HARMONIC FUNCTIONS

TSOGTGEREL GANTUMUR

Abstract. In these notes, we explore the fundamental properties of harmonic functions by using relatively elementary methods.

Contents

1. Introduction

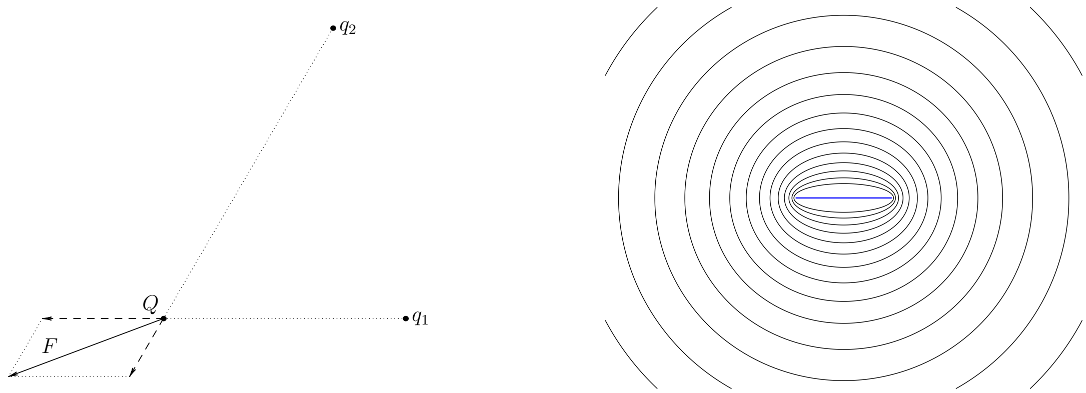
Newton's *law of universal gravitation*, first published in his *Principia* in 1687, asserts that the force exerted on a point mass Q at $x \in \mathbb{R}^3$ by the system of finitely many point masses q_i at $y_i \in \mathbb{R}^3$, $(i = 1, \ldots, m)$, is equal to

$$F = \sum_{i=1}^{m} \frac{Cq_iQ}{|x - y_i|^2} \frac{x - y_i}{|x - y_i|} \in \mathbb{R}^3, \tag{1}$$

with a constant $C < 0$ (like masses attract). Here Q and q_i are understood as real numbers that measure how much mass the corresponding points have, and $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ is the Euclidean length of the vector $a \in \mathbb{R}^3$. The same law of interaction between point charges was discovered experimentally by Charles Augustin de Coulomb and announced in 1785, now with $C > 0$ (like charges repel). Note that the numerical value of the constant C depends on the unit system one is using to measure force, mass (or charge), and distance.

It is convenient to view the force $F = F(x)$ as a vector function of x, that is, a vector field. This means that we fix the configuration of the point masses $\{q_i\}$, and think of Q as a test

Date: September 21, 2018.



(A) Electrostatic force acting on Q by the two charges q_1 and q_2, cf. (1).

(B) Contour lines of the potential produced by a charged wire, cf. Example 2.

Figure 1. Electrostatic (or gravitational) force and potential.

mass, that can be placed at any point in space to "probe the field." After the introduction of the function

$$u(x) = \sum_{i=1}^{m} \frac{Cq_i}{|x - y_i|}, \tag{2}$$

into the theory of gravitation by Daniel Bernoulli in 1748, Joseph-Louis Lagrange noticed in 1773 that

$$F(x) = -Q\nabla u(x), \qquad \text{at points} \quad x \neq y_i, \tag{3}$$

where $\nabla u = (\partial_1 u, \partial_2 u, \partial_3 u)$ is the *gradient* of u. It is remarkable that the force field F can be encoded in a scalar function u, which is called the *potential function*, or simply the *potential*. We see that the potential u and therefore the vector field[1] $E = -\nabla u$ do not depend on the test mass Q, and hence can be thought of as preexisting entities that characterize the gravitational (or electric) field generated by the point masses $\{q_i\}$. In fact, we call E the *gravitational field* (or the *electric field*).

