Introduction to Einstein Equations

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1 Abstract

In this note, I will first recall basics of General Relativity and Einstein equation. We aimed at setting out the basic mathematical framework for general relativity. Then we will briefly talk about Einstein equations and their solutions, regular ellipticity and hyperbolicity, Cauchy problem for Einstein equations and the decomposition of Einstein equations. Finally, if time permitted, we will talk about in details of rough solutions of the Einstein Constriant equations on compact manifold based on a nicely presented paper by David Maxwell.

2 Basics

The theory of General relativity is a theory that unified space, time and gravitation. Notice that our understanding of structure of space time has been renovated by Albert Einstein and others in early 20th century contrary to people's intuitive notion of absolute space and time. Therefore, since everything now is not absolute, it becomes more mathematically complicated. At first , special relativity is introduced, but it treated only uniform motion that we wish to develop a more advanced theory that could correct all logical inconsistency of SR. That is General Relativity, since the large structure is uncertain and locally, one may use SR. One naturally relates this to the theory of manifold.

Definition 1 *A spacetime manifold is a 4 dimensional oriented differentiable manifold M , endowed with a Lorentzian metric g.*

Definition 2 *A Lorentzian metric g is a continuous assignment of a non-degenerate quadratic form g_p, of index 1 , in T_pM at each p of M, where T_pM is the tangent space of M at p and non-degenerate means $g(X,Y) = 0 \forall Y \in T_pM \Rightarrow X = 0$.*

Definition 3 *A quadratic form g_p in T_pM is called Lorentzian if there exists a vector $V \in T_pM$ such that $g_p(V,V) < 0$ while setting*

$$\Sigma_V = \{X; g_p(X,V) = 0\},$$

$g_p|_{\Sigma_V}$ is positive definite.

Now I assume the reader have knowledge of tensors and differential forms. We recall

Definition 4 *The covariant derivative of vector V is given by*

$$\bigtriangledown_\mu \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda.$$

While for a one form ,it is given by

$$\bigtriangledown_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda.$$

In general, for a tensor, we have

$$\bigtriangledown_\sigma T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} = \partial_\sigma T^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} +$$

$$\Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}_{\nu_1\cdots\nu_l} + \Gamma^{\mu_2 T^{\mu_1\lambda\cdots\mu_k}_{\nu_1\cdots\nu_l}}_{\sigma\lambda} + \cdots$$
$$-\Gamma^\lambda_{\sigma\nu_1} T^{\mu_1\cdots\mu_k}_{\lambda\cdots\nu_l} - -\Gamma^\lambda_{\sigma\nu_2} T^{\mu_1\cdots\mu_k}_{\nu_1\lambda\cdots\nu_l} - \cdots.$$

where the Christoffel symbol is given as $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\beta\nu}) + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}$ Now we define Ricci tensors and scalars:

$$R^\alpha_{\mu\lambda\nu} = \partial_\lambda \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\lambda} + \Gamma^\alpha_{\beta\lambda}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\lambda} \tag{1}$$

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{2}$$

$$R = g^{\mu\nu} R_{\mu\nu} \tag{3}$$

The Einstein equation relates spacetime and matter. It interprets gravity as curvature of spacetime which gives :

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2T_{\mu\nu} \tag{4}$$

Here $T_{\mu\nu}$ is the energy momentum tensor of matter, $G_{\mu\nu}$ the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor and R the scalar curvature of the metric $g_{\mu\nu}$. Also from the Bianchi identity

$$\bigtriangledown_{\alpha} R_{\beta\gamma\delta\epsilon} + \bigtriangledown_{\beta} R_{\gamma\alpha\delta\epsilon} + \bigtriangledown_{\gamma} R_{\alpha\beta\delta\epsilon} = 0 \tag{5}$$

One obtains

$$\bigtriangledown^{\nu} G_{\mu\nu} = 0 \tag{6}$$

and the twice contracted Bianchi identity implies

$$\bigtriangledown^{\nu} T_{\mu\nu} = 0 \tag{7}$$

So the Einstein vacuum equations

$$G_{\mu\nu} = 0 \tag{8}$$

correspond to $T_{\mu\nu} = 0$ which are equivalent to $R_{\mu\nu} = 0$.
Denoting by P.P the principal part which is the part containing the highest derivatives of the metric , we get:

$$P.P\{R_{\mu\nu}\} = \frac{1}{2} g^{\alpha\beta} (\partial_{\mu}\partial_{\alpha} g_{\beta\nu} + \partial_{\nu}\partial_{\alpha} g_{\beta\mu} - \partial_{\mu}\partial_{\nu} g_{\alpha\beta} - \partial_{\alpha}\partial_{\beta} g_{\mu\nu}). \tag{9}$$

We shall now consider the *symbol* of an Einstein equation. The symbol is defined by replacing in the principal part $\partial_{\mu}\partial_{\nu} g_{\alpha\beta}$ by $\xi_{\mu}\xi_{\nu}\dot{g_{\alpha\beta}}$. We then obtain the symbol σ_{ξ} at point $p \in M$ and a covector $\xi \in T_p M^*$, for given metric g:

$$(\sigma_{\xi}, \dot{g})_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\xi_{\mu}\xi_{\alpha}\dot{g}_{\beta\nu} + \xi_{\nu}\xi_{\alpha}\dot{g}_{\beta\mu} - \xi_{\mu}\xi_{\nu}\dot{g}_{\alpha\beta} - \xi_{\alpha}\xi_{\beta}\dot{g}_{\mu\nu}). \tag{10}$$

We will denote

$$(i_{\xi}\dot{g})_{\nu} = g^{\alpha\beta}\xi_{\alpha}\dot{g}_{\beta\nu},$$

$$(\xi, \xi) = g^{\alpha\beta}\xi_{\alpha}\xi_{\beta},$$

$$(\xi \otimes \zeta)_{\mu\nu} = \xi_{\mu}\xi_{\nu},$$

$$g^{\alpha\beta}\dot{g}_{\alpha\beta} = tr\dot{g}.$$

Write

$$(\sigma_\xi \dot{\vdots} g) = \frac{1}{2}\{\xi \otimes i_\xi \dot{g} + i_\xi \dot{g} \otimes \xi - tr\dot{g}\xi \otimes \xi - (\xi, \xi)\dot{g}\}. \quad (11)$$

