STRUCTURE OF THE LAPLACIAN ON BOUNDED DOMAINS

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Contents

1. Introduction

In this chapter, we will be concerned with the *eigenvalue problem*

$$-\Delta u = \lambda u, \tag{1}$$

in a bounded open set $\Omega \subset \mathbb{R}^n$, with either the Dirichlet $u = 0$ or the Neumann $\partial_\nu u = 0$ condition on the boundary $\partial\Omega$. The unknown in the problem is the pair (u, λ) where u is a function and λ is a number. If (u, λ) is a solution then u is called an *eigenfunction*, and λ is called the *eigenvalue* associated to u. Let us note the following.

- Since the right hand side involves λu, the problem is *not* linear.
- If (u, λ) is a solution then so is $(\alpha u, \lambda)$ for any number α.
- We exclude the trivial solution $u = 0$ from all considerations.

We have studied the problem $-\Delta u + tu = f$ with the Dirichlet or Neumann boundary conditions, where $t \in \mathbb{R}$ and $f \in L^2(\Omega)$ are given. Since (1) is equivalent to $-\Delta u + tu = 0$ with $t = -\lambda$, we can give the following weak formulation for (1). Let V be either $H^1_0(\Omega)$ or $H^1(\Omega)$, depending on the boundary condition we wish to impose. Then the problem is to find $u \in V$ and $\lambda \in \mathbb{R}$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega uv \qquad \text{for all } v \in V. \tag{2}$$

We know that a unique weak solution $u \in V$ to $-\Delta u + tu = f$ exists if $t > t_0$, where $t_0 = 0$ for the Neumann case, and $t_0 < 0$ for the Dirichlet case. This shows that if (2) has a nontrivial solution, then we must have $-\lambda \leq t_0$. In particular, if they exist, the eigenvalues must satisfy $\lambda \geq 0$ for the Neumann case, and $\lambda > 0$ for the Dirichlet case. If $u \in V$ satisfies (2) then the regularity results imply that $u \in C^\omega(\Omega)$, and so in particular u is a classical solution of (1) in Ω, and moreover that u satisfies the desired boundary condition in the classical sense provided the boundary is regular enough. Finally, if $u \in H^1(\Omega)$ satisfies (2) with $\lambda = 0$, then putting $v = u$ necessitates that u must be locally constant. The dimension of the space of locally constant functions is equal to the number of connected components of Ω, meaning that the multiplicity of the Neumann eigenvalue $\lambda = 0$ is equal to the same number.

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Now we want to write (2) as an abstract operator eigenvalue problem. We introduce the linear operator $A : V \to V'$ and the bilinear form $a : V \times V \to \mathbb{R}$ by

$$\langle Au, v\rangle = a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \qquad u, v \in V, \tag{3}$$

where $\langle \cdot, \cdot\rangle$ is the duality pairing between V' and V. Recall that a is continuous

$$|a(u, v)| \leq \|u\|_{H^1}\|v\|_{H^1}, \qquad u, v \in V, \tag{4}$$

symmetric

$$a(u, v) = a(v, u), \qquad u, v \in V, \tag{5}$$

and satisfies

$$a(u, u) + t\langle u, u\rangle_{L^2} \geq \alpha\|u\|^2_{H^1}, \qquad u \in V, \tag{6}$$

for all $t > t_0$, with $\alpha > 0$ possibly depending on t. The operator A is called the *energy extension* of $-\Delta$ with the given boundary condition, in the sense that it is an extension of the classical Laplacian acting on a dense subset of V. We can check that it is bounded:

$$\|Au\|_{V'} = \sup_{v\in V} \frac{\langle Au, v\rangle}{\|v\|_{H^1}} = \sup_{v\in V} \frac{a(u, v)}{\|v\|_{H^1}} \leq \|u\|_{H^1}. \tag{7}$$

In terms of the operator A, the problem (2) can be written as

$$Au = \lambda Ju, \tag{8}$$

where the inclusion map $J : L^2(\Omega) \to V'$ is defined by

$$\langle Jf, v\rangle = \int_\Omega fv, \qquad v \in V. \tag{9}$$

Obviously, J is injective because $Jf = 0$ implies $f = 0$ for $f \in L^2(\Omega)$ by the du Bois-Reymond lemma. It is also continuous:

$$\|Jf\|_{V'} = \sup_{v\in V} \frac{\langle Jf, v\rangle}{\|v\|_{H^1}} \leq \sup_{v\in V} \frac{\|f\|_{L^2}\|v\|_{L^2}}{\|v\|_{H^1}} \leq \|f\|_{L^2(\Omega)}, \tag{10}$$

and hence J defines a continuous embedding of $L^2(\Omega)$ into V'. In what follows we will identify $L^2(\Omega)$ with a subspace of V' through J. So for instance, we write (8) simply as

$$Au = \lambda u. \tag{11}$$

For $u \in H^2(\Omega) \cap V$ and $v \in \mathscr{D}(\Omega)$, integration by parts yields

$$\langle Au, v\rangle = -\int_\Omega v\Delta u, \tag{12}$$

meaning that $Au = -J\Delta u$, or simply, $Au = -\Delta u$. It is in this sense that A is an extension of the Laplacian. On the other hand, if $Au \in L^2(\Omega)$ and if $\partial\Omega$ is smooth enough, we know from regularity theory of the Poisson equation that $u \in H^2(\Omega)$. Thus, if $\partial\Omega$ is smooth enough, $u \in V$ satisfies $Au \in L^2(\Omega)$ if and only if $u \in H^2(\Omega)$.

By the Riesz representation theorem, $A + tI$ is invertible for $t > t_0$, and $(A + tI)^{-1} : V' \to V$ is bounded. In what follows, we fix some $t > t_0$. Then adding tu to both sides of (11), and applying $(A + tI)^{-1}$, we get

$$u = (t + \lambda)(A + tI)^{-1}u. \tag{13}$$

At this point, we introduce the *resolvent*[1]

$$R_t = (A + tI)^{-1}|_{L^2(\Omega)} : L^2(\Omega) \to L^2(\Omega), \tag{14}$$

which is the restriction of $(A + tI)^{-1}$ to $L^2(\Omega)$. Hence if $u \in V$ and $\lambda \in \mathbb{R}$ satisfy (11), then

$$(t + \lambda)R_tu = u. \tag{15}$$

[1]The usual definition is $(A - tI)^{-1}$, but we are using the plus sign for convenience.

Conversely, if $u \in L^2(\Omega)$ and $\lambda \in \mathbb{R}$ satisfy (15), then by applying $A + tI$ on both sides, we derive (11), proving the equivalence of the two formulations.

Let us derive some straightforward properties of the resolvent.

- The resolvent is bounded as an operator $R_t : L^2(\Omega) \to V$, because
$$\|R_t f\|_V \leq c\|f\|_{V'} \leq c\|f\|_{L^2(\Omega)}, \tag{16}$$
where the constant c may have different values at its different occurrences.
- The resolvent is *positive*, in the sense that $\langle R_t f, f\rangle > 0$ for $f \neq 0$, where $\langle \cdot, \cdot \rangle$ is the L^2 inner product on Ω. To see this, let $f \in L^2(\Omega)$ and let $u = R_t f$, so that $f = (A + tI)u$. Then the strict coercivity property (6) gives
$$\langle R_t f, f\rangle = \langle u, (A + tI)u\rangle = a(u, u) + t\langle u, u\rangle \geq \alpha\|u\|^2_{H^1}. \tag{17}$$
- The resolvent is injective: If $R_t f = 0$ then $f = 0$.
- The resolvent is *symmetric*, in the sense that $\langle R_t f, g\rangle = \langle f, R_t g\rangle$ for $f, g \in L^2(\Omega)$. With $u = R_t f$ and $v = R_t g$, we have
$$\langle R_t f, g\rangle = \langle u, (A + tI)v\rangle = a(u, v) + t\langle u, v\rangle, \tag{18}$$
which clearly shows the claim.
- A function $u \in L^2(\Omega)$ is in the range of R_t if and only if $Au \in L^2(\Omega)$. To show this, first let u be such that $Au \in L^2(\Omega)$. Then $(A+tI)u \in L^2(\Omega)$, hence $u = R_t(A+tI)u$, which means that $u \in \mathrm{ran} R_t$. Second, let $u \in \mathrm{ran} R_t$, i.e., let $u = R_t f$ for some $f \in L^2(\Omega)$. It is obvious that $f = (A + tI)u$. From this, we have $Au = f - tu \in L^2(\Omega)$.

Remark 1. The natural setting for eigenvalue problems is to consider complex eigenvalues and complex valued eigenfunctions. To this end, we complexify the function spaces under consideration, as follows. Given a real function space X, such as $L^2(\Omega)$ or $H^1(\Omega)$, we define its complexification $X^{\mathbb{C}}$ by
$$X^{\mathbb{C}} = \{u + iv : u, v \in X\}. \tag{19}$$
For example, we have
$$L^2(\Omega, \mathbb{C}) \equiv L^2(\Omega)^{\mathbb{C}} = \{u + iv : u, v \in L^2(\Omega)\}. \tag{20}$$
The inner products must be extended accordingly to Hermitian inner products. For example, we define
$$\langle f, g\rangle = \int_\Omega f\bar{g}, \qquad f, g \in L^2(\Omega, \mathbb{C}). \tag{21}$$
and
$$\langle u, v\rangle_{H^1} = \int_\Omega (u\bar{v} + \nabla u \cdot \nabla\bar{v}), \qquad u, v \in H^1(\Omega, \mathbb{C}). \tag{22}$$
Note that with the identification that $L^2(\Omega)$ (or $H^1(\Omega)$) is the subspace of $L^2(\Omega, \mathbb{C})$ (or $H^1(\Omega, \mathbb{C})$) consisting of real valued functions, the Hermitian inner products reduce to the usual (real) inner products for real functions. Moreover, linear operators can be extended to act on these spaces by linearity:
$$A(u + iv) = Au + iAv, \qquad \text{and} \qquad R_t(u + iv) = R_t u + iR_t v, \tag{23}$$
et cetera. The resolvent is *self-adjoint*, in the sense that $\langle R_t f, g\rangle = \langle f, R_t g\rangle$ for $f, g \in L^2(\Omega, \mathbb{C})$.

Apart from these simple properties, a crucial property we would like to have for the resolvent is *compactness*. Since $R_t : L^2(\Omega) \to V$ is bounded, the resolvent sends bounded sets in $L^2(\Omega)$ into bounded sets in V. Therefore, if the embedding $V \hookrightarrow L^2(\Omega)$ is compact, that is, if bounded sets in V are relatively compact in $L^2(\Omega)$, then the resolvent as a map $R_t : L^2(\Omega) \to L^2(\Omega)$ would be a compact operator. Recall that a subset S of a metric space X is said to be *relatively compact* if any sequence $\{x_n\} \subset S$ has a subsequence that converges in X, and a linear operator is called *compact* if it sends bounded sets into relatively compact sets. As for the compactness of $V \hookrightarrow L^2(\Omega)$, we will prove in the next section that the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, provided Ω is bounded. Hence in the Dirichlet case, boundedness of Ω is sufficient for the resolvent to be compact.

For the Neumann case, however, the situation is a bit more involved, and one needs additional assumptions regarding the regularity of Ω.

Our strategy to solve the Laplace eigenvalue problem (11) will be through the equivalent formulation (15) in terms of the resolvent. The main feature that makes this formulation attractive is the fact that the resolvent is compact under some very mild assumptions on Ω.

2. Rellich's compactness lemma

In this section, we will establish compactness of the embeddings $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ and $H^1(\Omega) \hookrightarrow L^2(\Omega)$ under some assumptions on Ω. Either of these results is traditionally known as *Rellich's lemma*. More generally, compactness results on embeddings of the type $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ are called *Rellich-Kondrashov theorems*.