For a continuous distribution of mass (or charge) with density ρ, vanishing outside some bounded set, the formula (2) becomes

$$u(x) = C \int_{\mathbb{R}^3} \frac{\rho(y)\mathrm{d}y}{|x - y|}. \tag{4}$$

As observed by Pierre-Simon Laplace in 1782, whether it is given by (2) or by (4), the potential satisfies

$$\Delta u = 0 \qquad \text{in free space}, \tag{5}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the 3 dimensional *Laplacian*, and *free space* means a place where there is no mass (or charge). In other words, the field $E = -\nabla u$ is divergence free in free space. The equation $\Delta u = 0$ came to be known as the *Laplace equation*, and functions satisfying it are called *harmonic functions*. It should however be noted that the same equation had been considered by Lagrange in 1760 in connection with his study of fluid flow problems.

Exercise 1. Prove the statements in the observations of Lagrange and Laplace.

Example 2. Let us compute the potential produced by a uniformly charged straight line segment in $\mathbb{R}^3$. Suppose that the line segment is lying along the x-axis, with its endpoints

[1] The minus sign in $E = -\nabla u$ is by convention that E is the same as F with $Q = +1$.

given by $(x_1, 0, 0)$ and $(x_2, 0, 0)$. Then the potential at $(0, h, 0)$ is

$$u = c \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{x^2 + h^2}} = c \log\left(x + \sqrt{x^2 + h^2}\right)\Big|_{x_1}^{x_2} = c \log \frac{x_2 + r_2}{x_1 + r_1}, \tag{6}$$

where c is a constant that depends on the linear charge density of the wire and the unit system, and $r_i = \sqrt{x_i^2 + h^2}$ for $i = 1, 2$. See Figure 1.

In this chapter, we will study some of the fundamental properties of harmonic functions. To get a first impression about harmonic functions, let us consider harmonic functions that are polynomials. In 1 dimension, all harmonic functions are simply linear functions. Likewise in 2 dimensions, all linear and bilinear polynomials (i.e., $axy + bx + cy + d$) are harmonic. However, there are more harmonic polynomials in $\mathbb{R}^2$, such as $x^2 - y^2$ and $y^3 - 3x^2y$. Playing with some explicit examples will reveal that harmonic polynomials do not have any maximum or minimum points; for example, polynomials such as $x^4 + y^4$ can never be harmonic. If the gradient of a harmonic polynomial vanishes at some point, then this point is necessarily a saddle point, like the point $(0, 0)$ for $x^2 - y^2$. This is actually the tip of the iceberg known as *mean value property* and *maximum principles*, which hold for general harmonic functions.

A careful study of harmonic functions is invaluable in understanding the Laplace operator and in solution of the Laplace equation under various conditions. Obviously, we do not need to solve the Laplace equation if we know the distribution of charges: We would just use (2) or (4) to calculate the field. However, there are important situations where the charge distribution must be implied from some indirect information. For example, imagine a closed surface in space, with some charges distributed inside and possibly also outside of it. Then we pose the problem of replacing the charges inside the surface by charges at the surface, so that the potential *at* the surface remains the same. This amounts to finding a function u satisfying $\Delta u = 0$ inside the surface (as there is no charge there), that agrees with the old values of the potential at the surface, leading to the *Dirichlet problem*.

Laplace's result (5) was completed by his student Siméon Denis Poisson in 1813, when Poisson showed that

$$\Delta u = -4\pi C\rho \qquad \text{in } \mathbb{R}^3, \tag{7}$$

for ρ smooth enough and vanishing outside some bounded set[2]. This equation is called the *Poisson equation*, and is valid everywhere, as opposed to (5), which is only valid in free space. Note that in terms of E, (7) is simply $\nabla \cdot E = 4\pi C\rho$, which is called the *Gauss law* in its differential form.

Taking a slightly different viewpoint, if we started with the equation (7) for the unknown function u, with ρ sufficiently "nice," then the formula (4) gives a particular solution. Hence in a certain sense, (4) *inverts* the Laplace operator. A generalization of (4) to arbitrary dimensions leads to the concept of fundamental solutions.