We introduce the notion of the symbol of a system of Euler-Lagrange equations. Let us denote x, the independent variables , $x^\mu, m = 1, \cdots, n$ by q, the independent variables $q^\alpha, a = 1, \cdots, m$ by v. Then for the Lagrangian $L = L(x, q, v)$

We have a set of solution of the Euler Lagrange equations , if substituting

$$q^a = u^a(x),$$

$$v_\mu^\alpha = \frac{\partial u^a}{\partial x^\mu}(x). \Rightarrow$$

$$\frac{\partial}{\partial x^\mu}(\frac{\partial L}{\partial v_\mu^a}(x, u(x), \partial u(x))) - \frac{\partial L}{\partial q^a}(x, u(x), \partial u(x)) = 0. \quad (12)$$

Define $p_a^\mu = \frac{\partial L}{\partial v_\mu^a}$, $f_a = \frac{\partial L}{\partial q^a}$. Then , the Euler Lagrange equation become

$$\frac{\partial p_a^\mu}{\partial x_\mu} = f_\alpha \quad (13)$$

The principal part of the equation is

$$h_{ab}^{\mu\nu} \frac{\partial^2 u^b}{\partial x_\mu \partial x_\nu}(x, u(x), \partial u(x)),$$

where

$$h_{ab}^{\mu\nu} = \frac{\partial^2 L}{\partial v_\mu^a \partial v_\nu^b}(x, q, v).$$

The equation of variation, these are the linearized equations, has principal part is

$$h_{ab}^{\mu\nu}(x, u(x), \partial u(x)) \frac{\partial^2 \dot{u}^b}{\partial x^\mu x^\nu}$$

where we denote by $\dot{u}^a$, the variation of functions u^a .

Consider the oscillatory solutions $\dot{u}^a = \dot{w}^a e^{i\psi}$ of the equation of variation, writing $\frac{\psi}{\epsilon}$ in place of ψ and substitute back into the linearized equations and keeping only the leading terms as ϵ goes to zero, we obtain

$$h^{\mu\nu}_{ab}(x, u(x), \partial u(x))\dot{w}^b \frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu} = 0 \tag{14}$$

The left hand is the symbol $\sigma_\xi \dot{w}$ where $\xi_\mu = \frac{\partial\psi}{\partial x^\mu}$. Hence the symbol of the Euler Lagarange equations is in general given by

$$(\sigma_\xi \dot{u})^a = h^{\mu\nu}_{ab}\xi_\mu\xi_\nu \dot{(u)}^b = \ chi_{ab}(\xi)\dot{u}^b \tag{15}$$

where

$$\chi_{ab}(\xi) = h^{\mu\nu}_{ab}\xi_\mu\xi_\nu.$$

Globally, the x^μ are the local coordinates on an n dimensional manifold M and x denotes an arbitrary point , q^a are local coordinates on an m dimensional manifold N and q arbitrary point on N. The unknown u is a mapping from M to N and the function $(u^a(x), a = 1, \cdots, m)$ describes the mapping locally.

Definition 5 *Let M be an n-dimensional manifold. The characteristic subset $C^*_x \subset T^*_x M$ defined by*

$$\{C^*_x = \{\xi \neq 0 \in T^*_x : nullsp(\sigma_\xi) \neq 0\} = \{\xi \neq 0 \in T^*_x : \det \chi(\xi) = 0.\}$$

Example:
We see that the linear wave equation $g^{\mu\nu} \bigtriangledown_\mu (\partial_\nu u) = 0$ which arises from the Lagrangian $L = \frac{1}{2}g^{\mu\nu}v_\mu v_\nu$. The symbol is $\sigma_\xi \dot{u} = (g^{\mu\nu})\xi_\mu\xi_\nu$ and the character is

$$C^*_x = \{\xi \neq 0 \in T^*_x M : (\xi, \xi) = g^{\mu\nu}\xi_\mu\xi_\nu = 0\}$$

Now return to Einstein equations ,set

$$\dot{g} = \zeta \otimes \xi + \xi \otimes \zeta \tag{16}$$

for any $\xi \in T^*_x M$, then

$$i_\xi \dot{g} = (\zeta, \xi)\xi + (\xi, \xi)\zeta; tr\dot{g} = 2(\zeta, \xi) \tag{17}$$

We have that $\sigma_\xi \dot{g} = 0$ Therefore the null space of σ_ξ is nontrivial for every covector ξ . The degeneracy is due to that the equation are generally covariant. If g is a solution of the Einstein equation and f is a diffeomorphism of the manifold onto itself, then X generates a 1-parameter group $\{f_t\}$ of

diffeomorphisms of M and $L_X g = \frac{d}{dt} f_t^* g|_{t=0}$, the Lie derivative with respect to X of g , is a solution of the linearized equations.
Note that the Lie derivative of g with respect to a vector field X is given by

$$(L_X g)(Y, Z) = g(\bigtriangledown_Y X, Z) + g(\bigtriangledown_Z X, Y) \tag{18}$$

Let $Y = E_\mu$ and $Z = E_\nu$, where $\{E_\mu\}$ is an arbitrary frame , write

$$(L_X g)_{\mu\nu} = \bigtriangledown_\mu X_\nu + \bigtriangledown_\nu X_\mu \tag{19}$$

here $X_\mu = g_{\mu\lambda} X^\lambda$. Now the symbol of a Lie derivative is given by

$$\dot{g}_{\mu\nu} = \xi_\mu \zeta_\nu + \xi_\nu \zeta_\mu \tag{20}$$

where $\zeta_\mu = \dot{X}_\mu$. A simple analogue is given by Maxwell equations for the electromagnetic field $F_{\mu\nu}$,

$$\bigtriangledown^\nu F_{\mu\nu} = g^{\nu\lambda} \bigtriangledown_\lambda F_{\mu\nu} = 0, \tag{21}$$