Example 2. a) Let $\phi \in \mathscr{D}(B)$ be a nontrivial function, where $B = B_r(0)$ with $r > 0$, and let ϕ_k, $k = 1, 2, \ldots$, be translates of ϕ, with their supports not overlapping. Suppose that $\Omega \subset \mathbb{R}^n$ is a domain that contains $\operatorname{supp} \phi_k$ for all k. In particular, Ω is necessarily unbounded. Now, the sequence $\{\phi_k\} \subset H^1_0(\Omega)$ is bounded in $H^1_0(\Omega)$. However, since $\|\phi_j - \phi_k\|_{L^2} = 2\|\phi\|_{L^2} > 0$ whenever $j \neq k$, no subsequence of $\{\phi_k\}$ converges in $L^2(\Omega)$, meaning that the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is *not* compact. Furthermore, the sequence $f_k = (A+tI)\phi_k$, $k \in \mathbb{N}$, is bounded in $L^2(\Omega)$, as $\|f_k\|_{L^2} = \|(-\Delta + tI)\phi\|_{L^2}$, but $\phi_k = R_t f_k$, $k \in \mathbb{N}$, has no convergent subsequence in $L^2(\Omega)$. Thus the resolvent $R_t : L^2(\Omega) \to L^2(\Omega)$ is *not* compact.

b) Let $\Omega \subset \mathbb{R}^n$ be an open set with infinitely many connected components. Let us denote those components by Ω_k, $k \in \mathbb{N}$, and let $\phi_k = |\Omega_k|^{-1/2}$, $k \in \mathbb{N}$. It is clear that the sequence $\{\phi_k\}$ is bounded in $H^1(\Omega)$, but there is no subsequence that converges in $L^2(\Omega)$. Moreover, the resolvent $R_t : L^2(\Omega) \to L^2(\Omega)$ is *not* compact.

c) Let $0 < a < 1$, and let $U_j = (2^{-j}, 2^{-j} + a2^{-j}) \times [0, 1)$ for $j \in \mathbb{N}$. We form a bounded domain as $\Omega = (0, 1) \times (-1, 0) \cup U_1 \cup U_2 \cup \ldots$. Pick a nontrivial function $\psi \in \mathscr{D}(I)$, where $I = (0, 1)$, and define $\phi_j \in \mathscr{C}^\infty(U_j)$ by $\phi_j(x, y) = 2^{j/2}\psi(y)$, for $j \in \mathbb{N}$. We set $\phi_j = 0$ in $\Omega \setminus U_j$, which yields $\phi_j \in \mathscr{C}^\infty(\Omega)$. Moreover, we have

$$\|\phi_j\|^2_{L^2(\Omega)} = \int_{U_j} |\phi_j|^2 = a \int_I |\psi|^2. \tag{24}$$

The norms $\|\partial_y \phi_j\|_{L^2}$ and $\|\partial^2_y \phi_j\|_{L^2}$ are also constant, since $\partial_y \phi_j = 2^{j/2}\psi'$ and $\partial^2_y \phi_j = 2^{j/2}\psi''$. This means that the sequence $\{\phi_j\}$ is bounded in $H^1(\Omega)$, and the sequence $\{(-\Delta + tI)\phi_j\}$ is bounded in $L^2(\Omega)$, but no subsequence of $\{\phi_j\}$ is Cauchy in $L^2(\Omega)$. Thus, neither the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ nor the resolvent $R_t : L^2(\Omega) \to L^2(\Omega)$ is compact.

We shall approach the compactness problem from the metric space standpoint. The most basic compactness criterion for metric spaces is the following: A metric space is compact if and only if it is complete and totally bounded. By definition, a metric space is *totally bounded* if it admits a finite ε-cover for each $\varepsilon > 0$, and an *ε-cover* is an open cover consisting of sets of diameter not exceeding ε. The aforementioned criterion will be sufficient for our purposes, but we will spend a little additional effort and prove below in Lemma 4 a slightly more general criterion. If the reader is convinced of the total boundedness criterion, by skipping Lemma 4 they would not lose the main thread of the section.

Definition 3. A metric space X is said to be *precompact* if any sequence $\{x_n\} \subset X$ has a Cauchy subsequence. A subset S of a metric space X is said to be *relatively compact* if any sequence $\{x_n\} \subset S$ has a subsequence that converges in X.

Note that for a subset of a complete metric space, the two notions coincide. The total boundedness criterion basically says that compactness can be established by approximating the given set by finite sets. The following criterion extends this result to approximation by compact sets.

Lemma 4. *Let (X,ρ) be a metric space, and let $S\subset X$. Suppose that $\{X_n\}$ is a sequence of relatively compact subsets of X, satisfying*

$$\sup_{x\in S}\inf_{y\in X_n}\rho(x,y)\to 0 \qquad \text{as } n\to\infty. \tag{25}$$

Then S is relatively compact in X.

Proof. We will show that any sequence $\{x_k\}\subset S$ has a Cauchy subsequence. For each n, let

$$\varepsilon_n=\sup_{x\in S}\inf_{y\in X_n}\rho(x,y). \tag{26}$$

Without loss of generality, assume that $\varepsilon_n\geq\varepsilon_{n+1}$ for all n. Pick a sequence $\{x_k\}\subset S$, and let $\delta_1,\delta_2,\ldots$ be a sequence of positive reals converging to 0. Then for each k there is $y_k\in X_1$ such that $\rho(x_k,y_k)<\varepsilon_1+\delta_1$. Since X_1 is precompact, we have a Cauchy subsequence $\{y_{k_i}\}\subset\{y_k\}$. In particular, there exists N_1 such that for all indices $i\geq N_1$ and $j\geq N_1$, we have $\rho(y_{k_i},y_{k_j})<\varepsilon_1+\delta_1$. This means that

$$\rho(x_{k_i},x_{k_j})\leq\rho(x_{k_i},y_{k_i})+\rho(y_{k_i},y_{k_j})+\rho(y_{k_j},x_{k_j})\leq 3(\varepsilon_1+\delta_1), \tag{27}$$

for all $i\geq N_1$ and $j\geq N_1$. Let us denote the subsequence $\{x_{k_i}\}$ by $\{x_{1,i}\}$. Then by applying the above procedure to $\{x_{1,i}\}$, with X_1 replaced by X_2 and $\varepsilon_1+\delta_1$ by $\varepsilon_2+\delta_2$, we extract a subsequence $\{x_{2,j}\}\subset\{x_{1,i}\}$, with the property that

$$\rho(x_{2,i},x_{2,j})\leq 3(\varepsilon_2+\delta_2), \tag{28}$$

for all $i\geq N_2$ and $j\geq N_2$, where N_2 is some integer. We continue this recursively, and get nested sequences $\{x_k\}\supset\{x_{1,k}\}\supset\{x_{2,k}\}\supset\ldots$, such that

$$\rho(x_{n,i},x_{n,j})\leq 3(\varepsilon_n+\delta_n), \tag{29}$$

for all $i\geq N_n$ and $j\geq N_n$. Without loss of generality, we can take $N_1\leq N_2\leq\ldots$.

Now we consider the "diagonal" sequence $\{x_{i,i}\}$. This is obviously a subsequence of the original sequence $\{x_k\}$. Moreover, given any n and $i\geq n$, we have $x_{i,i}=x_{n,k}$ for some $k\geq i$. Hence we infer

$$\rho(x_{i,i},x_{j,j})\leq 3(\varepsilon_n+\delta_n), \tag{30}$$

for all $i\geq N_n$ and $j\geq N_n$, meaning that $\{x_{i,i}\}$ is a Cauchy sequence. □

The following general criterion is known as the Kolmogorov-Riesz criterion.

Theorem 5. *Let $\Omega\subset\mathbb{R}^n$ be a domain. A subset $S\subset L^2(\Omega)$ is totally bounded if and only if S satisfies the following conditions.*

- *Boundedness: There is $M<\infty$ such that $\|f\|_{L^2(\Omega)}\leq M$ for all $f\in S$,*
- *L^2-equicontinuity: $\|\Delta_h f\|_{L^2(\Omega_h)}\to 0$ uniformly in $f\in S$ as $h\to 0$,*
- *Uniform decay: $\|f\|_{L^2(\Omega\setminus K_j)}\to 0$ uniformly in $f\in S$ as $j\to\infty$, for some sequence $K_1\subset K_2\subset\ldots\subset\Omega$ of compact sets satisfying $\bigcup_j K_j=\Omega$.*

Proof. We first prove the "if" part. Let $\varepsilon>0$, and let j be so large that

$$\|f\|_{L^2(\Omega\setminus K_j)}<\varepsilon \qquad \text{for all} \quad f\in S, \tag{31}$$

and pick $\delta>0$ so small that

$$\|\Delta_h f\|_{L^2(\Omega_h)}<\varepsilon \qquad \text{for all} \quad h\in B_\delta, \quad f\in S, \tag{32}$$

where we recall $\Omega_h = \{x \in \Omega : [x, x+h] \subset \Omega\}$. Consider the partition

$$G_\lambda = \{\lambda a + [0, \lambda]^n : a \in \mathbb{Z}^n\}, \tag{33}$$

of $\mathbb{R}^n$, consisting of cubes of sidelength $\lambda > 0$, and collect the cubes $Q \in G_\lambda$ satisfying $Q \cap K_j \neq \varnothing$ into the collection $\{Q_1, \dots, Q_m\}$. We link λ to δ by $2\lambda n = \delta$, and lower the value of $\delta > 0$ if necessary, to ensure the following two conditions.

- All $Q_i \subset \Omega$, i.e., that $\Sigma = \bigcup_i Q_i$ satisfies $\Sigma \subset \Omega$.
- $\delta < \operatorname{dist}(\Sigma, \partial\Omega)$.

The first condition is satisfied if $\lambda > 0$ is sufficiently small. Then the second condition is granted by reducing $\delta > 0$ further, because Σ would only shrink as λ gets smaller. Note that $2\lambda n = \delta$ implies $[0, \lambda]^n \subset B_\delta$.

Next, let $X = \operatorname{span}\{\chi_{Q_i}\}$ be the space of piecewise constant functions subordinate to the lattice $\{Q_i\}$ and hence supported in Σ, and define the projector $P : L^2(\Omega) \to X$ by

$$Pf = \sum_i \left(\frac{1}{|Q_i|} \int_{Q_i} f \right) \chi_{Q_i}. \tag{34}$$

Then for $f \in S$, we have

$$\|Pf\|_{L^2}^2 = \sum_i \left(\frac{1}{|Q_i|} \int_{Q_i} f \right)^2 |Q_i| \leq \sum_i \int_{Q_i} |f|^2 = \|f\|_{L^2(\Sigma)}^2 \leq \|f\|_{L^2(\Omega)}^2, \tag{35}$$

where we have used the Cauchy-Schwarz inequality in the second step. This shows that the image $P(S)$ of S under the projection P is contained in the ball $X_M = \{\phi \in X : \|\phi\|_{L^2} \leq M\}$. Proceeding further, for $f \in S$, we have

$$\begin{aligned} \|f - Pf\|_{L^2(\Omega)} &\leq \|f - Pf\|_{L^2(\Omega \setminus \Sigma)} + \|f - Pf\|_{L^2(\Sigma)} \\ &= \|f\|_{L^2(\Omega \setminus \Sigma)} + \|f - Pf\|_{L^2(\Sigma)} \\ &\leq \varepsilon + \|f - Pf\|_{L^2(\Sigma)}, \end{aligned} \tag{36}$$

because $Pf = 0$ outside Σ, and $\Omega \setminus \Sigma \subset \Omega \setminus K_j$. The last term can be estimated as

$$\begin{aligned} \|f - Pf\|_{L^2(\Sigma)}^2 &= \sum_i \int_{Q_i} \left(f(x) - \frac{1}{|Q_i|} \int_{Q_i} f \right)^2 \mathrm{d}x \\ &= \sum_i \int_{Q_i} \left(\frac{1}{|Q_i|} \int_{Q_i} (f(x) - f(y)) \mathrm{d}y \right)^2 \mathrm{d}x \\ &\leq \sum_i \int_{Q_i} \frac{1}{|Q_i|} \int_{Q_i} |f(x) - f(y)|^2 \mathrm{d}y \, \mathrm{d}x, \end{aligned} \tag{37}$$

where we have invoked $\int_{Q_i} f(x) \mathrm{d}y = |Q_i| f(x)$ and applied the Cauchy-Schwarz inequality. At this point, we replace the domain Q_i of the inner integration by $B_\delta(x)$, and take into account the fact that $2\lambda n = \delta$, to get

$$\begin{aligned} \|f - Pf\|_{L^2(\Sigma)}^2 &\leq \sum_i \int_{Q_i} \frac{1}{|Q_i|} \int_{B_\delta(x)} |f(x) - f(y)|^2 \mathrm{d}y \, \mathrm{d}x \\ &= \frac{1}{\lambda^n} \sum_i \int_{Q_i} \int_{B_\delta} |f(x) - f(x+h)|^2 \mathrm{d}h \, \mathrm{d}x \\ &= \frac{1}{\lambda^n} \int_{B_\delta} \|\Delta_h f\|_{L^2(\Sigma)}^2 \mathrm{d}h \leq \frac{|B_\delta| \varepsilon^2}{\lambda^n} = |B_1| (2n)^n \varepsilon^2 \end{aligned} \tag{38}$$