The Poisson equation can also be formulated in a domain $\Omega \subset \mathbb{R}^n$. Note that the Poisson equation includes the Laplace equation as a special case, and the difference between two solutions (with the same ρ) of the Poisson equation is harmonic. There are tons of harmonic functions, meaning that the solutions of the Poisson equation are far from unique. In order to get uniqueness, i.e., as a convenient way of parameterizing the solution space of the Poisson equation, one introduces *boundary conditions*, which are conditions on the behaviour of u at the boundary $\partial\Omega$ of the domain. The common boundary conditions include

$$au + b\,\partial_\nu u = g, \qquad \text{on} \quad \partial\Omega, \tag{8}$$

[2]In Gaussian type unit systems, one sets up the units so that $C = \pm 1$, and hence the Newtonian/Coulomb potential (4) has a simple expression. In other systems such as SI, one has $C = \pm\frac{1}{4\pi}$, meaning that the Poisson equation (7) has a simple expression.

with various choices of the functions a and b on the boundary, where ∂_ν is the outward normal derivative at the boundary, cf. Figure 2. The case $a \equiv 1$ and $b \equiv 0$ is called the *Dirichlet*, $a \equiv 0$ and $b \equiv 1$ the *Neumann*, $a \equiv 1$ and $b > 0$ the *Robin*, and $a \equiv 1$ and $b < 0$ the *Steklov* boundary conditions. If we couple one of these boundary conditions with the Laplace (or Poisson) equation, we get the Dirichlet problem, the Neumann problem, the Robin problem, and the Steklov problem[3], respectively. Moreover, any of the aforementioned boundary conditions is said to be *homogeneous* if $g \equiv 0$, and *inhomogeneous* otherwise.

2. Green's identities

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set with $\mathscr{C}^1$ boundary, and let $F : \bar{\Omega} \to \mathbb{R}^n$ be a vector field that is continuously differentiable in Ω, with all its derivatives continuous up to the boundary, i.e., each component of F is in $\mathscr{C}^1(\bar{\Omega})$. Then the *divergence theorem* asserts that

$$\int_\Omega \nabla \cdot F = \int_{\partial\Omega} F \cdot \nu, \tag{9}$$

where ν is the outward pointing unit normal to the boundary $\partial\Omega$, see Figure 2. This can be thought of as an extension of the fundamental theorem of calculus to multidimensions. In physics terms, it says that the flux of the vector field F through the boundary of Ω is equal to the total divergence of F in Ω. We remark that the regularity condition on $\partial\Omega$ can be considerably weakened, e.g., to include surfaces that consist of finitely many $\mathscr{C}^1$ pieces. We will not discuss those issues here, as they are not necessary for our purposes. The same holds for the regularity conditions on F. The divergence theorem first appeared in Lagrange's 1762 work, and was proved in a special case by Gauss in 1813. The general 3-dimensional case was treated by Mikhail Vasilievich Ostrogradsky in 1826.

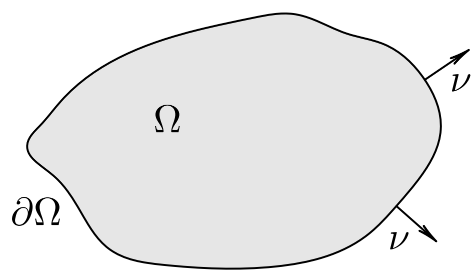


FIGURE 2. The setting of the divergence theorem.

Thinking of F as the electric (or gravitational) field, in view of the Gauss law $\nabla \cdot F = 4\pi C\rho$, this formula says that the *flux* of the electric field F through $\partial\Omega$ is equal to the total charge in Ω multiplied by $4\pi C$. Thus we obtain the integral form of the Gauss law.

In a preliminary section of his groundbreaking 1828 *Essay*, George Green proved several reductions of 3-dimensional volume integrals to surface integrals, similar in spirit to the divergence theorem, and independently of Ostrogradsky. Nowadays, those are called *Green's identities* and best viewed as consequences of the divergence theorem. As a warmup, let $\varphi \in \mathscr{C}^2(\bar{\Omega})$, and apply the divergence theorem to $F = \nabla\varphi$. Then we have $\nabla \cdot F = \Delta\varphi$ and $\nu \cdot F = \partial_\nu\varphi$, the latter denoting the (outward) normal derivative of φ, implying what can be called *Green's zeroth identity*[4]

$$\int_\Omega \Delta\varphi = \int_{\partial\Omega} \partial_\nu\varphi. \tag{10}$$

[3] In the literature, the distinction between the Robin and Steklov boundary conditions is occasionally blurred, and the terms are used interchangeably. Note also that the Steklov problem often designates a problem which is different from but related to the meaning we ascribe to it here.