Recall that $F = dA$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the electromagnetic potential , a 1-form. The Maxwell equations are the Euler Lagrange equations of Lagrangian

$$L = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{22}$$

where $F^{\mu\nu} = g^{\mu\kappa} g^{\nu\lambda} F_{\kappa\lambda}$. The symbol of these equation is

$$(\sigma_\xi \dot{A})_\mu = g^{\nu\lambda} \xi_\lambda (\xi_\mu \dot{A}_\nu - \xi_\nu \dot{A}_\mu) \tag{23}$$

That is ,

$$(\sigma_\xi \dot{A}) = (\xi, \dot{A})\xi - (\xi, \xi)\dot{A} \tag{24}$$

By looking at the variation,

$$\dot{A} = \lambda \xi.$$

for any real $\lambda \Rightarrow$

$$\sigma_\xi \dot{A} = 0$$

we also have degeneracy here and the null space of σ_ξ is nontrivial for all $\xi \in T_x^* M$. Notice that this is due to the gauge invariance of the Maxwell equations. If A is a solution , so is

$$\tilde{A} = A + df$$

In fact, $\tilde{A}$ is considered to be equivalent to A. By linearity, $\dot{A}_\mu = \partial_\mu f$ is a solution of the linearized equations, for any function f. To remove degeneracy, we shall factor out these trivial equations. At the level of the symbol , the gauge transformation is

$$\tilde{\dot{A}}_\mu = \dot{A}_\mu + \xi_\mu \dot{f} \tag{25}$$

we then obtain a relation $\dot{A}_1 \sim \dot{A}_2$ if and only if

$$\dot{A}_2 = \dot{A}_1 + \lambda\xi.$$

$Case.1$: $(\xi, \xi) \neq 0$

$$\sigma_\xi \dot{A} = 0 \Rightarrow \dot{A} = \lambda\xi$$

where $\lambda = \frac{(\xi, \dot{A})}{(\xi, \xi)}$ That is $\dot{A} \sim 0$

$Case.2$ $(\xi, \xi) = 0$ In this case ,we may find $\bar{\xi}$ in the same component of the null cone such that $(\bar{\xi}, \bar{\xi}) = -2$. Therefore there exist a unique representative $\dot{A}$ in each equivalence class in Q such that

$$(\bar{\xi}, \dot{A}) = 0.$$

If we take another element $\dot{A}'$ out of equivalence class of $\dot{A}$, i.e $\dot{A}' = \dot{A} + \lambda\xi$ for some λ any real number. Then

$$0 = (\bar{\xi}, \dot{A}') = (\xi, \dot{A}) - 2\lambda.$$

so that $\dot{A}$ is the unique represntation of equivalent class with $(\xi, \dot{A})?0$ The null space of σ_ξ consist of the spacelike 2 dimensional plane, the g-orthogonal complement of the timelike plane π spanned by ξ and $\bar{\xi}$. So, π is the space of the degrees of freedom of electromagnetic field at a point.

Come back to the symbol for Einstein equations, the symbol for the Lie derivative

$$(L_X g)_{\mu\nu} = \bigtriangledown_\mu X_\nu + \bigtriangledown_\nu X_\mu \tag{26}$$

as

$$\xi_\mu \dot{X}_\nu + \xi_\nu \dot{X}_\nu, \tag{27}$$

where $\dot{X}_\mu$ are the component of an arbitrary covector. For the equivalence relation $\dot{g}_1 \sim \dot{g}_2$ implies that

$$\dot{g}_2 = \dot{g}_1 + \zeta \otimes \xi + \xi \otimes \zeta + \ xi \otimes \zeta$$

which gives a quotient space.
Again we distinguish the two cases according as to answer whether the covector ξ satisfies $(\xi, \xi) \neq 0$ or $(\xi, \xi) = 0$
Case.1 $(\xi, \xi) \neq 0$ then $\sigma_\xi \vdots g = 0$ implies that

$$\dot{g} = \zeta \otimes \xi + \xi \otimes \zeta.$$

where $\zeta = \frac{(i_\xi \dot{g} - \frac{1}{2} tr\dot{g}\xi)}{(\xi,\xi)}$ thus $\sigma_\xi$has only trivial null space.
Case.2 $(\xi, \xi) = 0$
now we can choose ξ in the same component of the null cone N_x^* in T_x^*M such that $(\xi, \xi) = -2$. There is a unique representative $\dot{g}$ in each equivalence class $\{\dot{g}\}$ such $i_\xi \dot{g} = 0$. So,

$$\sigma_\xi \vdots g = 0$$

implies thar

$$\xi \otimes i_\xi \dot{g} + i_\xi \dot{g} \otimes \xi - \xi \otimes \xi tr\dot{g} = 0$$

By taking inner product with ξ we see that $(i_\xi \dot{g}, \xi) = (i_\xi \dot{g}, \xi) = 0$, thus

$$-2i_\xi \dot{g} + 2\xi tr\dot{g} = 0.$$

Taking inner product again with ξ, we get

$$-4tr\dot{g} = 0$$

That is $tr\dot{g} = 0$, which yields $i_\xi \dot{g} = 0$. Conversely, $i_\xi \dot{g} = 0$ and $tr\dot{g} = 0$ implies that $\dot{g}$ is in the null space of σ_ξ. Therefore , if $\xi \in N_x^*$, then the null space of trace-free quadratic forms on the 2 dimensional spacelike plane π, the g-orthogonal complement of the linear span of ξ and $\bar{\xi}$, which is the space of gravitational degrees of freedom at a point .