To conclude, given any $\varepsilon > 0$, and any $f \in S$, there is a finite dimensional subspace $X \subset L^2(\Omega)$, and $g \in X$ with $\|g\|_{L^2(\Omega)} \leq M$ such that $\|f - g\|_{L^2(\Omega)} < \frac{\varepsilon}{2}$. Then any $(\frac{\varepsilon}{2})$-cover of the ball $\{g \in X : \|g\|_{L^2(\Omega)} \leq M\}$ induces an ε-cover of S.

Now we prove the "only of" part. Total boundedness of S trivially implies boundedness. To prove L^2-equicontinuity and uniform boundedness, note that

$$\|\Delta_h g\|_{L^2(\Omega_h)} \to 0 \quad \text{as } h \to 0, \qquad \text{and} \qquad \|g\|_{L^2(\Omega \setminus K_j)} \to 0 \quad \text{as } j \to \infty, \tag{39}$$

for any $g \in L^2(\Omega)$, where $K_1 \subset K_2 \subset \ldots \subset \Omega$ is any sequence of compact sets satisfying $\bigcup_j K_j = \Omega$. These properties can be "planted uniformly" on S with the help of total boundedness as follows. Let $\{K_j\}$ be a sequence of compact sets with the aforementioned properties, and let $\varepsilon > 0$. Suppose that $S_1, \ldots, S_m \subset L^2(\Omega)$ is an ε-cover of S, and pick $g_k \in S_k$ for each k. We can choose j so large, and $\delta > 0$ so small that

$$\|g_k\|_{L^2(\Omega \setminus K_j)} < \varepsilon, \qquad \text{and} \qquad \|\Delta_h g_k\|_{L^2(\Omega_h)} < \varepsilon \quad \text{for all} \quad h \in B_\delta, \tag{40}$$

for all $k = 1, \ldots, m$. Now, for any given $f \in S$, there exists j such that $\|f - g_j\|_{L^2(\Omega)} \leq \varepsilon$. Thus we have

$$\|f\|_{L^2(\Omega \setminus K_j)} \leq \|f - g_j\|_{L^2(\Omega \setminus K_j)} + \|g_k\|_{L^2(\Omega \setminus K_j)} < 2\varepsilon, \tag{41}$$

and

$$\|\Delta_h f\|_{L^2(\Omega_h)} \leq \|\Delta_h (f - g_j)\|_{L^2(\Omega_h)} + \|\Delta_h g_k\|_{L^2(\Omega_h)} < 3\varepsilon, \tag{42}$$

for all $h \in B_\delta$, concluding the proof. □

Corollary 6. *Let $\Omega \subset \mathbb{R}^n$ be an open set. A bounded subset $S \subset H^1(\Omega)$ is relatively compact in $L^2(\Omega)$ if and only if it satisfies the uniform decay condition.*

Proof. Boundedness of S in $H^1(\Omega)$ trivially implies boundedness in $L^2(\Omega)$. Recall that

$$\|\Delta_h f\|_{L^2(\Omega_h)} \leq |h| \|\nabla f\|_{L^2(\Omega)}, \qquad h \in \mathbb{R}^n, \quad f \in H^1(\Omega), \tag{43}$$

where $\Omega_h = \{x \in \Omega : [x, x+h] \subset \Omega\}$. Since S is bounded in the $H^1(\Omega)$, we infer

$$\|\Delta_h f\|_{L^2(\Omega_h)} \leq M|h|, \qquad f \in S, \tag{44}$$

for some constant $M > 0$, yielding L^2-equicontinuity of S. Given that we have boundedness and L^2-equicontinuity, the uniform decay condition is equivalent to total boundedness. □

Corollary 7. *The embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, if $\Omega \subset \mathbb{R}^n$ is a bounded domain.*

Proof. Let $E : \mathscr{D}(\Omega) \to \mathscr{D}(\mathbb{R}^n)$ be the extension operator defined by

$$E\varphi = \begin{cases} \varphi, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{45}$$

Since $\|E\varphi\|_{H^1(\mathbb{R}^n)} = \|\varphi\|_{H^1(\Omega)}$ for $\mathscr{D}(\Omega)$, this operator can be uniquely extended to an isometry $E : H^1_0(\Omega) \to H^1(\mathbb{R}^n)$. Now let $S \subset H^1_0(\Omega)$ be a set bounded in $H^1(\Omega)$. Then $E(S)$ is bounded in $H^1(\mathbb{R}^n)$. Moreover, if $r > 0$ is sufficiently large, then $\operatorname{supp} f \subset B_r$ for all $f \in E(S)$, and hence $E(S)$ trivially satisfies the uniform decay condition with respect to $\mathbb{R}^n$. Thus $E(S)$ is relatively compact in $L^2(\mathbb{R}^n)$. Finally, since the restriction $L^2(\mathbb{R}^n) \to L^2(\Omega)$ is continuous, S is relatively compact in $L^2(\Omega)$. □

To adapt the extension approach to $H^1(\Omega)$, suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain, and that *there exists* a bounded extension operator $E : H^1(\Omega) \to H^1(\mathbb{R}^n)$. Let $\phi \in \mathscr{D}(\mathbb{R}^n)$ be such that $\phi \equiv 1$ in Ω. Then we can define a bounded extension operator $\tilde{E} : H^1(\Omega) \to H^1_0(U)$ by $\tilde{E}u = \phi E u$, where U is a bounded domain containing $\operatorname{supp} \phi$. Since the embedding $H^1_0(U) \hookrightarrow L^2(U)$ is compact, it would show the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

It is a fundamental result in analysis that any bounded Lipschitz domain Ω admits an extension operator that is bounded as a map $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. These operators give rise to a streamlined approach to the theory of Sobolev spaces on domains, but in many cases one can obtain better results by working intrinsically. Here is an example where the extension approach would not work.

Example 8. Let $\Omega = \{(x, y) : 0 < x < 1,\ 0 < y < x^4\}$ and $u(x, y) = \frac{1}{x}$. We have

$$\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} = \int_0^1 \big(\frac{1}{x^2} + \frac{1}{x^4}\big)x^4 \mathrm{d}x < \infty, \tag{46}$$

meaning that $u \in H^1(\Omega)$. Suppose that there is an extension of u satisfying $\tilde{u} \in H^1(\mathbb{R}^2)$. Then we must have $\gamma\tilde{u} \in L^2(\mathbb{R})$, where $\gamma : H^1(\mathbb{R}^2) \to L^2(\mathbb{R})$ denotes the trace map onto the line $\{y = 0\}$. However, this contradicts with

$$\|\gamma\tilde{u}\|^2_{L^2(\mathbb{R})} \geq \|\gamma\tilde{u}\|^2_{L^2(\varepsilon,1)} = \|\gamma u\|^2_{L^2(\varepsilon,1)} \geq \int_\varepsilon^1 \frac{\mathrm{d}x}{x^2} \to \infty \qquad \text{as} \quad \varepsilon \to 0, \tag{47}$$

and so we conclude that there is no bounded extension operator $E : H^1(\Omega) \to H^1(\mathbb{R}^2)$.

The domain in the preceding example has a continuous boundary, in the following sense.

Definition 9. An open set $\Omega \subset \mathbb{R}^n$ is said to have a *continuous boundary*, or to be of *class* $\mathscr{C}$, if for each $y \in \partial\Omega$ there is an open neighbourhood U of y, $B_r \subset \mathbb{R}^{n-1}$ with $r > 0$, and a continuous function $\phi \in \mathscr{C}(B_r)$, such that under a rigid transformation of the coordinate system, we have $\Omega \cap U = \{(x', x_n) : y' \in B_r,\ x_n > \phi(x')\} \cap U$.

Note that by choosing $r > 0$ smaller, and shrinking U if necessary, we can always assume that $\phi \in \mathscr{C}(\bar{B}_r)$, and that

$$U_h := \{(x', x_n) : y' \in \bar{B}_r,\ \phi(x') < x_n < \phi(x') + h\} = \Omega \cap U, \tag{48}$$

for some $h > 0$.

Theorem 10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with a continuous boundary. Then the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Proof. In view of Corollary 6, we only need to prove the uniform decay condition. Without loss of generality, let $u \in \mathscr{D}(\mathbb{R}^n)$. We will work locally, i.e., we assume (48). For $x = (x', x_n) \in U_h$ and $0 \leq s \leq h$, we have

$$u(x) = u(x', s) + \int_s^{x_n} \partial_n u(x', t)\mathrm{d}t, \tag{49}$$

which, upon squaring and invoking the Cauchy-Schwarz inequality, yields

$$|u(x)|^2 \leq 2|u(x', s)|^2 + 2h \int_{\phi(x')}^{\phi(x')+h} |\partial_n u(x', t)|^2 \mathrm{d}t. \tag{50}$$

Now we integrate over $s \in [\phi(x'), \phi(x') + h]$, to get

$$h|u(x)|^2 \leq 2 \int_{\phi(x')}^{\phi(x')+h} |u(x', s)|^2 \mathrm{d}s + 2h^2 \int_{\phi(x')}^{\phi(x')+h} |\partial_n u(x', t)|^2 \mathrm{d}t. \tag{51}$$

Finally, an integration over $x_n \in [\phi(x'), \phi(x') + \delta]$, followed by integration over $x' \in B_r$, give

$$\int_{U_\delta} |u|^2 \leq \frac{2\delta}{h} \int_{U_h} |u|^2 + 2h\delta \int_{U_h} |\nabla u|^2, \tag{52}$$

where $\delta > 0$ is small, meaning that $\|u\|_{L^2(U_\delta)} \leq c\delta\|u\|_{H^1(\Omega)}$ for some constant c. This readily implies the uniform decay property by a covering argument. □

The estimate $\|u\|^2_{L^2(U_\delta)} \leq M\delta$ from the preceding proof suggests that the theorem should hold under much weaker conditions. The following is one possible improvement.

