} \tilde{v} \to \tau_y \tilde{v}$ in $\mathscr{D}$. The Riemann sums of the expressions, with respect to volume segments $\Delta \nu$ are

$$< u, \sum_i \tau_{y_i} \tilde{v} \cdot \phi(y_i) \Delta \nu_i > = \sum_i < u \tau_{y_i}, \tilde{v} > \phi(y_i) \Delta \nu_i$$

which is true by *linearity*. Expressions converge to the limit indicated above. Thus,

$$< u * v, \phi > = \int < u, \tau_y \tilde{u} > \phi(y)\ dy, \qquad \forall \phi \in \mathscr{D}$$

hence $(v * u)(y) = w(y) = < u, \tau_x \tilde{v} >$ which proves a). To show $v * u \in \mathscr{E}$, it suffices to show $v * u \in C^\infty$. Consider the definition of the derivative:

$$w(y+h) - w(y) = < u, \tau_{y+h} \tilde{v} - \tau_y \tilde{v} > = < u, [\tau_x v](y+h) - [\tau_v](y) >$$

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$$\implies \partial^\alpha w = < u, \partial_y^\alpha \tau_y \tilde{v} >$$

and therefore $w \in C^\infty$. To prove $c)$,

$$\partial_y^\alpha \tau_y \tilde{v} = \partial_y^\alpha v(y-x) = [\partial^\alpha v](y-x) = \tau_y \widetilde{\partial^\alpha v}$$

$$\implies \partial^\alpha (v * u) = < u, \tau_y \widetilde{\partial^\alpha v} > = (\partial^\alpha v) * u.$$

On the other hand,

$$[\partial^\alpha u](\tau_y \tilde{v}) = (-1)^{|\alpha|} < u, \partial^\alpha \tau_y \tilde{v} > . \tag{1}$$

The function in the RHS is explicitly,

$$\partial_x^\alpha v(y-x) = (-1)^{|\alpha|}[\partial^\alpha v](y-x) = (-1)^{|\alpha|} \tau_y \widetilde{\partial^\alpha v}(x)$$

and so carrying on from RHS of (1)

$$(-1)^{|\alpha|} < u, \partial^\alpha \tau_y \tilde{v} > = < u, \tau_y \widetilde{\partial^\alpha v} >$$

$$\implies v * \partial^\alpha u = [\partial^\alpha v] * u$$

hence $c)$. To show $b)$ we have

$$w(y) = u(\tau_y \tilde{v}) = 0 \quad if \quad supp\ u \bigcap supp \underbrace{\tau_y \tilde{v}}_{=v(y-x)} = \emptyset.$$

Finally to show continuity of $C_u : \mathscr{D} \to \mathscr{E}$, it suffices to shoe $C_u : \mathscr{D}_K \to \mathscr{E}$ is continuous for any K compact. Let $\| \cdot \|_{C^l(K)}$ be a seminorm on $\mathscr{E}$ and $v \in \mathscr{D}(K)$. For $y \in K'$ compact,

$$\tau_y \tilde{v} = v(y-x) \in \mathscr{D}(K' - K)$$

using the fact that $u : \mathscr{D}(K' - K) \to \mathbb{R}$ is continuous,

$$\underbrace{| < u, \tau_y \tilde{v} > |}_{=(v*u)(y)} \leq C \|\tau_y \tilde{v}\|_{C^m(K'-K)} = C \sup_{\substack{|\alpha| \leq m \\ x \in K'-K}} |\partial^\alpha v(y-x),$$

for some m. Noting that by $a)$ and using the previous estimate,

$$\begin{aligned} \|v * u\|_{C^l(K')} &= \sup_{y \in K'} \{| < u, \partial_y^\beta \tau_y \tilde{v} > | : |\beta| \leq l\} \\ &\leq C \sup_{\substack{y \in K' \\ x \in K'-K}} \{|\partial_y^\beta \partial_x^\alpha v(y-x)| : |\alpha| \leq m,\ |\beta| \leq l\} \end{aligned}$$

$$\begin{aligned} & while\ y \in K', x \in K' - K \implies y - x \in K, \\ & \leq C \|v\|_{C^{m+l}(K)}. \end{aligned}$$

□

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The following computation reveals another definition:

$$\begin{aligned} < C_u v, \phi > &=< u, \tilde{v} * \phi > \\ &=< u, \widetilde{v * \tilde{\phi}} > \\ &=< \tilde{u}, v * \tilde{\phi} > \\ &=< C_{\tilde{u}} \phi, v >, \end{aligned}$$

thus we may define:

$$< \tilde{u}, \phi >=< u, \tilde{\phi} > .$$

Definition 1. *Let* $u \in \mathscr{D}'$ *and* $v \in \mathscr{E}'$*.*

$$< v * u, \phi >=< v, \phi * \tilde{u} >=< v, C_{\tilde{u}} \phi >, \qquad \phi \in \mathscr{D}.$$

Moreover, $v * u \in \mathscr{D}'$

Example

$$< \delta * u, \phi >=< \delta, C_{\tilde{u}} \phi >=< \delta, \tilde{u}(\tau_y \tilde{\phi}) >= \tilde{u}(\tilde{\phi}) = u(\phi).$$

$$\implies \delta * u = u.$$

Define : $u * v = v * u$.

Fact:

$$< u * v, \phi >=< u, C_{\tilde{v}} \phi >, \qquad v \in \mathscr{E}'$$

$$\begin{aligned} < \partial^\alpha (v * u), \phi > &= (-1)^{|\alpha|} < v * u, \partial^\alpha \phi > && (2) \\ &= (-1)^{|\alpha|} < v, \partial^\alpha \phi * \tilde{u} > && (3) \\ &= (-1)^{|\alpha|} < v, \phi * \partial^\alpha \tilde{u} > && (4) \\ &=< v, \phi * \widetilde{\partial^\alpha u} > && (5) \end{aligned}$$

Constant Coefficient Operators

Consider the finite sum

$$P(\xi) = \sum_\alpha a_\alpha \xi^\alpha.$$

We define the linear differential operator:

$$P(\partial) = \sum_\alpha a_\alpha \partial^\alpha.$$

Definition 2. *$E \in \mathscr{D}'(\Omega)$ is called a fundamental solution of $P(\delta)$ if*

$$P(\partial)E = \delta.$$

Examples

- Take $P(\xi) = \xi_1^2 + \cdots + \xi_n^2$.

  $$P(\partial) = \Delta.$$

  $$E(x) = \frac{1}{(2-n)|S^{n-1}||x|^{n-2}}$$

  is a fundamental solution since

  $$< \Delta E, \phi >=< E, \Delta\phi >= \phi(0).$$

- $f \in \mathscr{E}'$. $u = f * E$

  $$P(\partial)u = P(\partial)(f * E) = f * P(\partial E) = f * \delta = f.$$

Definition 3. *$P(\partial)$ is* ***hypoelliptic*** *if the following property holds:*

$$\mathcal{U} \subset \text{ open and } P(\partial)u \in C^\infty(\mathcal{U}) \implies u \in C^\infty(\mathcal{U}).$$

$P(\partial)$ is hypoelliptic, $P(\partial)E = \delta$ then $E \in C^\infty(\mathbb{R}^n \backslash \{0\})$.

Theorem 2 (Schwartz)**.** *If there exists $E \in \mathscr{D}' \cap C^\infty(\mathbb{R}^n \backslash \{0\})$ such that $P(\partial)E = \delta$, then $P(\delta)$ is hypoelliptic.*