2.1 Regular ellipticity and hyperbolicity

Continuing the notation as in the previous section, we have introduced the quadratic form

$$h_{ab}^{\mu\nu} = \frac{\partial^2}{\partial v_\mu^\alpha \partial v_\nu^b}(x, q, v).$$

in general context of Lagrangian theory of mappings $u : M \to N$. If u is a background solution, $x \in M$, $q = u(x)$ and let $\xi \in T_x^*M$ and $Q \in T_qN$.

Q is a variation in position. The corresponding variation in velocity $\dot{v}$ is a linear map from T_xM to T_qN. For any $X \in T_xM$, the components of the vector $Q = \dot{v}\dot{X} \in T_qN$ are $Q^a = \dot{v}^a_\mu X^\mu$, where $X^\mu$are the component of X and v^μ_a component of of $\dot{v}$. The space $S_2(L(T_xM, T_qN))$ of quadratic forms on $L(T_xM, T_qN)$ splits into the direct sum:

$$S_2 = S_{2+} \oplus S_{2-}$$

where S_{2+} consists of the even quadratic forms and S_{2-} of odd quadratic forms. Thus , a quadratic form h on $L(T_xM, T_qN)$ decomposes into

$$h = h_+ + h_-,$$

where h_+ and h_- are , the even and odd part of h respectively. Componentwise, we have

$$h^{\mu\nu}_{ab} = h^{\mu\nu}_{+ab} + h^{\mu\nu}_{-ab},$$

where

$$h^{\nu\mu}_{ba} = h^{\mu\nu}_{ab}$$

and

$$h^{\nu\mu}_{+ab} = h^{\mu\nu}_{+ab} = h^{\mu\nu}_{+ab},$$
$$h^{\nu\mu}_{-ab} = h^{\mu\nu}_{-ba} = -h^{\mu\nu}_{-ab}.$$

We begin by

Definition 6 *Rank-1 elements of $L(T_xM, T_qN)$ are the elements $\dot{v}$ of the form*

$$\dot{v} = \xi \otimes Q, \xi \in T^*_xM, Q \in T_qN,$$

which means: $\dot{v}\dot{X} = (\xi\dot{X})Q$ for all X in T_xM.

For now, we consider the quadratic form $h(\dot{v}, \dot{v}) = h^{\mu\nu}_{ab}\dot{v}^a_\mu \dot{v}^b_\nu$.
Regular ellipticity (Lengendre-Hadamard condition). A Lagrangian L is called regular elliptic at (x, q, v) if the quadratic form $h = \frac{\partial^2 L}{\partial v^2}$ on $L(T_xM, T_qN)$ is positive definite on the set of rank-1-elements $\dot{v}^a_\mu = \xi_\mu Q^\alpha$ with $\xi \in T^*_xM$ and $Q \in T_qN$.
Notice that L and L' two Lagrangians satisfy the same Lagrangian equation, then the difference $h - h'$ of the corresponding quadratic forms is an odd quadratic form. Also, the definition of regular ellipticity is independent of the choice of Lagrangian for the same Euler -Lagrange equations because odd quadratic forms is 0 on the set of rank-1-elements.

Definition 7 *A Lagrangian L is called regularly hyperbolic at (x, q, v) if the quadratic form $h = \frac{\partial^2 L}{\partial v^2}(x, q, v)$ on $L(T_x M, T_q N)$ has the following property: There exists a pair (ξ, X) in $T_x^* M \times T_x M$ with $\xi \dot{X} > 0$ such that 1.h is negative definite on the space*

$$L_\xi = \{\xi \otimes Q; Q \in TqN\},$$

2. h is positive definite on the set of rank-1-elements of the subspace

$$\Sigma_X = \{\dot{v} \in L(T_x M, T_q N); \dot{v}\dot{X} = 0\}.$$

Note that the definition is also independent of the choice of Lagrangian giving rise to the same Euler-Lagrange equation.

Definition 8 *For quadratic form h on $L(T_x M, T_q N)$, and a pair (ξ, X) in $T_x^* M \times T_x M$ with $\xi \dot{X} > 0$, we define a new quadratic form*

$$m(\xi, X)(\dot{v}_1, \dot{v}_2) = (\xi \dot{X}) h(\dot{v}_1, \dot{v}_2) - h(\xi \otimes \dot{v}_1 \dot{X}, \dot{v}_2) - h(\dot{v}_1, \xi \otimes \dot{v}_2 \dot{X}).$$

We call this the Noether transform of h.

Proposition 9 *A Langrangian L is regularly hyperbolic at (x, q, v) if and only if there exists a pair (ξ, X) in $T_x^* M \times T_x M$ with $\xi \dot{X} > 0$ such that the Noether transform $m(\xi, X)$ of h corresponding to (ξ, X) is positive definite on the following set*

$$R_\xi = \{\xi \otimes P + \zeta \otimes Q : \forall \zeta \in T_x^* M, \forall P, Q \in T_q N.\}$$

Remark 10 *If h an odd quadratic form, then the Noether transform of h vanishes on R_ξ*

Given h and a nonzero ξ, we define

$$\chi(\xi)(Q_1, Q_2) = h(\xi \times Q_1, \xi \otimes Q_2).$$

i.e

$$\chi_{ab}(\xi) = h_{ab}^{\mu\nu} \xi_\mu \xi_\nu.$$

Then the characteristic subset C_x^* of $T_x^* M$ is defined by

$$C_x^* = \{\xi \neq 0 \in T_x^* M : \chi(\xi) singular\} \tag{28}$$

We can also define:

$$\psi(Q)(\xi_1, \xi_2) = h(\xi_1 \otimes Q, \xi_2 \otimes Q) \tag{29}$$

i.e

$$\psi^{\mu\nu}(Q) = h^{\mu\nu}_{ab} Q^a Q^b.$$

Next we define,

$$\Lambda(\xi) = \{\psi(Q)\dot{\xi} : Q \neq 0 \in nullsp(\chi(\xi))\}.$$

a subset of

$$\Sigma_\xi = \{X \in T_x M : \xi \dot{X} = 0\}$$

so that we can consider $\psi(Q)$ a linear map of $T^*_x M \to T_x M$

$$\xi_\mu \to \psi^{\mu\nu}(Q)\xi_\nu.$$

Note $\Lambda(\xi)$ is a positive cone in Σ_ξ. This means for $X \in \Lambda(\xi)$, $\lambda > 0$, λX lies in $\Lambda\xi$. Also,

$$\Lambda(\mu\xi) = \Lambda(\xi), \forall \mu > 0.$$

For a regular point ξ of C^*_x, the null space of $\chi(\xi)$ has dimension 1 and $\Lambda(\xi)$ is a ray. Else, the maximal dimension of $\Lambda(\xi)$ is $dim\Sigma_\xi = n - 1$.