Corollary 11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with a quasi-continuous boundary, which means by definition that Ω is a finite union of domains with continuous boundaries. Then the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Proof. The underlying phenomenon here is the fact that compactness of embeddings is stable under finite union of domains. Suppose that $\Omega = \Omega_1 \cup \Omega_2$, where both $H^1(\Omega_1) \hookrightarrow L^2(\Omega_1)$ and $H^1(\Omega_2) \hookrightarrow L^2(\Omega_2)$ are compact. Let $\{u_k\} \subset H^1(\Omega)$ be a bounded sequence. Then $\{u_k|_{\Omega_1}\}$ is bounded in $H^1(\Omega_1)$, and hence there is a subsequence $\{u_{k_j}\}$ converging in $L^2(\Omega_1)$. Now, $\{u_{k_j}|_{\Omega_2}\}$ is bounded in $H^1(\Omega_2)$, which means that we can extract a subsequence that converges in $L^2(\Omega_2)$. This new subsequence is clearly Cauchy in $L^2(\Omega)$. □

Example 12. By the preceding corollary, $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact for simple slit domains, such as the unit disk with the line segment $[0,1) \times \{0\}$ removed.

3. Spectral theory of compact self-adjoint operators

In this section, we will prove the spectral theorem for compact, symmetric operators in a real Hilbert space H. This theorem will then be applied to the resolvent of the Laplacian in the next section. In the following, H will denote a real Hilbert space.

Lemma 13. *With $D \subset H$, let $A : D \to H$ be a symmetric operator, in the sense that $\langle Au, v\rangle = \langle u, Av\rangle$ for all $u, v \in D$. Then we have the following.*

a) If $u, v \in D \setminus \{0\}$ and $\lambda, \mu \in \mathbb{R}$ satisfy $Au = \lambda u$, $Av = \mu v$, and $\lambda \neq \mu$, then $\langle u, v\rangle = 0$.

b) Let $\{u_n\} \subset D$ be a complete orthogonal basis of H, such that $Au_n = \lambda_n u_n$ for each n. Suppose that $u \in D$ satisfies $Au = \lambda u$ with $\lambda \notin \{\lambda_n\}$. Then $u = 0$.

Proof. a) By symmetry, we have $\lambda\langle u, v\rangle = \langle Au, v\rangle = \langle u, Av\rangle = \mu\langle u, v\rangle$, hence $(\lambda - \mu)\langle u, v\rangle = 0$.

b) By completeness, we can write

$$u = \sum_n \langle u, u_n\rangle u_n, \qquad \text{with the convergence in } H, \tag{53}$$

but since $\lambda \notin \{\lambda_n\}$, the preceding paragraph shows that $\langle u, u_n\rangle = 0$ for all n. □

Exercise 14. Let H be a complex Hilbert space, and with $D \subset H$, let $A : D \to H$ be a self-adjoint operator, in the sense that $\langle Au, v\rangle = \langle u, Av\rangle$ for all $u, v \in D$. Prove the following.

(a) If $u \in D \setminus \{0\}$ and $\lambda \in \mathbb{C}$ satisfy $Au = \lambda u$, then $\lambda \in \mathbb{R}$.

(b) If $u, v \in D \setminus \{0\}$ and $\lambda, \mu \in \mathbb{C}$ satisfy $Au = \lambda u$, $Av = \mu v$, and $\lambda \neq \mu$, then $\langle u, v\rangle = 0$.

(c) Let $\{u_n\} \subset D$ be a complete orthonormal basis of H, such that $Au_n = \lambda_n u_n$ for each n. Suppose that $u \in H$ satisfies $Au = \lambda u$ with $\lambda \notin \{\lambda_n\}$. Then $u = 0$.

Lemma 15. *Let $B : H \to H$ be a bounded operator.*

a) Any eigenvalue $\lambda \in \mathbb{R}$ of B satisfies $|\lambda| \leq \|B\|$.

b) If B is a positive operator, in the sense that $\langle Bu, u\rangle > 0$ for all $u \in H \setminus \{0\}$, then all real eigenvalues of B are positive.

c) If B is symmetric, then the norm of B satisfies

$$\|B\| = \sup_{u \in H} \frac{|\langle Bu, u\rangle|}{\|u\|^2}. \tag{54}$$

Proof. We will only prove c). For any $u \in H$ we have

$$|\langle Bu, u\rangle| \leq \|Bu\|\|u\| \leq \|B\|\|u\|^2, \tag{55}$$

which shows that

$$\mu = \sup_{u\in H} \frac{|\langle Bu, u\rangle|}{\|u\|^2} \leq \|B\|. \tag{56}$$

On the other hand, from the parallelogram identity

$$4\langle Bu, v\rangle = \langle B(u+v), u+v\rangle - \langle B(u-v), u-v\rangle, \tag{57}$$

we infer

$$4\langle Bu, v\rangle \leq \mu\|u+v\|^2 + \mu\|u-v\|^2 = 2\mu\big(\|u\|^2 + \|v\|^2\big), \tag{58}$$

for $u, v \in H$, and putting $v = \alpha Bu$ with $\alpha > 0$ yields

$$2\alpha\|Bu\|^2 \leq \mu\big(\|u\|^2 + \alpha^2\|Bu\|^2\big), \qquad u \in H. \tag{59}$$

We choose $\alpha = \|u\|/\|Bu\|$ when $Bu \neq 0$, and, say, $\alpha = 1$ when $Bu = 0$. In either case, we get

$$\|Bu\| \leq \mu\|u\|, \tag{60}$$

which implies that $\|B\| \leq \mu$. □

Lemma 16. *Let $K : H \to H$ be a compact operator. Then we have the following.*

a) Each nonzero eigenvalue has a finite multiplicity.

b) The only possible accumulation point of the set of eigenvalues is 0*.*

Proof. a) Suppose that there is an eigenvalue $\mu \neq 0$ with infinite multiplicity, i.e., let $\{v_k\}$ be a countable orthonormal set of vectors satisfying

$$Kv_k = \mu v_k, \qquad k = 1, 2, \ldots. \tag{61}$$

We can interpret the latter as $\{v_k\}$ being the image of the set $\{\mu^{-1}v_k\}$ under K. Since $\{\mu^{-1}v_k\}$ is a bounded set, the set $\{v_k\}$ is relatively compact, meaning that after passing to a subsequence, v_k converges to some element of H. However, we have $\|v_j - v_k\|^2 = 2$ for $j \neq k$, which leads to a contradiction.

b) Note that this part contains a) as a special case. If $\alpha \neq 0$ is an accumulation point of eigenvalues, then there exists a countable orthonormal set $\{v_k\}$ of eigenvectors with corresponding eigenvalues μ_k satisfying $\inf_k |\mu_k| > 0$. The latter condition ensures that $\{\mu_k^{-1}v_k\}$ is a bounded set, and the argument we have used in a) leads to a contradiction. □

We are ready to prove the main result of this section.

Theorem 17. *Let H be a real Hilbert space, and let $K : H \to H$ be a compact, symmetric operator. Then K admits a countable set of eigenvectors $\{u_n\}$ that form an orthonormal basis of $(\ker K)^\perp$. Moreover, assuming, without loss of generality, that the corresponding eigenvalues are arranged as $|\mu_1| \geq |\mu_2| \geq \ldots > 0$, we have the variational characterization*

$$|\mu_n| = \sup_{u\in H_{n-1}} \frac{\|Ku\|}{\|u\|} = \sup_{u\in H_{n-1}} \frac{|\langle Ku, u\rangle|}{\|u\|^2}, \qquad \text{for } n = 1, 2, \ldots, \tag{62}$$

where H_{n-1} is the orthogonal complement of $\operatorname{span}\{u_1, \ldots, u_{n-1}\}$ *in H.*

Proof. If $K = 0$, then $\ker K = H$, so the theorem is trivial. Suppose that $K \neq 0$, and let $\{v_i\}$ be a sequence in H such that $\|v_i\| = 1$ and

$$|\langle Kv_i, v_i\rangle| \to \|K\| > 0. \tag{63}$$

Possibly passing to a subsequence, we may assume that

$$\langle Kv_i, v_i\rangle \to \mu_1, \tag{64}$$

where μ_1 is a number satisfying $|\mu_1| = \|K\|$. Since the set $\{Kv_i\}$ is relatively compact, passing to a subsequence if necessary, we have $Kv_i \to w$ in H. On the other hand, we have

$$\begin{aligned} 0 \leq \|Kv_i - \mu_1 v_i\|^2 &= \|Kv_i\|^2 + \mu_1^2 - 2\mu_1 \langle Kv_i, v_i\rangle \\ &\leq 2\mu_1 (\mu_1 - \langle Kv_i, v_i\rangle) \to 0, \qquad \text{as } i \to \infty, \end{aligned} \tag{65}$$

which shows that $\mu_1 v_i \to w$, that is, $v_i \to u_1 := \mu_1^{-1} w$ in H, hence $\|u_1\| = 1$. It is easy to check that $Ku_1 = w = \mu_1 u_1$, because

$$\|Ku_1 - w\| \leq \|Ku_1 - Kv_i\| + \|Kv_i - w\| \leq \|K\|\|u_1 - v_i\| + \|Kv_i - w\| \to 0. \tag{66}$$

Now let $H_1 = \{v \in H : \langle v, u_1\rangle = 0\}$, which is a closed linear subspace of H. Moreover, H_1 is invariant under K, because

$$\langle Kv, u_1\rangle = \langle v, Ku_1\rangle = \mu_1 \langle v, u_1\rangle = 0, \qquad \text{for } v \in H_1. \tag{67}$$

So if K acts on H_1 is nontrivially, we can construct as above an element $u_2 \in H_1$ with $\|u_2\| = 1$ and a number μ_2 satisfying $|\mu_2| = \|K|_{H_1}\| > 0$, such that $Ku_2 = \mu_2 u_2$. By induction, we have a (finite or infinite) sequence $\{(\mu_n, u_n)\} \subset \mathbb{R} \times H$ with $\{u_n\}$ orthonormal, satisfying $Ku_n = \mu_n u_n$ and the formula (62). This sequence is finite only if either H_m is trivial, or $K|_{H_m} = 0$, for some m. In the former case, we have $H = \operatorname{span}\{u_1, \ldots, u_m\}$. In the latter case, we infer $\ker K = H_m$, and so $\operatorname{span}\{u_1, \ldots, u_m\} = (\ker K)^\perp$.

It remains to show that $\{u_n\}$ is a basis of $(\ker K)^\perp$, in the case the sequence $\{u_n\}$ is infinite. Note that we have $\mu_n \to 0$ as $n \to 0$. Let $u \in H$ be in $\operatorname{span}\{u_n\}^\perp$, i.e., let $\langle u, u_n\rangle = 0$ for all n. Then $u \in H_n$ for each n, and hence $\|Ku\| \leq |\mu_n|\|u\| \to 0$ as $n \to 0$. This shows that $u \in \ker K$, or $\operatorname{span}\{u_n\}^\perp \subset \ker K$. On the other hand, if $u \in \ker K$, then

$$\mu_n \langle u, u_n\rangle = \langle u, Ku_n\rangle = \langle Ku, u_n\rangle = 0, \tag{68}$$

for all n, meaning that $\ker K \subset \operatorname{span}\{u_n\}^\perp$. We conclude that $\ker K = \operatorname{span}\{u_n\}^\perp$. □

Corollary 18 (Positive operators). *If K is positive, then $\mu_n > 0$ for all n, $\mu_n \to 0$ as $n \to \infty$, and the eigenvectors $\{u_n\}$ form an orthonormal basis of H.*

Corollary 19 (Spectral decomposition). *For any $u \in H$, we have*

$$Ku = \sum_n \mu_n \langle u, u_n\rangle u_n, \tag{69}$$

with the convergence in H.

