Math 581: Partial Differential Equations 2 Notes

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Lecture 1

Distributions (Generalized Functions)

The existence of non-differentiable functions posses difficulty when subjected to calculus operators such as linear differential operators. The theory of distributions was developed to overcome those problems. This is done by extending the class of continuous functions into a larger class which preservers results from the classical formulation, while overcoming problematic situations due to points of non-differentiability. Such extension procedures are abundant in mathematics: from integers to rational numbers, from rationals to reals, from reals to complex numbers etc.

One of the goals is to have an extension $\mathcal{D}' \supset C$ such that any differential operator with smooth coefficients

$$a_\alpha \partial^\alpha : \mathcal{D}' \to \mathcal{D}' \quad \textit{is continuous.}$$

In the case

$$a_\alpha \partial^\alpha : C^k \to C^{k-m}$$

we have linearity, where we note the spaces are not identical. A theorem tells us

$$a_\alpha \partial^\alpha : X \to X \text{ and } X \text{ normable} \implies \text{operator not continuous},$$

hence if such an extension $\mathcal{D}'$ exists, then it must be not normable. As a consequence, we at least need to be concerned with general metric spaces, or better, with topological vector spaces.

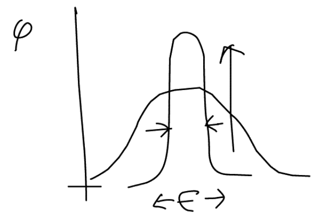
The first step to construct such an extension, we first need to reinterpret the notion of a function map. In particular instead of considering the function f as taking values $x \in \mathbb{R}^n$ to the value $f(x)$, we consider it as an assignment which maps suitable 'test' functions to their integrals against f.

Let $f \in C(\Omega)$ and define $T_f : C_o^\infty(\Omega) \to \mathbb{R}$ by

$$T_f(\phi) = \int_\Omega f\phi$$

Notice if we infinitesimally concentrate ϕ around x and we recover the value $f(x)$.

The *functional* T_f exists for any continuous or locally integrable *function f*. The integral for $T_f(\phi)$ defines a *linear functional* on $C_o^\infty(\Omega)$, with values in $\mathbb{R}$. Moreover, the values of T_f for varying ϕ determine the function $f(x)$ uniquely when f is continuous [FRITZ], p.90.



Notations and Definitions

Let Ω be an open set in $\mathbb{R}^n$ and let $K \subset \Omega$ compact. We define Schwartz notation:

$$\mathcal{D}(\Omega) = C_o^\infty(\Omega) \qquad \mathcal{E}(\Omega) = C^\infty(\Omega) \qquad \mathcal{D}(K) = C_o^\infty(K).$$

$\mathcal{D}'(\Omega)$ 'continuous' linear functionals on $\mathcal{D}(\Omega)$.

$C(\Omega) \subset \mathcal{D}'(\Omega)$.

$$f \in L^1_{loc}(\Omega) = \{u : \ u|_K \in L^1(K), \ \forall K \subset \Omega \ compact\}.$$

Note: a natural consequence: $T_f(\phi) = \int_\Omega f \cdot \phi$ induces $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$.

$\delta(\phi) = \phi(0), \ \delta \in \mathcal{D}'(\Omega)$.

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n : \ \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < \infty, \ \forall \alpha, \beta\}$$

$\mathcal{S}$ is known as the *Schwatrz class.*

Definition 1 (Convergence)**.** $\phi_j \to 0$ *in* $\mathcal{E}(\Omega)$ *if and only if*

$$P_{m,K}(\phi_j) = \|\phi_j\|_{C^m(K)} \to 0, \qquad \forall K \ compact, \forall m.$$

$f \in \mathcal{E}'(\Omega)$ *is and only if* $f : \mathcal{E}(\Omega) \to \mathbb{R}$ *linear, and*

$$\phi_j \to 0 \implies f(\phi_j) \to 0$$

$\phi_j \to 0$ *in* $\mathcal{D}(\Omega)$ *if and only if*

$$\exists K \subset \Omega \ s.t \ \phi_j \in \mathcal{D}(K) \ and \ \phi_j \to 0 \ \in \mathcal{D}(K)$$

$f \in \mathcal{D}'(\Omega)$ *if and only if* $f : \mathcal{D}(\Omega) \to \mathbb{R}$ *linear,*

$$\phi_j \to 0 \ in \ \mathcal{D}(\Omega) \implies f(\phi_j) \to 0$$

$\mathcal{D}'(\Omega)$-distributions.
$\mathcal{S}'(\mathbb{R}^n)$-tempered distributions.
$\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega)$-compactly supported.

Topological Vector Spaces

Let X be a set. We define a collection τ of subsets of X a **Topology** on X should τ satisfy the following properties:

- $\emptyset, X \in \tau$
- $a, b \in \tau \implies a \cap b \in \tau$
- $A_\alpha \in \tau \implies \cup_\alpha A_\alpha \in \tau$

An element of τ is said to be an *open set*.

Definition 2. *$f : X \to Y$ continuous if and only if $U \subset Y$ open $\implies f^{-1}(U)$ open.*

Topological Vector spaces: vector spaces with a topology such that addition and scalar multiplications are continuous. Usually assume the *Hausdroff property* (any two distinct points in X have disjoint open nbhds).

Lemma 1. *$T_a : X \to X$ and $M_\lambda : X \to X$ respectively defined by*

$$x \mapsto a + x,$$

and

$$x \mapsto \lambda x$$

are homeomorphisms of X onto X, $\forall x \in X, \lambda \in \mathbb{R} \backslash \{0\}$.

Corollary 1. *τ is translation invariant:*

$$U \in \tau \implies T_a U \in \tau, \ \forall a \in X.$$

Some terminology:

Base σ : $\forall U \in \tau$ is the union of elements from σ.

Local base σ : $\forall U \in \tau$ is the union of translates of elements from σ.

Types of topological vector spaces:

$A \subset X$ is **bounded** if and only if $\forall$ open nbhd U of 0, $\exists \lambda \in \mathbb{R}$ such that $A \subset \lambda U$.

X is **locally convex** if $\exists$ local base whose elements are convex.

X is **locally bounded** if 0 has a bounded nbhd.

X is **locally compact** if 0 has a nbhd whose closure is compact.

The following are some facts:

- locally bounded (LB) $\implies \exists$ countable local base.
- metrizable $\Leftrightarrow \exists$ countable local base.

- normable $\Leftrightarrow$ locally bounded (LB) and locally convex (LC).
- dim$< \infty \Leftrightarrow$ locally compact.
- locally bounded (LB) and Heine-Borel (HB) $\implies$ dim$< \infty$.
- $\mathcal{E}, \mathcal{S}, \mathcal{D}$ have HB property. $\implies$ not normable.

References

[RUDIN] Walter Rudin, *Functional Analysis*, McGraw-Hill Inc. Second Edition (1991).

[FRITZ] Fritz John, *Partial Differential Equations*, Springer-Verlag New York Inc. Fourth Edition (1981).