Definition 11 *The characteristic subset C_x of $T_x M$ is defined:*

$$C_x = \bigcup_{\xi \in C^*_x} \Lambda(\xi)$$

3 Cauchy Problem

We will now explore, Cauchy problem for Einstein equations: local in time , existence and uniqueness of solutions. We shall discuss some work of Y.Choquet Bruhat, which based on reduction of Einstein equation to wave equations.

Definition 12 *Let (M, g) be a Riemannian manifold, a function is called harmonic if*

$$\Delta_g \Phi = 0$$

where $\Delta_g \Phi = g^{\mu\nu} \bigtriangledown_\mu (\partial_\nu \Phi)$.

Now our problem is : Given a coordinate chart (U, x) with $x = (x^0, x^1, x^2, x^3)$, aimed at finding functions $\Phi^\mu, \mu = 0, 1, 2, 3$, each of which is a solution of the wave equation in U ,

$$\Delta_g \Phi = 0$$

in U, and such that, setting

$$\bar{x}^\mu = \Phi^\mu(x^0, x^1, x^2, x^3).$$

so that we get a diffeomorphism of the range V in R^4 onto another domain $\bar{V}$ in R^4. We thus have another chart $(U, \bar{x})$ with domain U, another system of local coordinates. The equation $\Delta_g \Phi = 0$ in an arbitrary system of local coordinates:

$$\Delta_g \Phi = g^{\mu\nu}(\frac{\partial^2 \Phi}{\partial x^\mu \partial x^\nu} - \Gamma^\alpha_{\mu\nu} \frac{\partial \Phi}{\partial x^\alpha}) = 0 \tag{30}$$

Suppose now that we use the function $(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ as local coordinates in U.That is we express thins locally in new coordinate chart. Setting Φ equal to each one of the $\bar{x}^\beta, \beta = 0, 1, 2, 3$, we have a solution of above equation . Since

Dropping the bars we say that the system of local coordinates is harmonic if and only if the connection coefficients in these coordinates satisfy the condition.

$$\Gamma^\alpha := g^{\mu\nu} \Gamma^\alpha_{\mu\nu} = 0 \tag{31}$$

Let us set $\Gamma_\alpha := g_{\alpha\beta} \Gamma^\beta$. Thus , we can write

$$\Gamma_\mu = g^{\alpha\beta} \partial_\alpha g_{\beta\mu} - \frac{1}{2} g^{\alpha\beta} \partial_\mu g_{\alpha\beta}.$$

Consider the principal part of $\partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu$, which is :

$$P.P\{\partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu\} = g^{\alpha\beta}\{\partial_\alpha \partial_\mu g_{\beta\nu} + \partial_\alpha \partial_\nu g_{\beta\nu} - \partial_\mu \partial_\nu g_{\alpha\beta}\}$$

We will define

$$H_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu \tag{32}$$

with the principal part

$$P.P\{H_{\mu\nu}\} = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \partial_\beta g_{\mu\nu}.$$

and now

$$H_{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} g_{\mu\nu} + B_{\mu\nu}^{\alpha\beta\kappa\lambda\rho\sigma} \partial_\alpha g_{\kappa\lambda} \partial_\beta g_{\rho\sigma}. \tag{33}$$

where B is a rational function of the metric g of degree -2, the ratio of a homogeneous polynomial in g of degree 6. Replacing the Einstein equations

$$R_{\mu\nu} = 0. \Rightarrow H_{\mu\nu} = 0$$

which is a system of non-linear wave equations for the metric component $g_{\mu\nu}$. Choquet-Bruhat studied the Cauchy problem for these reduced equations. By writing

$$R_{\mu\nu} = H_{\mu\nu} + \frac{1}{2} S_{\mu\nu}.$$

$$S_{\mu\nu} = \partial_{\mu\nu} = \partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu.$$

We have

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = H_{\mu\nu} - \frac{1}{2} g_{\mu\nu} H + \frac{1}{2}(S_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S), \tag{34}$$

and

$$\bigtriangledown^\nu (S_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S) = g^{\nu\lambda} \bigtriangledown_\lambda \hat{S}_{\mu\nu}. \tag{35}$$

Once we have a solution of the reduced equations, then by twice contracting Bianchi identities.

$$\bigtriangledown^\nu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0 \tag{36}$$

and the solution also satisfies

$$\bigtriangledown^\nu \hat{S}_{\mu\nu} = 0 \tag{37}$$

Since $S = 2\partial^\nu \Gamma_\nu$,

$$\hat{S}_{\mu\nu} = \partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu - g_{\mu\nu} \partial^\lambda \Gamma_\lambda.$$

Also,

$$P.P\{\bigtriangledown^\nu \hat{S}_{\mu\nu}\} = \partial_\mu(\partial^\nu \Gamma_\nu) + \partial^\nu \partial_\nu \Gamma_\mu - \partial_\mu(\partial^\lambda \Gamma_\lambda) = g^{\alpha\beta} \partial_\alpha \partial_\beta \Gamma_\mu.$$