Proof. It is obvious that the map $P_j : H \to \operatorname{span}\{u_1, \ldots, u_j\}$ defined by

$$P_j u = \textstyle\sum_{n=1}^j \langle u, u_n\rangle u_n, \tag{70}$$

is the orthogonal projector onto $\operatorname{span}\{u_1, \ldots, u_j\}$. In particular, we have $w_j = u - P_j u \in H_j$. In view of

$$Kw_j = Ku - \textstyle\sum^j \mu_n \langle u, u_n\rangle u_n, \tag{71}$$

we need to show that $Kw_j \to 0$ as $j \to \infty$. But this follows from

$$\|Kw_j\| \leq |\mu_{j+1}|\|w_j\| \leq |\mu_{j+1}|\|u\| \to 0, \tag{72}$$

since H_j is invariant under K and $|\mu_{j+1}| = \|K|_{H_j}\|$. □

Corollary 20 (Fredholm alternative). *Let $\mu \in \mathbb{R} \setminus \{0\}$ and let $B = \mu I - K$. Then we have $\dim\ker B < \infty$, and $\operatorname{ran} B = (\ker B)^\perp$. In particular, one and only one of the following is true.*

- *$B : H \to H$ is surjective.*
- *$B : H \to H$ is noninjective.*

Proof. For $u \in H$, Corollary 19 gives

$$Bu = (\mu I - K)\sum_n \langle u, u_n\rangle u_n = \sum_n (\mu - \mu_n)\langle u, u_n\rangle u_n. \tag{73}$$

This makes it clear that

$$\ker B = \operatorname{span}\{u_n : \mu_n = \mu\}, \qquad \operatorname{ran} B = \operatorname{span}\{u_n : \mu_n \neq \mu\}, \tag{74}$$

establishing the claim. □

4. Application to the Laplace eigenproblems

It is time to apply the general spectral theory to the Laplace eigenvalue problems. To this end, it will be convenient to introduce an intermediate level of abstraction, as follows. Let V and H be Hilbert spaces, with the embedding $V \hookrightarrow H$ compact, and V dense in H. Keep in mind that our main examples are $H = L^2(\Omega)$, and $V = H^1(\Omega)$ or $V = H^1_0(\Omega)$. It is easy to see that H is continuously embedded into V' through the injection $J : H \to V'$ given by

$$\langle Jf, v\rangle = \langle f, v\rangle_H, \qquad f \in H,\, v \in V, \tag{75}$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between V' and V, and $\langle\cdot,\cdot\rangle_H$ is the inner product in H. We will identify H with a subspace of V', and write $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_H$ if there is no risk of confusion. Let $a : V \times V \to \mathbb{R}$ be a symmetric, continuous bilinear form, and define $A : V \to V'$ by

$$\langle Au, v\rangle = a(u, v), \qquad u, v \in V. \tag{76}$$

Then we assume that the operator $A + tJ : V \to V'$ is invertible, whenever $t > t_0$, where t_0 is some constant. In what follows, we fix a value $t > t_0$, and introduce the resolvent

$$R_t = (A + tJ)^{-1}|_H : H \to H, \tag{77}$$

which is the restriction of $(A + tJ)^{-1}$ to H. Since $(A + tJ)^{-1}|_H : H \to V$ is continuous and $V \hookrightarrow H$ is compact, the resolvent is a compact operator.

Remark 21. In all applications that will follow, we have $H = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ an open set. Moreover, as we know, the main examples of V are

- $V = H^1_0(\Omega)$ for the Dirichlet boundary condition, and
- $V = H^1(\Omega)$ for the Neumann boundary condition.

Rellich's lemma guarantees the compactness of the embedding $V \hookrightarrow H$ if Ω is bounded in the Dirichlet case (Corollary 7), and if Ω is bounded and has a quasi-continuous boundary in the Neumann case (Corollary 11). In both cases, $a : V \times V \to \mathbb{R}$ is given by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v, \qquad u, v \in V. \tag{78}$$

We can take $t_0 = 0$ for the Neumann case, and $t_0 < 0$ for the Dirichlet case.

Getting back to the abstract setting, one can show as in §1 that R_t is symmetric and positive. Therefore, the results from the preceding section imply the existence of an orthonormal basis of H consisting of eigenvectors of R_t. Moreover, each eigenvalue has a finite multiplicity, they are all positive and accumulate at 0. Now if $R_t u = \mu u$ or $(A + tJ)^{-1}u = \mu u$ with $u \in H$, then $u = \mu(Au + tu)$, yielding $Au = (\frac{1}{\mu} - t)u$. This leads us to the following result, which can be regarded as an abstract version of the *Hodge theorem*.

Theorem 22. *There exists an orthonormal basis $\{u_k\}$ of H consisting of eigenvectors of A. Each eigenvalue has a finite multiplicity, and with $\{\lambda_k\}$ denoting the eigenvalues, we have $\lambda_k > -t$ for all k, and $\lambda_k \to \infty$ as $k \to \infty$.*

Furthermore, the eigenvectors $\{u_k\}$ *form an orthogonal basis of* V *with respect to the inner product* $\langle\cdot,\cdot\rangle_V = a(\cdot,\cdot) + t\langle\cdot,\cdot\rangle_H$*. In particular, for* $u \in V$*, the expansion*

$$u = \sum_k \langle u, u_k\rangle u_k, \tag{79}$$

converges in V*.*

Proof. Let $\{u_k\}$ be an orthonormal basis of H consisting of eigenvectors of R_t, and denote by $\{\mu_k\}$ the corresponding eigenvalues. Then by the above discussion, the vectors $\{u_k\}$ are also eigenvectors of A, with the eigenvalues $\{\lambda_k\}$ given by

$$\lambda_k = \frac{1}{\mu_k} - t. \tag{80}$$

Positivity of μ_k implies $\lambda_k > -t$, and $\mu_k \to 0$ implies $\lambda_k \to \infty$.

For the second part of the theorem, orthogonality of the eigenfunctions with respect to the inner product $\langle\cdot,\cdot\rangle_V$ follows from

$$a(u_j, u_k) = \langle Au_j, u_k\rangle = \lambda_j \delta_{jk}. \tag{81}$$

For $u \in V$, we have

$$a(u_j, u) = \langle Au_j, u\rangle = \lambda_j \langle u_j, u\rangle, \quad \text{so that} \quad \langle u, u_j\rangle_V = (\lambda_j + t)\langle u, u_j\rangle. \tag{82}$$

This means that the H-orthogonal projector

$$P_m u = \textstyle\sum_{k=1}^m \langle u, u_k\rangle u_k, \tag{83}$$

is also a V-orthogonal projector sending V onto $\mathrm{span}\{u_1, \ldots, u_m\}$, as

$$\langle u - P_m u, u_j\rangle_V = \langle u, u_j\rangle_V - \textstyle\sum_{k=1}^m \langle u, u_k\rangle\langle u_k, u_j\rangle_V = 0. \tag{84}$$

Hence we get the Pythogarian identity

$$\|u - P_m u\|_V^2 + \|P_m u\|_V^2 = \|u\|_V^2, \qquad u \in V, \tag{85}$$

which implies Bessel's inequality

$$\|P_m u\|_V^2 = \textstyle\sum_{k=1}^m |\langle u, u_k\rangle|^2 \|u_k\|_V \le \|u\|_V^2, \qquad u \in V. \tag{86}$$

Bessel's inequality in turn guarantees that $P_m u \to v$ in V as $m \to \infty$ for some $v \in V$. However, since $P_m u \to u$ in H, we conclude that $v = u$. □

Remark 23. In the Dirichlet case, since we can take $t = 0$, all eigenvalues are *strictly positive*. In the Neumann case, we can take any $t > 0$, hence all eigenvalues are nonnegative. The latter estimate is sharp, since $\lambda = 0$ is a Neumann eigenvalue with the eigenfunction $u \equiv 1$. In both cases, by the regularity theory of Poisson's equation (with a lower order term), the eigenfunctions are real analytic in Ω, and smooth up to the boundary if $\partial\Omega$ is smooth.

As discussed in the introduction, the Neumann eigenfunctions corresponding to the eigenvalue $\lambda = 0$ are locally constant, and hence the multiplicity of $\lambda = 0$ is equal to the number of connected components of Ω. Since the eigenfunctions are pairwise orthogonal, all the other Neumann eigenfunctions (i.e., the ones corresponding to nonzero eigenvalues) must change sign in at least one of the connected components of Ω. We will prove later that on a bounded domain, the first Dirichlet eigenfunction does not change sign, which means by orthogonality that all the other eigenfunctions must change sign.

Example 24 (Periodic boundary condition)**.** Classically, a periodic boundary condition on an interval $[0, 2\pi]$ is the collection of requirements

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad \ldots, \quad u^{(k)}(0) = u^{(k)}(2\pi), \tag{87}$$

for some k. In a variational setting, there are two equivalent ways to realize this:

- Make sense of the space $H^1(S^1)$, where $S^1 \subset \mathbb{R}^2$ is the unit circle.
- Consider the subspace $V \subset H^1(\mathbb{R})$, satisfying the property $u(x+2\pi) = u(x)$.

It can be argued that the first approach is a natural way (in the technical sense), while the second one is an essential way to impose the boundary condition. We will choose the second approach. Let $L^2_{\text{per}}(\mathbb{R}) = \{f \in L^2_{\text{loc}}(\mathbb{R}) : \tau^*_{2\pi} f = f\}$, where τ_h is the translation operator $\tau_h(x) = x + h$, and let $H^k_{\text{per}}(\mathbb{R}) = H^k_{\text{loc}}(\mathbb{R}) \cap L^2_{\text{per}}(\mathbb{R})$. A function $f \in L^2_{\text{per}}(\mathbb{R})$ is completely determined by its restriction on a given interval (a, b), as long as $b - a \geq 2\pi$, since f can be recovered by the formula $f(x + 2\pi m) = f(x)$ for $m \in \mathbb{Z}$. It can be shown that $H^k_{\text{per}}(\mathbb{R})$ is a Hilbert space for each $k \geq 0$, with $H^0_{\text{per}}(\mathbb{R}) = L^2_{\text{per}}(\mathbb{R})$, and that

$$\langle u, v\rangle_{L^2} = \int_a^b uv, \qquad \text{and} \qquad \langle u, v\rangle_{H^k} = \int_a^b \big(uv + u^{(k)}v^{(k)}\big), \tag{88}$$

are inner products in $L^2_{\text{per}}(\mathbb{R})$ and in $H^k_{\text{per}}(\mathbb{R})$, respectively, whenever $b - a \geq 2\pi$. Finally, we let $V = H^1_{\text{per}}(\mathbb{R})$, and define $A : V \to V'$ by

$$\langle Au, v\rangle = \int_0^{2\pi} u'v', \qquad u, v \in V. \tag{89}$$

It is obvious that $A + tI : V \to V'$ is invertible for any $t > 0$. Moreover, the embedding $V \hookrightarrow L^2_{\text{per}}(\mathbb{R})$ is compact, and hence the resolvent $R_t = (A + tI)|_{L^2} : L^2_{\text{per}}(\mathbb{R}) \to L^2_{\text{per}}(\mathbb{R})$ is compact. This guarantees an orthogonal basis of $L^2_{\text{per}}(\mathbb{R})$, consisting of eigenfunctions of A. To find these functions, we consider the problem

$$-u'' = \lambda u, \qquad u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \tag{90}$$

on $(0, 2\pi)$, whose solutions are

$$\phi_n(x) = \cos nx, \qquad \psi_n(x) = \sin nx, \qquad \lambda_n = n^2, \qquad n = 0, 1, \ldots. \tag{91}$$

Here the eigenvalue $\lambda_0 = 0$ is a simple eigenvalue with the eigenfunction $\phi_0(x) = 1$, while $\lambda_n = n^2$ for each $n \in \mathbb{N}$ is a double eigenvalue with the eigenfunctions ϕ_n and ψ_n. The system $\{\phi_n, \psi_n\}$ forms an orthogonal basis, because by the Weierstrass approximation theorem, trigonometric polynomials are dense in $\mathscr{C}_{\text{per}}(\mathbb{R}) = \{f \in \mathscr{C}(\mathbb{R}) : f(x + 2\pi) = f(x)\, \forall x \in \mathbb{R}\}$, which implies that $\operatorname{span}\{\phi_n, \psi_n\}$ is dense in $L^2_{\text{per}}(\mathbb{R})$.