[4] This name is not standard.

If we think of φ as the electrostatic potential, this leads to the integral form of the Gauss law expressed in terms of φ.

Similarly, letting $u \in \mathscr{C}^1(\bar{\Omega})$ and $\varphi \in \mathscr{C}^2(\bar{\Omega})$, and applying the divergence theorem to $F = u\nabla\varphi$, we get *Green's first identity*

$$\int_\Omega \nabla u \cdot \nabla\varphi + u\Delta\varphi = \int_{\partial\Omega} u\partial_\nu\varphi. \tag{11}$$

Interchanging the roles of u and φ in this identity, and subtracting the resulting identity from (11), we infer *Green's second identity*

$$\int_\Omega u\Delta\varphi - \varphi\Delta u = \int_{\partial\Omega} u\partial_\nu\varphi - \varphi\partial_\nu u, \qquad u, \varphi \in \mathscr{C}^2(\bar{\Omega}). \tag{12}$$

Note that (10) follows from (11) by putting $u \equiv 1$. The identities (11) and (12) can be considered as instances of, and are often called, *integration by parts* in n-dimensions.

Let us look at some simple consequences of Green's identities. First, the identity (10) gives a necessary condition for existence of a solution of the *Neumann problem*. For example, thinking of the Laplace equation, any solution $u \in \mathscr{C}^2(\bar{\Omega})$ of $\Delta u = 0$ must satisfy

$$\int_{\partial\Omega} \partial_\nu u = 0. \tag{13}$$

In physics terms, there is no charge in Ω, so the total flux of electric field through the boundary $\partial\Omega$ must vanish. Hence if u were to satisfy the Neumann boundary condition $\partial_\nu u = g$, the Neumann datum g should have the constraint that g has mean zero on the boundary $\partial\Omega$. Similarly, one can deduce that if the Poisson equation $\Delta u = f$ has a solution $u \in \mathscr{C}^2(\bar{\Omega})$ with the homogeneous Neumann boundary condition $\partial_\nu u = 0$, then f necessarily has mean zero in the domain Ω. In other words, if the total flux of electric field through the boundary vanish, then the total charge in Ω must be 0.

Second, the identity (11) implies several *uniqueness theorems* for the Poisson equation. By linearity, the issue of uniqueness for the Poisson equation reduces to the study of the Laplace equation with homogeneous boundary conditions. By putting $u \equiv \varphi$ and $\Delta u = 0$ in (11), we have

$$\int_\Omega |\nabla u|^2 = \int_{\partial\Omega} u\partial_\nu u, \tag{14}$$

for $u \in \mathscr{C}^2(\bar{\Omega})$ with $\Delta u = 0$ in Ω. For example, if $u = 0$ on $\partial\Omega$, then the left hand side is zero, implying that $u = \text{const}$ in Ω. Using the condition $u = 0$ on the boundary once again makes $u \equiv 0$ in Ω, hence we get uniqueness of solutions to the Dirichlet problem for the Poisson equation. On the other hand, if $\partial_\nu u = 0$ on $\partial\Omega$, then by the same argument we have $u = \text{const}$, implying that any two solutions (with suitable regularity) of the Neumann problem for the Poisson equation differ by a constant. This cannot be strengthened, since one can explicitly check that if u is a solution of $\Delta u = 0$ with $\partial_\nu u = 0$ on the boundary, then so is $u + \alpha$ for any constant α.

The aforementioned uniqueness results indicate that the *Cauchy problem* for the Laplace (or Poisson) equation, where we specify both u and $\partial_\nu u$ at the boundary of the domain, is in general *not* solvable. It would be analogous to overdetermined equations, where one has more equations than unknowns.