In fact,

$$\bigtriangledown^\nu \hat{S}_{\mu\nu} = g^{\alpha\beta} \partial_\alpha \partial_\beta \Gamma_\mu + \dot{A}_\mu^{\alpha\beta} \partial_\alpha \Gamma_\beta.$$

for A some linear form in ∂g which are homogeneous functions of g . Therefore the equations

$$\bigtriangledown^{\nu} \hat{S}_{\mu\nu} = 0$$

contains a systemof homogeneous linear wave equations for Γ_{μ}. Consequently, Γ_{μ} vanish identically provided the initial condition vanish, i.e

$$\Gamma_{\mu}|_{\Sigma_0} = 0$$

$$\partial_0 g_{\mu\nu}|_{\Sigma_0} = 0$$

where Σ_0 is the initial hypersurface $x^0 = 0$. Given now initial data for the Einstein equations

$$R_{\mu\nu} = 0 = \hat{R}_{\mu\nu}.$$

here $\hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, we will construct initial data $g_{\mu\nu}|_{\Sigma_0}$ and $\partial_0 g_{\mu\nu}|_{\Sigma_0}$ for the reduced equations $H_{\mu\nu} = 0$ such that

$$\Gamma_{\mu}|_{\Sigma_0} = 0$$

$$\partial_0 g_{\mu\nu}|_{\Sigma_0} = 0$$

are satisfied. According to above, the $g_{\mu\nu}$ of the Cauchy problem for the reduced equations also satisfy the condition $\Gamma_{\mu} = 0$,therefore shall be the solution of the original Einstein equation .

Initial data for the Einstein equation consist of a pair $(\bar{g_{ij}}, k_{ij})$, where $\bar{g}_{ij}$ is a Riemman metric and k_{ij} a 2 covariant symmetric tensor field on the 3 manifold $\bar{M}$, which is to be identified with the initial data hypersurface Σ_0. Once we have a solution (M, g) with $M = [0, T] \times \Sigma_0$ and $\Sigma_0 = \bar{M}$. Then $\bar{g}_{ij}$ and k_{ij} will be the first and second fundamental form of $\Sigma_0 = \{0\} \times \Sigma_0$ in (M, g). That is ,

$$\bar{g}_{ij} = g_{ij}|_{\Sigma_0}$$

for $i, j = 1, 2, 3$.

We choose the coordinates to be Gaussian normal along Σ_0, that is

$$g_{i0}|_{\Sigma_0} = 0 \tag{38}$$

$$g_{00}|_{\Sigma_0} = -1 \tag{39}$$

Then

$$\partial_0 g_{ij}|_{\Sigma_0} = 2k_{ij} \tag{40}$$

We then choose $\partial_0 g_{0i}|_{\Sigma_0}$,$\partial_0 g_{00}|\Sigma_0$ so that the condition below is satisfied.

$$\Gamma_\mu|_{\Sigma_0} = 0.$$

A short calculation will show that

$$\partial_0 g_{0i}|_{\Sigma_0} = \bar{\Gamma}_i.$$

and

$$\partial_0 g_{00}|_{\Sigma} 2tr(k)$$

where $\bar{\Gamma}_i$ induced by metric g_{ij}. Note that $tr(k) = \bar{g}^{ij}k_{ij}$. This is all about specification of initial data for the reduced equations. Now we wish to consider that for a solution of the reduced equations,

$$\hat{R}_{0i}|_{\Sigma} = \frac{1}{2}\hat{S}_{0i}|_{\Sigma_0} = \frac{1}{2}\{\partial_0\Gamma_i + \partial_i\Gamma_0 - g_{0i}\partial^\lambda\Gamma_\lambda|_{\Sigma_0}\} = \frac{1}{2}\partial_0\Gamma_i|_{\Sigma_0}.$$

and

$$\hat{R}_{00}|_{\Sigma_{00}} = \frac{1}{2}\hat{S}_{00}|_{\Sigma_0} = \frac{1}{2}\{2\partial_0\Gamma_0 - g_{00}\partial^\lambda\Gamma_\lambda|_{\Sigma_0}\} = \frac{1}{2}\partial_0\Gamma_0|_{\Sigma_0}$$

Then, if the initial data $(\bar{g}_{ij}, k_{ij})$ verify the constraint equations.

$$\hat{R}_{0i}|_{\Sigma_0} = 0 \tag{41}$$

$$\hat{R}_{00}|_{\Sigma_0} = 0 \tag{42}$$

then the conditions $\partial_0\Gamma_\mu|_{\Sigma_0} = 0$ are also satisfied.
In the original work of Choquet-Bruhat, a local problem was posed and the initial data is given on a domain $\Omega \subset \Sigma_0$. The first step is to extend the initial condition to the whole plane in such way that it becomes trivial outside a larger domain Ω' containing Ω, with compact closure in R^3
The next step in construct a solution is based on the domain of dependence theorem, to be formulated below. Let (M, g) be the known spacetime, where $M = [0, T] \times \Sigma_0$.

Definition 13 *The domain of dependence of Ω in the spacetime (M, g) is the subset of M for which Ω is a (incomplete) Cauchy hypersurface, denoted by $D(\Omega)$.*