Example 25 (Intervals)**.** It is easy to see that on the interval $I = (0, \pi)$, the sequences

$$\psi_n(x) = \sin nx, \qquad \mu_n = n^2, \qquad n = 1, 2, \ldots. \tag{92}$$

and

$$\phi_n(x) = \cos nx, \qquad \nu_n = n^2, \qquad n = 0, 1, \ldots, \tag{93}$$

respectively solve the Dirichlet and Neumann eigenproblems. Their completeness can be seen as follows. Take the Dirichlet case, and for $f \in L^2(I)$, extend it so that it is odd and periodic, that is, let $\tilde{f} \in L^2_{\text{per}}(\mathbb{R})$ be defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in (0, \pi), \\ -f(-x) & \text{for } x \in (-\pi, 0). \end{cases} \tag{94}$$

By the preceding example, $\tilde{f}$ can be approximated by trigonometric polynomials in $L^2_{\text{per}}(\mathbb{R})$. Since $\tilde{f}$ is an odd function, the resulting series will involve only the sine terms, which shows that $\operatorname{span}\{\psi_n\}$ is dense in $L^2(I)$. The completeness of $\{\phi_n\}$ can be established similarly.

Example 26 (Rectangles)**.** Consider the Dirichlet eigenvalue problem on $Q = (0,a) \times (0,b)$. Looking for eigenfunctions of the form $u(x,y) = \phi(x)\psi(y)$ yields

$$u_{m,n}(x,y) = \phi_m(x)\psi_n(y) := \sin\Big(\frac{\pi m x}{a}\Big)\sin\Big(\frac{\pi n y}{b}\Big), \tag{95}$$

with the corresponding eigenvalues

$$\mu_{m,n} = \Big(\frac{\pi m}{a}\Big)^2 + \Big(\frac{\pi n}{b}\Big)^2, \qquad m,n \in \mathbb{N}. \tag{96}$$

To show completeness, working as in the proof of Theorem 22, for $u \in H_0^1(Q)$, we get the expansion of u in terms of $u_{m,n}$ converging to some $v \in H_0^1(Q)$ in the H^1-norm. It then remains to be seen that $v = u$. Because v is constructed so that its component along $u_{m,n}$ matches with the corresponding component of u, we have the orthogonality

$$\int_Q w u_{m,n} = 0 \qquad \text{for all} \quad m,n, \tag{97}$$

where $w = u - v$. Now Fubini's theorem gives

$$\int_0^a \Big(\int_0^b w(x,y)\psi_n(y)\mathrm{d}y\Big)\phi_m(x)\mathrm{d}x = 0, \tag{98}$$

and on account of the completeness of $\{\phi_m\}$ in $L^2(0,a)$, for each n, it implies that

$$G_n(x) := \int_0^b w(x,y)\psi_n(y)\mathrm{d}y = 0, \qquad \text{for a.e. } x \in (0,a). \tag{99}$$

Hence $\bigcup_n \{x : G_n(x) \neq 0\}$ is a null set, meaning that for almost every $x \in (a,b)$ and for all n, we have $G_n(x) = 0$. By completeness, this in turn implies that for almost every $x \in (a,b)$, the set $\{y : w(x,y) = 0\}$ is of full measure. In other words,

$$\int_0^b \chi(x,y)\mathrm{d}y = 0, \qquad \text{for a.e. } x \in (0,a), \tag{100}$$

where χ is the characteristic function of $Z = \{(x,y) : w(x,y) = 0\}$. An application of Fubini's theorem finally guarantees that Z is of full measure.

Thus the full set of eigenvalues of $Q = (0,a) \times (0,b)$ is given by (96). We do not have an exact formula for the k-th eigenvalue, but we can derive a good estimate. Let

$$N(r^2) = \#\{\mu_{m,n} : \mu_{m,n} \leq r^2\}, \tag{101}$$

denote the number of eigenvalues not exceeding r^2. Then it is easy to see that

$$\frac{\pi(r-\delta)^2}{4} \leq \frac{\pi}{a}\cdot\frac{\pi}{b}\cdot N(r^2) \leq \frac{\pi r^2}{4}, \tag{102}$$

where $\delta = \sqrt{(\frac{\pi}{a})^2 + (\frac{\pi}{b})^2}$. This leads to the asymptotic formula

$$N(r^2) = \frac{ab}{\pi}r^2 + O\big((a+b)r\big) = \frac{|Q|}{\pi}r^2 + O\big(|\partial Q|r\big) \qquad \text{as} \quad r \to \infty, \tag{103}$$

which gives, upon "inverting"

$$\mu_k = \frac{4\pi k}{|Q|} + O\big(\sqrt{k}\,\big) \qquad \text{as} \quad k \to \infty. \tag{104}$$

The exact same asymptotic is true for the Neumann eigenvalues. Moreover, for the n-dimensional rectangle $Q = (0,a_1) \times \ldots \times (0,a_n)$, we have

$$N(r^2) = c_n|Q|r^n + O\big(|\partial Q|r^{n-1}\big) \qquad \text{as} \quad r \to \infty, \tag{105}$$

with $c_n = \frac{|S^{n-1}|}{n(2\pi)^n} = \frac{1}{(2\sqrt{\pi})^n\Gamma(\frac{n}{2}+1)}$.

Remark 27 (Rectangles are extension domains)**.** The Neumann eigenproblem on the rectangle $Q = (0, a) \times (0, b)$ is solved by

$$u_{m,n}(x, y) = \cos\Big(\frac{\pi m x}{a}\Big) \cos\Big(\frac{\pi n y}{b}\Big), \tag{106}$$

and

$$\nu_{m,n} = \Big(\frac{\pi m}{a}\Big)^2 + \Big(\frac{\pi n}{b}\Big)^2, \qquad m, n \in \mathbb{N}_0. \tag{107}$$

By the general theory, given any $v \in H^1(Q)$, we have $v_j \to v$ in $H^1(Q)$, where

$$v_j = \sum_{m+n\leq j} \langle v, u_{m,n}\rangle u_{m,n}. \tag{108}$$

Now, the fact that $u_{m,n} \in \mathscr{C}^\infty(\mathbb{R}^2)$ and hence $v_j \in \mathscr{C}^\infty(\mathbb{R}^2)$ suggests us a simple way to extend v to some $\tilde{v} \in H^1(\mathbb{R}^2)$. Namely, let $\chi \in \mathscr{D}(\mathbb{R}^2)$ be a nonnegative function satisfying $\chi \equiv 1$ in a neighbourhood of Q, and define $\tilde{v}_j = \chi v_j$. Then we have

$$\|\tilde{v}_j - \tilde{v}_k\|_{H^1(\mathbb{R}^2)} = \|\chi(v_j - v_k)\|_{H^1(\mathbb{R}^2)} \leq C\|v_j - v_k\|_{H^1(Q)}, \tag{109}$$

because v_j in $\mathbb{R}^2$ is made out of translations and reflections of $v_j|_Q$. This implies that $\tilde{v}_j \to \tilde{v}$ in $H^1(\mathbb{R}^2)$ for some $\tilde{v} \in H^1(\mathbb{R}^2)$, with $\|\tilde{v}\|_{H^1(\mathbb{R}^2)} \leq C\|v\|_{H^1(Q)}$ and $\tilde{v}|_Q = v$.

Remark 28 (Disjoint unions)**.** Let $U_1, \ldots, U_m$ be bounded, disjoint open sets in $\mathbb{R}^n$, and let $(\tilde{\mu}_{i,k})_k$ be the Dirichlet eigenvalues of U_i. Then the whole collection $(\tilde{\mu}_{i,k})_{i,k}$ coincides with the collection $(\mu_k)_k$ of the Dirichlet eigenvalues of the union $U = \bigcup_i U_i$. Indeed, each $\tilde{\mu}_{i,k}$ is an eigenvalue of U. Moreover, let $\{u_{i,k}\}_k$ be an orthonormal basis of eigenfunctions on U_i. We can extend $u_{i,k} \in H^1_0(U_i)$ to $u_{i,k} \in H^1_0(U)$ by assigning $u_{i,k} = 0$ on all U_j with $j \neq i$. Then it is clear that $\{u_{i,k}\}_{i,k}$ is complete in $H^1_0(U) = \bigoplus_i H^1_0(U_i)$, and thus $(\tilde{\mu}_{i,k})_{i,k}$ are precisely the Dirichlet eigenvalues of U. The same result is true for the Neumann case, provided that the corresponding resolvents are compact.

5. Courant's minimax principle

Keeping the abstract setting from the preceding section intact, we want to derive variational characterizations of the eigenvalues in terms of the *Rayleigh quotient*

$$\rho(u) = \frac{a(u, u)}{\|u\|^2}, \qquad u \in V, \tag{110}$$

where $\|\cdot\| = \|\cdot\|_H$ is the norm in H. In what follows, these characterizations will provide a basic device with which we extract precise spectral information.

Lemma 29. *Suppose that the eigenvalues are ordered so that $\lambda_1 \leq \lambda_2 \leq \ldots$, counting multiplicities. For each k, we have*

$$\lambda_k = \min_{H_{k-1}} \rho, \tag{111}$$

where $H_{k-1} = \{u \in V : \langle u, u_j\rangle = 0,\ j = 1, \ldots, k-1\}$, and if $u \in H_{k-1}$ satisfies $\rho(u) = \lambda_k$ then $Au = \lambda_k u$. We also have

$$\lambda_k = \max_{\mathrm{span}\{u_1,\ldots,u_k\}} \rho, \tag{112}$$

with the maximum attained only by the eigenfunctions corresponding to λ_k.

Proof. For $u \in H$, the series $\sum_k \langle u, u_k\rangle u_k$ converges in H to u. This in combination with continuity of the inner product implies *Plancherel's identity*

$$\|u\|^2 = \langle u, u\rangle = \big\langle \textstyle\sum_k \langle u, u_k\rangle u_k, u\big\rangle = \sum_k |\langle u, u_k\rangle|^2. \tag{113}$$

Similarly, for $u \in V$, by continuity of $a : V \times V \to \mathbb{R}$ we get

$$a(u, u) = a\big(\textstyle\sum_k \langle u, u_k\rangle u_k, u\big) = \sum_k \langle u, u_k\rangle a(u_k, u) = \sum_k \lambda_k |\langle u, u_k\rangle|^2, \tag{114}$$

where we have used $a(u_k, u) = \lambda_k \langle u_k, u\rangle$.

If in addition, $u \in H_{j-1}$, i.e., if $u \perp \operatorname{span}\{u_1, \dots, u_{j-1}\}$, then we have

$$a(u,u) = \sum_{k\geq j} \lambda_k |\langle u, u_k\rangle|^2 \geq \lambda_j \sum_{k\geq j} |\langle u, u_k\rangle|^2 = \lambda_j \|u\|^2, \tag{115}$$

with the equality occurring if and only if u is in the eigenspace of λ_j. We have established (111), and the fact that $u \in H_{j-1}$ satisfies $\rho(u) = \lambda_j$ if and only if $Au = \lambda_j u$.

The characterization (112) follows from the fact that for $u \in \operatorname{span}\{u_1, \dots, u_j\}$, we have

$$a(u,u) = \sum_{k\leq j} \lambda_k |\langle u, u_k\rangle|^2 \leq \lambda_j \sum_{k\leq j} |\langle u, u_k\rangle|^2 = \lambda_j \|u\|^2, \tag{116}$$

with the equality occurring if and only if u is in the eigenspace of λ_j. □

Remark 30 (Jackson and Bernstein inequalities)**.** Let $P_k : H \to E_k$ be the orthogonal projection onto $E_k = \operatorname{span}\{u_1, \dots, u_k\}$. Since $a(u, u - P_k u) = 0$, the inequality (111) implies

$$\|u - P_k u\|^2 \leq \frac{a(u,u)}{\lambda_{k+1}}, \qquad u \in V. \tag{117}$$

In the context of Fourier series (Example 24), this gives the Jackson inequality

$$\inf_{v\in\Sigma_n} \|u - v\|_{L^2} \leq n^{-1} \|u'\|_{L^2}, \qquad u \in H^1_{\text{per}}(\mathbb{R}), \tag{118}$$

where $\Sigma_n = \operatorname{span}\{1, \dots, \cos nx, \sin nx\}$. On the other hand, (112) can be written as

$$a(v,v) \leq \lambda_k \|v\|^2, \qquad v \in E_k, \tag{119}$$

yielding the Bernstein inequality

$$\|v'\|_{L^2} \leq n \|v\|_{L^2}, \qquad v \in \Sigma_n, \tag{120}$$

for trigonometric polynomials.

