1 Fundamental Matrix Solution $e^{P(\xi)t}/e^{At}$

[1]

Motivation

The term $e^{P(D_x)t}f$ or $e^{P(\xi)t}f$ appears often. These terms essentially defines a notation to express the solution of systems of differential equation. The notation is motivated by the following:

Consider the Cauchy problem for the ODE for scalar function $u : \mathbb{R} \to \mathbb{R}$,

$$\begin{cases} \frac{du}{dt} = au \quad t \in \mathbb{R} \\ u(0) = c \end{cases} \tag{1}$$

Then the solution has the form $u(t) = e^{at}c$. Analogously, we write the solution of the Cauchy problem for the linear system of ODEs for a vector valued function $u : \mathbb{R} \to \mathbb{R}^m$,

$$\begin{cases} \dot{u} = Au \\ u(0) = g \end{cases} \qquad \textbf{Sol}: \ \mathbf{u(t) = e^{At}g} \tag{2}$$

However although e^{At} is not 'defined' by taking $m \times m$ matrix as an exponent, it however possesses the following properties, and can be manipulated in a manner described below.

Definition 1. *The* ***matrix exponential*** e^{At} *of a* $A \in M_{m\times m}(\mathbb{C})$ *is an* $m \times m$ *matrix defined formally by the series*

$$e^{At} \equiv I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!} + \cdots \tag{3}$$

The reason for this definition is motivated not only by the structure resemblance indicated in (2), but also since this holds for all $t \in \mathbb{R}$ by virtue of the absolute convergence of the Taylor series for e^{at} on the entire real line.

1.1 Properties

- The absolute convergence of e^{At} for all square matrices A and all $t \in \mathbb{R}$ justifies

$$\frac{d}{dt}e^{At} = Ae^{At} \tag{4}$$

  which can be verified by differentiation each term in the series definition of e^{At}.

- The matrix exponential e^{At} shares technical similarities with scalar e^{at}:

$$(e^{At})^{-1} = e^{-At} \quad \text{and} \quad e^{A(t+s)} = e^{At}e^{st} \tag{5}$$

  The verification for those are possible, however complicated (especially for the inverse) as they require using the series definition. Note that $e^{At}e^{tB} = e^{(A+B)t} \iff AB = BA$.

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Fundamental Matrix Solution e^{At}

Consider the following for constant vector $g \in \mathbb{R}^m$ and $A \in M_{m\times m}(\mathbb{C})$, a simple computation reveals if $u = e^{At}g$ then

$$\dot{u} = \frac{d}{dt}e^{At}g = Ae^{At}g = Au.$$

It follows that $e^{At}g$ solves the system of ODEs $\dot{u} = Au$.

Consider the following

$$e^{At}g = e^{(A-\lambda I)t}e^{\lambda I}g \tag{6}$$

$$\begin{aligned} e^{\lambda It}g &= \left[I + \lambda It\frac{\lambda^2 I^2 t^2}{2!}g + \cdots\right]g \\ &= \left[1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots\right]g = e^{\lambda t}g \end{aligned}$$

It follows that

$$e^{At}g = e^{\lambda t}e^{(A-\lambda I)t}g. \tag{7}$$

It is important to note that the factor $e^{\lambda t}$ on the left is a scalar function which comes up when solving ODEs of the form $\dot{u} = Au$ for scalar and matrix A. It is left to examine $e^{(A-\lambda I)t}g$.

From linear algebra if g satisfies $(A-\lambda I)^p g = 0$ for some positive integer p, then $(A - I\lambda)^{p+l}g = 0$ for all $l \geq 0$. In particular, suppose this for some p, then by Definition (1) the series terminates after the first p terms:

$$e^{(A-\lambda I)t}g = g + t(A-\lambda I)g + \cdots + \frac{t^{p-1}}{(p-1)!}(A-\lambda I)^{p-1}g.$$

$$\begin{aligned} e^{At}g &= e^{\lambda t}e^{(A-\lambda I)t}g \\ &= e^{\lambda t}\left[g + t(A-\lambda I)g + \cdots + \frac{t^{p-1}}{(p-1)!}(A-\lambda I)^{p-1}g\right] \in \mathbb{C}^m. \end{aligned} \tag{8}$$

Where we note that g is an eigenvector of A with eigenvalue λ.

Theorem 1. *[BRAUN] Let $U(t)$ be a **fundamental matrix solution** of a differential equation $\dot{u} = Au$. Then,*

$$e^{At} = U(t)G^{-1}(0), \tag{9}$$

equivalently, we can consider $U(t)$ as solution vector

$$U(t) = e^{At}G(0). \tag{10}$$

In other words, the product of any fundamental matrix solution of $\dot{u} = Au$ with inverse at $t = 0$ must yield e^{At}.

It is easy to see now that $e^{P(\xi)t}$ defines the fundamental matrix solution described above for the ODE system (in Fourier space) in a more general setting, meanwhile $e^{P(D_x)t}$ defines the Fourier inverse of the solution $e^{P(\xi)t}$.

1.2 Canonical Examples

- Not diagonalized/Jordan Canonical form/ Repeated roots:

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \implies e^{At}g = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} g$$

- Non defected matrix/no repeated roots:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \implies e^{At}g = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} g = e^{\lambda_1 t} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g + e^{\lambda_2 t} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} g$$

(Recall Spectral Projectors.)

References

[BRAUN] Matrin Braun, *Differential Equations and Their Applications*, Springer Inc. Forth Edition.