The domain $D(\Omega)$ of M is the set of points $p \in M$ such that each past directed casual curve in M through p intersects Ω. In $D(\Omega)$, the solution depends only on the initial data in Ω. In particular, since the constraint equations are satisfied in Ω, we have that Γ_μ and $\partial_0 \Gamma_\mu$ all vanish in Ω. By the domain of dependence theorem applied to the (linear homogeneous) wave equations for Γ_μ, it vanish throughout $D(\Omega)$. Therefore, the solution of the reduced equations is a solution of the Einstein equations

$$R_{\mu\nu} = 0$$

in $D(\Omega)$. If the 3-manifold $\bar{M}$ is compact, one can cover $\bar{M}$ with a finite number of coordinate charts and construct a local time solution by combining together the local solutions. For , suppose Ω_1 and Ω_2 are two such coordinate charts with $\Omega_1 \cap \Omega_2 \neq 0$. Note that we are given initial data $(\bar{g}, k)$ on the whole 3-manifold $\bar{M}$, the representations of these data given by two charts, are related by the diffeomorphism in the overlap. Thus there exists diffeomorphism f of $\Omega_1 \cap \Omega_2$ onto itself such that

$$\bar{g}_2 = f^* \bar{g}_1, k_2 = f^* k_1.$$

After the transformation, we may assume that the initial data coincide in $\Omega_1 \cap \Omega_2$. For g_1 and g_2 are the two solutions of reduced equations, corresponding to the initial conditions in Ω_1 and Ω_2 respectively, the domain the domain of dependence theorem asserts that g_1 and g_2 coincide in the domain of dependence of $\Omega_1 \cap \Omega_2$ relative to either g_1 or g_2.
We can then extend either solution $(D(\Omega_1, g_1)$, (D_{Ω_2}, g_2) to the union , the domain of dependence of $\Omega_1 \cup \Omega_2$.

Definition 14 *Given initial data $(\Omega, \bar{g}, k)$, where completeness is not required , we say that a space time (U, g) is a development of this data, if Ω is a Cauchy hypersurface for U. So Ω is the past boundary of U and $\bar{g}$ and k are respectively the first and second fundamental forms of the hypersurfaces Ω in (U, g). Moreover, g satisfies the Einstein equations*

$$R_{\mu\nu} = 0.$$

Simlarly, if U_1 and U_2 are development of the initial data, then we can define a development with domain $U_1 \cup U_2$ which extends the corresponding metric g_1 and g_2. Hence, the union of all developments of given initial data is also a development of the same data, the maximal development of that data.

Theorem 15 *(Y.Choquet-Bruhat, R. Geroch) Any initial data set* $(\bar{M}, \bar{g}, k)$ *where completeness is not assumed, satisfying the constraint equations, gives to a unique maximal development.*

We shall now give an give a presentation of domain of independence theorem in general Lagragian setting. Recall the Lagrangian of mapping $u : M \to N$. Given a background solution u_0, we define the quadratic form $h = \frac{\partial^2 L}{\partial v^2}(v_0)$ with $h(\dot{v}, \dot{v}) = h^{\mu\nu}_{ab} \dot{v}^a_\mu \dot{v}^b_\nu$ and $v_0 = du_0(x)$. Let us denote by $\{L\}$, the equivalence class of Lagrangians giving the same Euler-Lagrange equations

Definition 16 *We call* $\{L\}$ *regularly hyperbolic at* v_0 *if the quadratic form* h *satisfy: 1.There is a covector* $\xi \in T^*_x M$ *such that* h *is negative definite on*

$$L_\xi := \{\xi \otimes Q : Q \in T_q N\} \subset L(T_x M, T_q N).$$

that is , the elements of the form $\dot{v}^a_\mu = \xi_\mu Q^a$.
2. There is a vector $X \in T_x M$ *with* $\xi \dot{X} > 0$ *such that* h *is positive definite on the set* Σ^1_X *of rank -1 -elements of the subspace*

$$\Sigma_X = \{\dot{v} \in L(T_x M, T_q N) : \dot{v}\dot{X} = 0\}$$

That is , the elements of the form $\dot{v}^a_\mu = \zeta_\mu P^a$ *where* $\zeta_\mu X^\mu = 0$.

Definition 17 *Let* h *be regularly hyperbolic. Set*

$$I^*_x = \{\xi \in T^*_x M : h \textit{negative definite on} L_\xi\}$$

and

$$J_x = \{X \in T_x M : h \textit{positive definite on} \Sigma^1_X\}$$

Proposition 18 I^*_x *and* J_x *are open cones each of which has two components* $I^*_x = I^{*+}_x \cup I^{*-}_x$ *and* $J_x = J^+_x \cup J^-_x$. *where* I^{*-}_x *and* J^-_x *are the sets of opposites of elements in* I^{*+}_x *and* J^+_x. *Further each component is convex. The boundary* ∂I^*_x *is a component (the inner component) of* C^*_x*, the characteristic in* $T^*_x M$ *and* ∂J_x *is a component (the inner component) if* C_x*, the characteristic in* $T_x M$.

Recall the definition of Noether transform $m(\xi, X)$ if h corresponding to a pair $(\xi, X) \in T^*_x(M) \times T_x M$ with $\xi \cdot \dot{X} > 0$:

$$m(\xi, X)(\dot{v}_1, \dot{v}_2) = (\xi, X) h(\dot{v}_1, \dot{v}_2) - h(\xi \otimes \dot{v}_1 \cdot X, \dot{v}_2) - h(\dot{v}_1, \xi \otimes \dot{v}_2 \dot{X}).$$

Proposition 19 *Let $U_x^+ \subset T_x^*M \times T_xM$ be given such that*

$$U_x^+ = \{(\xi, X) : \xi \cdot X > 0\}$$

Consider the subset of U_x^+ contains of those (ξ, X) with $m(\xi, X)$ positive definite on $R_\xi = \{\xi \otimes P + \zeta \otimes Q : \forall \zeta \in T_x^*M, forall P, Q \in T_qN\}$. Then this subset is given by

$$(I_x^{*+} \times J_x^+) \cup (I_x^{*-} \times J_x^-).$$

Furthermore, on the boundary of this set $m(\xi, X)$ has nullity.