Remark 31 (Friedrichs inequality)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the Dirichlet eigenvalue problem on Ω. Then the best constant c in the inequality

$$\|u\|_{L^2(\Omega)} \leq c \|\nabla u\|_{L^2(\Omega)} \qquad \text{for} \quad u \in H_k, \tag{121}$$

is $c = 1/\sqrt{\lambda_{k+1}}$. As $H_1 = V = H^1_0(\Omega)$, the case $k = 0$ is simply the Friedrichs inequality.

Remark 32 (Poincaré inequality)**.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain, and that its Neumann Laplacian has a compact resolvent. We then have the same characterization of the sharp constants for the inequalities (121) as in the previous remark, except of course now $V = H^1(\Omega)$ and λ_k are the Neumann eigenvalues. Since $\lambda_1 = 0$, the first nontrivial inequality occurs when $k = 1$, with the sharp constant $c = 1/\sqrt{\lambda_2}$. Keeping in mind that $u_1 \equiv 1$, let $P : H \to E_1$ be the orthogonal projection onto the constants $E_1 = \operatorname{span}\{u_1\}$. Note that Pu is simply the average of u over Ω. By construction, we have $u - Pu \in H_1$ for $u \in H^1(\Omega)$, and so

$$\|u - Pu\|_{L^2(\Omega)} \leq c \|\nabla u\|_{L^2(\Omega)} \qquad \text{for} \quad u \in H^1(\Omega). \tag{122}$$

This is called the *Poincaré inequality*.

Note that the variational characterizations

$$\lambda_k = \max_{u\in\operatorname{span}\{u_1,\dots,u_k\}} \rho(u) = \min_{u\in H_{k-1}} \rho(u), \tag{123}$$

given by Lemma 29 involve the eigenfunctions $\{u_j\}$, so it is not very convenient if, e.g., one is only interested in the eigenvalues. In any case, it is possible to remove the dependence on eigenfunctions altogether by adding one more layer of extremalization, because the space $\operatorname{span}\{u_1, \dots, u_k\}$ is positioned in an optimal way inside the manifold of k dimensional subspaces of V (this manifold is called the k-th Grassmannian of V).

Theorem 33 (Courant's minimax principle)**.** *We have*

$$\lambda_k = \min_{X\in\Phi_k} \max_X \rho, \tag{124}$$

where $\Phi_k = \Phi_k(V) = \{X \subset V \text{ linear subspace} : \dim X = k\}$*.*

Proof. The first equality in (123) shows that

$$\lambda_k \geq \min_{X\in\Phi_k} \max_{u\in X} \rho(u). \tag{125}$$

On the other hand, if $X \in \Phi_k$, then $X \cap H_{k-1}$ is nontrivial by dimensional considerations. This means that there is a nonzero $v \in X \cap H_{k-1}$, hence $\rho(v) \geq \lambda_k$ by the second characterization in (123), that is, $\max\limits_X \rho \geq \lambda_k$. As $X \in \Phi_k$ was arbitrary, we conclude

$$\lambda_k \leq \min_{X\in\Phi_k} \max_{u\in X} \rho(u), \tag{126}$$

establishing the theorem. □

In what follows, the Dirichlet and Neumann eigenvalues of Ω are denote by

$$\mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots, \qquad \text{and} \qquad \nu_1(\Omega) \leq \nu_2(\Omega) \leq \ldots, \tag{127}$$

respectively, whenever they make sense.

Since $H^1_0(\Omega)$ is a subspace of $H^1(\Omega)$, it is clear that $\Phi_k(H^1_0(\Omega)) \subset \Phi_k(H^1(\Omega))$. This observation leads to the following simple inequality between Dirichlet and Neumann eigenvalues.

Corollary 34 (Basic Dirichlet-Neumann comparison)**.** *If* Ω *is a bounded domain with a quasi-continuous boundary, then we have* $\nu_k(\Omega) \leq \mu_k(\Omega)$ *for all* k*.*

The next corollary is based on the observation that if $\Omega_1 \subset \Omega_2$ then $H^1_0(\Omega_1) \subset H^1_0(\Omega_2)$.

Corollary 35 (Domain monotonicity for Dirichlet eigenvalues)**.** *If* $\Omega_1 \subset \Omega_2$ *are bounded domains, then we have* $\mu_k(\Omega_2) \leq \mu_k(\Omega_1)$ *for all* k*.*

Proof. For $u \in H^1_0(\Omega_1)$, let us denote by $\tilde{u} \in L^2(\Omega_2)$ the extension of u by 0 outside Ω_1. We claim that $\tilde{u} \in H^1_0(\Omega_2)$ with $\|\tilde{u}\|_{H^1(\Omega_2)} = \|u\|_{H^1(\Omega_1)}$. If the claim is true, $H^1_0(\Omega_1)$ can be considered as a subspace of $H^1_0(\Omega_2)$, and

$$\rho(u, \Omega_2) = \frac{\|\nabla u\|^2_{L^2(\Omega_2)}}{\|u\|^2_{L^2(\Omega_2)}} = \frac{\|\nabla u\|^2_{L^2(\Omega_1)}}{\|u\|^2_{L^2(\Omega_1)}} = \rho(u, \Omega_1), \tag{128}$$

for $u \in H^1_0(\Omega_1)$, where extension of u by 0 outside Ω_1 is understood in necessary places.

To see that the claim is true, let $\{\phi_k\} \subset \mathscr{D}(\Omega_1)$ be a sequence converging to u in $H^1(\Omega_1)$. Passing to a subsequence if necessary, we can arrange that ϕ_k converges almost everywhere in Ω_1 to u. Then we extend each ϕ_k by 0 outside Ω_1, and note that ϕ_k converges almost everywhere in Ω_2 to $\tilde{u}$. Now the equality $\|\phi_j - \phi_k\|_{H^1(\Omega_1)} = \|\phi_j - \phi_k\|_{H^1(\Omega_2)}$ and the completeness of $H^1_0(\Omega_2)$ imply that ϕ_k converges in the $H^1(\Omega_2)$ norm to some $v \in H^1_0(\Omega_2)$. Again passing to a subsequence if necessary, the convergence is almost everywhere in Ω_2. Hence $v = \tilde{u}$ almost everywhere in Ω_2, which means that $\tilde{u} \in H^1_0(\Omega_2)$. □

As it turns out, domain monotonicity does *not* hold for Neumann eigenvalues.

Example 36. The Neumann eigenvalues of the rectangle with sides a and b are

$$\nu_{k,\ell} = \frac{(\pi k)^2}{a^2} + \frac{(\pi \ell)^2}{b^2}, \qquad k, \ell \in \mathbb{N}_0. \tag{129}$$

So assuming that $a > b$, the first 3 eigenvalues are

$$\nu_1 = 0, \qquad \nu_2 = \frac{\pi^2}{a^2}, \qquad \text{and} \qquad \nu_3 = \min\Big\{\frac{2\pi^2}{a^2}, \frac{\pi^2}{b^2}\Big\}. \tag{130}$$

We pick $1 < a < \sqrt{2}$, and choose $b > 0$ small, so that the rectangle can be placed inside the unit square. For the unit square, the first 3 Neumann eigenvalues are

$$\nu_1' = 0, \qquad \nu_2' = \pi^2, \qquad \text{and} \qquad \nu_3' = \pi^2. \tag{131}$$

Since $a > 1$, we have $\nu_2 < \nu_2'$, which could not happen if domain monotonicity were true.

Even though domain monotonicity is not true in the strict sense, a very weak form of domain monotonicity holds for the Neumann eigenvalues.

Corollary 37 (Weak domain monotonicity for Neumann eigenvalues)**.** *Suppose that Ω_1 is a bounded domain having the H^1 extension property, that is, there exists a bounded extension operator $E : H^1(\Omega_1) \to H^1(\mathbb{R}^n)$. Let Ω_2 be another bounded domain such that $\Omega_1 \subset \Omega_2$. Then there exists a constant c such that $\nu_k(\Omega_2) \le c\nu_k(\Omega_1)$ for all k.*

With an extension operator $\tilde{E} : H^1(\Omega_1) \to H^1(\Omega_2)$ playing the role of an injection, the proof of the preceding corollary is similar to that of Corollary 35.

6. Weyl's asymptotic law

Recall from Example 26 that for an n-dimensional rectangle Ω, we have

$$N(r^2) = c_n|\Omega|r^n + O\big(|\partial\Omega|r^{n-1}\big) \qquad \text{as} \quad r \to \infty, \tag{132}$$

where $N(r^2)$ denotes the number of eigenvalues (either Dirichlet or Neumann) not exceeding r^2, and c_n is a constant depending only on n. Inspired by this and other examples, the asymptotic law of the form $N(r^2) = c_n|\Omega|r^n + o\big(r^n\big)$ for general domains was conjectured by Arnold Sommerfeld and Hendrik Antoon Lorentz in 1910, and proved by Hermann Weyl in 1911. This law is now called *Weyl's law*, which we will prove here.

Remark 38. Before proving the precise asymptotic, let us establish the behavior $N(r^2) \sim r^n$. Note that this gives the expectation $\mu_k \sim \nu_k \sim k^{2/n}$ for the eigenvalues. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and consider a rectangle Q_1 contained in Ω, and a rectangle Q_2 containing Ω. Then by the domain monotonicity of Dirichlet eigenvalues, we immediately infer the existence of two constants $\alpha > 0$ and $\beta < \infty$, such that

$$\alpha k^{2/n} \le \mu_k \le \beta k^{2/n} \qquad \text{for all } k. \tag{133}$$

In addition, suppose that the resolvent of the Neumann Laplacian is compact. As Remark 27 indicates, there is a bounded extension operator $E : H^1(Q_1) \to H^1(\Omega)$, and hence by the weak domain monotonicity, there exists $\beta' < \infty$ such that

$$\nu_k \le \beta' k^{2/n} \qquad \text{for all } k. \tag{134}$$

Moreover, *if* there is a bounded extension operator $E : H^1(\Omega) \to H^1(\mathbb{R}^n)$, we can compare the eigenvalues of Ω with those of Q_2, meaning that there exists $\alpha' > 0$ such that

$$\nu_k \ge \alpha' k^{2/n} \qquad \text{for all large } k. \tag{135}$$

Example 39. An intuitive reason behind Weyl's law is that high frequency oscillations in different "pieces" of Ω are "decoupled," and $N(r^2)$ is proportional to the volume $|\Omega|$ simply because more volume contains more "pieces." Let Ω be a finite disjoint union of rectangles, i.e., Ω is given by the interior of $\bar{Q}_1 \cup \ldots \cup \bar{Q}_m$, where $Q_1, \ldots, Q_m$ are pairwise disjoint rectangles in $\mathbb{R}^n$. Let $(\tilde{\mu}_{i,k})_k$ and $(\tilde{\nu}_{i,k})_k$ be the Dirichlet and Neumann eigenvalues of Q_i. We denote the nondecreasing arrangement of $(\tilde{\mu}_{i,k})_{i,k}$ and $(\tilde{\nu}_{i,k})_{i,k}$, respectively, by

$$\tilde{\mu}_1 \le \tilde{\mu}_2 \le \ldots, \qquad \text{and} \qquad \tilde{\nu}_1 \le \tilde{\nu}_2 \le \ldots. \tag{136}$$

In light of Remark 28, these are exactly the Dirichlet and Neumann eigenvalues of the union $U = Q_1 \cup \ldots \cup Q_m$. Since $H^1_0(U) \subset H^1_0(\Omega) \subset H^1(\Omega) \subset H^1(U)$, we have

$$\tilde{\nu}_k \leq \nu_k \leq \mu_k \leq \tilde{\mu}_k \qquad \text{for all } k, \tag{137}$$

where as usual (μ_k) and (ν_k) denote respectively the Dirichlet and Neumann eigenvalues of Ω. This means that the corresponding eigenvalue counting functions satisfy

$$N_{\tilde{\mu}} \leq N_\mu \leq N_\nu \leq N_{\tilde{\nu}}. \tag{138}$$

It is straightforward to estimate $N_{\tilde{\mu}}$ and $N_{\tilde{\mu}}$ as

$$\begin{aligned} N_{\tilde{\mu}}(r^2) &= \#\{\tilde{\mu}_k \leq r^2\} = \sum_i \#\{\tilde{\mu}_{i,k} \leq r^2\} \\ &= \sum_i \Big(c_n |Q_i| r^n + O\big(|\partial Q_i| r^{n-1}\big)\Big) \\ &= c_n |\Omega| r^n + O\big(\textstyle\sum_i |\partial Q_i| r^{n-1}\big), \end{aligned} \tag{139}$$

and similarly for $N_{\tilde{\nu}}$, implying the same asymptote for both N_μ and N_ν.