Exercise 3. Prove a uniqueness theorem for the Robin problem for the Poisson equation. What if one specifies a Dirichlet condition on part of the boundary, and a Neumann condition on the rest?

3. Fundamental solutions

As mentioned in the introduction, Poisson showed in 1813 that the function

$$u(y) = C \int_{\mathbb{R}^3} \frac{\rho(x)\mathrm{d}x}{|x-y|}, \tag{15}$$

satisfies the equation

$$\Delta u = -4\pi C\rho, \tag{16}$$

at least when ρ is "nice" enough, cf. (4) and (7). Observe that (15) and (16) together imply

$$u(y) = \int_{\mathbb{R}^3} E(x-y)\Delta u(x)\mathrm{d}x, \qquad \text{with} \quad E(x) = -\frac{1}{4\pi|x|}. \tag{17}$$

In other words, the operation of sending ρ to u in (15) *inverts* the action of Δ, up to a constant multiple. We want to find functions E with analogous behaviour in dimensions other than $n=3$. The following definition formalizes what we mean.

Definition 4. A *fundamental solution* (or *elementary solution*) of the Laplacian in n dimensions is a function $E \in \mathscr{C}^2(\mathbb{R}^n \setminus \{0\})$ satisfying

$$u(0) = \int_{\mathbb{R}^n} E(x)\Delta u(x)\mathrm{d}x, \tag{18}$$

for all compactly supported functions $u \in \mathscr{C}^2(\mathbb{R}^n)$.

A few comments are in order. First, recall that the *support* of u is the closure of the set on which u is nonzero, and so u is compactly supported iff u vanishes in the complement of a bounded set. Second, if E is a fundamental solution, then by translation

$$u(y) = \int_{\mathbb{R}^n} E(x-y)\Delta u(x)\mathrm{d}x, \tag{19}$$

for all $y \in \mathbb{R}^n$ and for all compactly supported functions $u \in \mathscr{C}^2(\mathbb{R}^n)$. Third, $E(x)$ may have a singularity at $x=0$, which is certainly the case for $n=3$, meaning that the integral in (18) should be interpreted either as an absolutely convergent improper Riemann integral, or as an ordinary Lebesgue integral, cf. Problem 5. Lastly, since E in (17) is harmonic in the complement of the origin, in general dimensions we may have also wanted to require that $\Delta E = 0$ in $\mathbb{R}^n \setminus \{0\}$, but this is actually already included in (18). Indeed, suppose that $\Delta E(y) \neq 0$ for some $y \in \mathbb{R}^n \setminus \{0\}$. Then by continuity, ΔE does not change sign in a sufficiently small ball $B_\varepsilon(y)$ with $\varepsilon > 0$ and $0 \notin B_\varepsilon(y)$, and so for any nontrivial nonnegative function $u \in \mathscr{C}^2(\mathbb{R}^n)$ whose support is contained in $B_\varepsilon(y)$, we are led to the contradiction

$$0 = u(0) = \int_{B_\varepsilon(y)} E(x)\Delta u(x)\mathrm{d}x = \int_{B_\varepsilon(y)} u(x)\Delta E(x)\mathrm{d}x \neq 0, \tag{20}$$

due to Green's second identity (12). Here and in the following, $B_r(z) \subset \mathbb{R}^n$ denotes the open ball of radius r centred at z, and $B_r = B_r(0)$.

Remark 5. If E_1 and E_2 are two fundamental solutions, then

$$\int_{\mathbb{R}^n} \big(E_1(x) - E_2(x)\big)\Delta u(x)\mathrm{d}x = u(0) - u(0) = 0, \tag{21}$$

for all compactly supported functions $u \in \mathscr{C}^2(\mathbb{R}^n)$. Hence by the argument that lead to (20), the difference $E_1 - E_2$ must be everywhere harmonic in $\mathbb{R}^n$, that is, it must be *entire* harmonic. This means that if we have one fundamental solution, then all the others can be generated by adding entire harmonic functions to it.