We will now quickly discuss the idea of variation of a mapping $u_0 : M \to N$. A variation of u_0, namely $\dot{u}$ is a section of u_0^*TN, (the pullback by u_0 of TN). Generally, for B a bundle over N and $u_0 : M \to N$, denote by u_0^*B, the pullback bundle, namely the following bundle over M:

$$u_0^*B = \bigcup_{x \in M} \{x\} \times B_{u_0(x)},$$

where B_q is the fibre of B over $q \in N$.
Hence, a variation $\dot{u}$ maps $x \in M \to \dot{u}(x) \in T_{u_0(x)}N$. Now given $x \in M$, $\dot{u}(x)$ is the tangent vector at $u_0(x)$ of the curve $t \to u_t(x)$ in N, where u_t is a differentiable 1-parameter family of mappings $u_t : M \to N$, $\dot{u}(x) = \frac{du_t(x)}{dt}|_{t=0}$.
For $J_x \subset T_xM$: J_x is the set of possible values at $x \in M$ of a vector field X on M such that the reduced equations form a regular elliptic system. On the other hand , the subsets $I_x^* \subset T_x^*M$ defines a spacelike hypersurface.

Definition 20 *A hypersufrace H in M is called spacelike , if at each $x \in H$, the double ray $\{\lambda\xi : \lambda \neq 0 \in R\}$, defined by hyperplane T_xH in TxM.,*

is contained in I_x^*

Definition 21 *$\bar{I}_x$, the casual subset of T_xM, is the set of all vectors $X \in T_xX$ such that $\xi \cdot X \neq 0$ for all $\xi \in I_x^*$*

$\bar{I}_x$ is a closed subset of T_xM with $\bar{I}_x^+ \cup \bar{I}_x^-$, where $\bar{I}_x^-$ is the set of opposites of the elements in $\bar{I}_x^+$. One can see easily that each component is convex.
If $X \in \partial\bar{I}_x$, then there isa a covector $\xi \in \partial I_x^*$ such that $\xi \cdot X = 0$. It follows that each component I_x^{*+} and I_x^{*-} lies to one side of the plane

$$\Pi_X = \{\xi \in T_x^*M : \xi \cdot X = 0\}.$$

and it contains a ray of ∂I_x^{*+} and ∂I_x^{*-} respectively.

Definition 22 *A causal curve γ in M is a curve in M whose tangent vector $\dot{\gamma}(t)$ at each point of $\gamma(t)$ belongs to $\bar{I}_{\gamma(t)}$.*

The following statements are valid for the future and past components of $\bar{I}_x$ and J_x separately. Generally, $J_x \subset \bar{I}_x$.

Definition 23 *Let R be a domian in M in which a solution u of the Euler Lagrange equations is defined. Consider a domain $D \subset R$ and a hypersurface Σ in R, which is spacelike relative to du. 1. We say that D is a development of Σ if we can express $D = \cup_{t\in[0,T]}\Sigma_t$, where $\{\cup_{t\in[0,T]}\Sigma_t\}$ is a foliation and where each Σ_t is a spacelike hypersurface in R homologous to $\Sigma_0 = \Sigma$. In particular, $\partial\Sigma_t = \partial\Sigma$ for all $t \in [0,T]$.*
2.D is a development of Σ if each causal curve in R through any point of D intersects Σ at a single point.

One can show that if D_1 and D_2 are developments of Σ then $D_1 \cup D_2$ is also a development. So given a domain of definition R and a spacelike hypersurface Σ, we can define the domain of dependence of Σ in R relative to du to be the maximal development.
To state the domain of dependence theorem in a precise manner, first we formulate rigourously the general Lagragian setup. Consider maps $u : M \to N$ and the configuration space $C = M \times N$. The velocity space V is a bundle over C :

$$V = \cup_{(x,q)\in C} L(T_xM, T_qN).$$

we have a projection

$$\pi : V \to C.$$

$$v \in L(T_xM, T_qN) \to (x,q) \in C,$$

where we write $\pi_{V,M}$ for $\pi_1 \circ \pi : V \to C \to M$ and we write $\pi_{V,N}$ for $\pi_2 \circ \pi : V \to C \to N$.
The notation will be: TM : tangent bundle $\Gamma_r M$: bundle of (fully antisymmetric) r-forms on M S_2M: bundle of quadratic (symmetric bilinear) forms on M $\Gamma_n M$: bundle of top-degree-forms,
So, $\Gamma_n M$ is a bundle over M and :

$$\pi_{V,M} : V \to M$$

the projection from above. Then the pullback bundle is

$$\pi^*_{V,M}\Gamma_n M,$$

a bundle over V. An element of this is $\omega \in (\Gamma_n M)_x$ with $x \in M$ and ω is attached to an element $v \in L(T_xM, T_qN)$. The Lagragian $L$is a smooth section of $\pi^*_{V,M}\Gamma_n M$ over V. Thus, the map $v \to L(v) \in (\Gamma_n M)_x$. It is

$$L(v)(Y_1, \cdots, Y_n)$$

with $Y_1, \cdots, Y_n \in T_xM$ a n-linear fully antisymmetric form on T_xM.

Definition 24 *The action of a map u will be defined in a domain R in M, corresponding to a subdomain $D \subset R$ is*

$$A[u; D] = \int_D L \circ du$$

$L \circ du$ is a section of $\Gamma_n M$ over R. Notice that $(L \circ du) = L(du(x)) \in (\Gamma_n M)_x$. $dimM = n$, $dimN = m$, $dimC = n + m$, $dimV = n + m + mn$. Suppose that M is oriented and ϵ is a smooth volume form on M. Then given a smooth function L^* on V, we define the corresponding Lagrangian L by:

$$L(v)(Y_1, \cdots, Y_n) = L^*(V)\epsilon(Y_1, \cdots, Y_n)$$

with $v \in L(T_xM, T_qN)$ and $Y_1, \cdots, Y_n$ in T_xM.

Finally, we state the domain of dependence theorem.

Theorem 25 *Let u_0 be a C^2 solutiom of the Euler-Lagrange equations corresponding to a smooth Lagrangian L; and u_0 defined in a domain R in M. Let Σ be a hypersurface in R, which is spacelike relative to du_0. Then let u_1 be another solution of the Euler-Lagrange equations defined and C^1 on R. Suppose that*

$$du_0|_\Sigma = du_1|_\Sigma.$$

Then u_1 coincides with u_0 in the domain of dependence of Σ in R relative to du_0.

4 Reference

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