The second part of the following result is due to Richard Courant.

Theorem 40 (Weyl 1911, Courant 1920)**.** *For $\Omega \subset \mathbb{R}^n$ a bounded open set, the Dirichlet eigenvalue counting function satisfy*

$$N_\mu(r^2) = c_n |\Omega| r^n + o(r^n) \qquad \text{as} \quad r \to \infty. \tag{140}$$

In addition, if Ω has a quasi-Lipschitz boundary, in the sense that it is a finite union of Lipschitz domains, then we have

$$N_\mu(r^2) = c_n |\Omega| r^n + O(r^{n-1} \log r) \qquad \text{as} \quad r \to \infty. \tag{141}$$

Proof. Given any $\delta > 0$, we can construct domains Ω^- and Ω^+ of the form considered in the preceding example (that is, finite disjoint unions of rectangles) satisfying $\Omega^- \subset \Omega \subset \Omega^+$ and $|\Omega^+ \setminus \Omega^-| < \delta$. In particular, for each $\delta > 0$, there exists C_δ such that

$$c_n(|\Omega| - \delta) r^n - C_\delta r^{n-1} \leq N_\mu(r^2) \leq c_n(|\Omega| + \delta) r^n + C_\delta r^{n-1}, \tag{142}$$

for all $r > 0$. Let $\varepsilon > 0$ be arbitrary. Then we pick $\delta > 0$ so small that $2c_n\delta < \varepsilon$, and choose $r_* > 0$ so large that $2C_\delta < r_*\varepsilon$. This guarantees that

$$c_n |\Omega| r^n - \varepsilon r^n \leq N_\mu(r^2) \leq c_n |\Omega| r^n + \varepsilon r^n, \qquad r > r_*, \tag{143}$$

establishing the first part of the theorem.

To prove the second part of the theorem, we need some control on C_δ. In order for this, we consider the cubical mesh

$$G(\lambda) = \{\lambda a + [0, \lambda]^n : a \in \mathbb{Z}^n\}, \tag{144}$$

of $\mathbb{R}^n$, consisting of cubes of sidelength $\lambda > 0$ small enough. Our zeroth step is to construct Ω_0^- as the union of the cubes contained in Ω, and Ω_0^+ as the union of the cubes that intersect Ω nontrivially. In the next step, we consider $G(\lambda/2)$, and construct Ω_1^- and Ω_1^+ accordingly, by using the cubes that are twice smaller than before. As the error term in (139) grows with the total surface area of the rectangles, we want to keep the number of cubes in the decomposition of Ω_1^- and Ω_1^+ into cubes minimal. To this end, we will keep the original decomposition of Ω_0^- into cubes from $G(\lambda)$, and only decompose the differences $\Omega_1^- \setminus \Omega_0^-$ and $\Omega_1^+ \setminus \Omega_0^-$ into cubes from $G(\lambda/2)$. It is obvious that at the k-th step, the number of cubes in $G(\lambda/2^k)$ that are contained in $\Omega_k^+ \setminus \Omega_{k-1}^-$ is proportional to the number of cubes in $G(\lambda/2^{k-1})$ that intersect $\partial\Omega$. By the quasi-Lipschitz property of $\partial\Omega$, this number is bounded by a constant multiple of $2^{-(n-1)k}$. Thus we get

$$|\Omega_k^+ \setminus \Omega_k^-| = O(2^{-k}), \tag{145}$$

and that the total surface area of the cubes in $G(\lambda/2^k)$ that are contained in $\Omega_k^+ \setminus \Omega_{k-1}^-$ is bounded by a constant (independent of k). The latter means that the total surface area of the cubes in Ω_k^+ (and hence in Ω_k^-) is bounded by a constant multiple of k. Therefore, in view of (139), we infer

$$c_n(|\Omega| - A2^{-k})r^n - Ckr^{n-1} \leq N_\mu(r^2) \leq c_n(|\Omega| + A2^{-k})r^n + Ckr^{n-1}, \tag{146}$$

for all $r > 0$, with some constants A and C independent of k. We pick $k \in \mathbb{N}$ depending on $r > 0$, such that $2^k \leq r < 2^{k+1}$, that is, $k = [\log_2 r]$. This guarantees that $2^{-k}r = O(1)$ and $k = O(\log r)$, completing the proof. □

7. Problems and exercises

1. Produce a bounded domain $\Omega \subset \mathbb{R}^n$ such that the embedding $H^k(\Omega) \hookrightarrow L^2(\Omega)$ is not compact for all $k \in \mathbb{N}$.

2. Show that the embedding $H^1(B) \hookrightarrow L^q(B)$ is *not* compact, where $B \subset \mathbb{R}^n$ is an open ball, and $q = \frac{2n}{n-2}$.

3 (Kolmogorov-Riesz criterion)**.** Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $1 \leq p < \infty$. Prove that a subset $S \subset L^p(\Omega)$ is relatively compact if and only if S satisfies the following conditions.

- Boundedness: There is $M < \infty$ such that $\|f\|_{L^p(\Omega)} < M$ for all $f \in S$,
- L^p-equicontinuity: $\|\Delta_h f\|_{L^p(\Omega_h)} \to 0$ uniformly in $f \in S$ as $h \to 0$,
- Uniform decay: $\|f\|_{L^p(\Omega \setminus K_j)} \to 0$ uniformly in $f \in S$ as $j \to \infty$, for some sequence $K_1 \subset K_2 \subset \ldots \subset \Omega$ of compact sets satisfying $\bigcup_j K_j = \Omega$.

4. Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain, and let $1 \leq p < \infty$. Show that a bounded set $S \subset W^{1,p}(\Omega)$ satisfying the uniform decay condition is relatively compact in $L^p(\Omega)$. Is there an unbounded domain $\Omega \subset \mathbb{R}^n$ for which the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact? What about the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$? Here $W_0^{1,p}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in $W^{1,p}(\Omega)$.

5. Let $L^2_{\mathrm{per}}(\mathbb{R}) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}) : \tau_{2\pi}^* f = f\}$, where τ_h is the translation operator $\tau_h(x) = x + h$, and let $H^k_{\mathrm{per}}(\mathbb{R}) = H^k_{\mathrm{loc}}(\mathbb{R}) \cap L^2_{\mathrm{per}}(\mathbb{R})$.

(a) Show that $H^k_{\mathrm{per}}(\mathbb{R})$ is a Hilbert space for each $k \geq 0$, with $H^0_{\mathrm{per}}(\mathbb{R}) = L^2_{\mathrm{per}}(\mathbb{R})$, and that

$$\langle u, v \rangle_{L^2} = \int_a^b uv, \qquad \text{and} \qquad \langle u, v \rangle_{H^k} = \int_a^b \big(uv + u^{(k)} v^{(k)}\big), \tag{147}$$

are inner products in $L^2_{\mathrm{per}}(\mathbb{R})$ and in $H^k_{\mathrm{per}}(\mathbb{R})$, respectively, whenever $b - a \geq 2\pi$.

(b) Show that $\mathscr{C}^\infty_{\mathrm{per}}(\mathbb{R}) = \{f \in \mathscr{C}^\infty(\mathbb{R}) : f(x) = f(x + 2\pi)\}$ is dense in $H^k_{\mathrm{per}}(\mathbb{R})$ for each $k \geq 0$.

(c) Show that the embedding $H^1_{\mathrm{per}}(\mathbb{R}) \hookrightarrow L^2_{\mathrm{per}}(\mathbb{R})$ is compact.

6. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the *Poincaré inequality*

$$\int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2, \tag{148}$$

that is hypothesized to hold for all $u \in H^1(\Omega)$ with $\int_\Omega u = 0$, where $C = C(\Omega)$ is a constant.

(a) Is there a bounded domain Ω for which C is infinite?

(b) Characterize the best constant $C > 0$ appearing in the Poincaré inequality via an eigenvalue problem.

(c) Find the best constant when Ω is the rectangle $\Omega = (0, a) \times (0, b)$. Exhibit a function u that achieves the equality in (148).

7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary, and let

$$a(u, v) = \int_\Omega (a_{ij}\partial_i u \partial_j v + buv),$$

where the repeated indices are summer over, and the coefficients a_{ij} and b are smooth functions in $\bar{\Omega}$, with a symmetric matrix $[a_{ij}]$ satisfying the uniform ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \qquad \xi \in \mathbb{R}^n, \quad x \in \bar{\Omega},$$

for some constant $\alpha > 0$. We define the map $\tilde{A} : H_0^1(\Omega) \to [H_0^1(\Omega)]'$ by $\langle \tilde{A}u, v\rangle = a(u, v)$ for $u, v \in H_0^1(\Omega)$, and then we let A be the unbounded operator in $L^2(\Omega)$ that is given by the restriction of $\tilde{A}$ onto $L^2(\Omega)$, i.e., $Au = \tilde{A}u$ for $u \in \mathrm{Dom}(A)$ where

$$\mathrm{Dom}(A) = \{u \in H_0^1(\Omega) : \tilde{A}u \in L^2(\Omega)\}.$$

Consider the generalized eigenvalue problem

$$Au = \lambda\rho u,$$

where ρ is a (fixed) positive smooth function in $\bar{\Omega}$. Prove the following, by using the spectral theorem for compact self-adjoint positive operators where possible.

(1) The eigenvalues $\{\lambda_k\}$ are countable and real, and that $\lambda_k \to \infty$ as $k \to \infty$. Each eigenvalue has a finite multiplicity.
(2) The eigenfunctions $\{u_k\}$ form a complete orthonormal system in $L^2(\Omega)$, with respect to a suitable inner product.
(3) The system $\{u_k\}$ is complete and orthogonal in $H_0^1(\Omega)$, with respect to the inner product $a(u, v) + t\int_\Omega \rho uv$, where t is a suitably chosen constant.
(4) The eigenfunctions are smooth in Ω, and are smooth up to the boundary if $\partial\Omega$ is smooth.

8. Give a complete proof of Corollary 37.

9 (Maximin principle)**.** Show that

$$\lambda_k = \max_{X \in \Phi_{k-1}} \inf_{u \in X^\perp} \rho(u), \tag{149}$$

where $X^\perp$ is understood as $\{u \in V : u \perp_{L^2} X\}$.