Now taking inspiration from the 3-dimensional case, we look for a radial fundamental solution. First, we derive an expression for the Laplacian of a spherically symmetric function. Let $\phi \in \mathscr{C}^2(I)$ with $I \subset (0, \infty)$ an open interval, and let $\Phi(x) = \phi(|x|)$ for $x \in \mathbb{R}^n$ with $|x| \in I$. Then we apply Green's zeroeth identity (10) in $A = B_s \setminus \bar{B}_t$ with $s, t \in I$ and $s > t$, to get

$$\int_A \Delta\Phi = \int_{\partial B_s} \partial_r \Phi - \int_{\partial B_t} \partial_r \Phi, \tag{22}$$

where ∂_r is the radial derivative centred at the origin. Since Φ is spherically symmetric, so are $\Delta\Phi$ and $\partial_r\Phi$. We have $\partial_r\Phi = \phi'$ for the latter, and introducing the notation $\lambda(r) = \Delta\Phi(x)$ for $|x| = r$, (22) becomes

$$\int_t^s \lambda(r) r^{n-1} \mathrm{d}r = \phi'(s)s^{n-1} - \phi'(t)t^{n-1}, \tag{23}$$

since, for instance

$$\int_{\partial B_t} \partial_r \Phi = |S^{n-1}| t^{n-1} \phi'(t), \tag{24}$$

where $|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. For $n = 1$, it is understood that $|S^0| = 2$. Then the fundamental theorem of calculus yields

$$\Delta\Phi = \lambda(r) = r^{1-n}(r^{n-1}\phi'(r))' = \phi''(x) + \frac{n-1}{r}\phi'(r), \tag{25}$$

in $\{x \in \mathbb{R}^n : |x| \in I\}$, where $r = |x|$. In particular, we have

$$\Delta\Phi = 0 \quad \text{in } \{x \in \mathbb{R}^n : |x| \in I\} \qquad \Longleftrightarrow \qquad \phi'(r)r^{n-1} = \text{const.} \tag{26}$$

This can easily be integrated, to yield

$$\phi(r) = \begin{cases} \frac{a}{2-n} r^{2-n} + c & \text{for } n \neq 2, \\ a \log r + c & \text{for } n = 2. \end{cases} \tag{27}$$

Note that *any spherically symmetric harmonic function* in a spherically symmetric domain must be of this form.

Therefore, if a radial fundamental solution exists, it must be of the form $E(x) = \phi(|x|)$ for $x \in \mathbb{R}^n \setminus \{0\}$ with ϕ given by (27). By construction, E is harmonic in $\mathbb{R}^n \setminus \{0\}$. Moreover, with reference to (18), we can ascertain that the integral $\int E\Delta u$ over a bounded region is absolutely convergent whenever $u \in \mathscr{C}^2(\mathbb{R}^n)$, as

$$\int_{B_R} |E\Delta u| \leq M \int_0^R \phi(r) r^{n-1} \mathrm{d}r < \infty, \tag{28}$$

where M is a constant.

Since constants are entire harmonic, we can add any constant to a fundamental solution to produce a new fundamental solution. Thus in the following, we will take $c = 0$ for convenience. On the other hand, the value of a is determined if we invoke the full content of the condition (18). Let $u \in \mathscr{C}^2(\mathbb{R}^n)$ be a function whose support is contained in B_R. To compute the integral $\int E\Delta u$, we excise a small ball B_ε of radius $\varepsilon > 0$ from B_R, resulting in $\Omega_\varepsilon = B_R \setminus \bar{B}_\varepsilon$. Then an application of Green's second identity (12) gives

$$\int_{\Omega_\varepsilon} E\Delta u = \int_{\Omega_\varepsilon} u\Delta E + \int_{\partial B_R} (E\partial_\nu u - u\partial_\nu E) + \int_{\partial B_\varepsilon} (u\partial_r E - E\partial_r u), \tag{29}$$

where we have taken into account that the boundary of Ω_ε consists of two parts ∂B_R and ∂B_ε, and ∂_r is the radial derivative centred at the origin. Now, the first two integrals in the right hand side vanish, because $\Delta E = 0$ in Ω_ε and the support of u is contained in B_R. The remaining integral can be estimated term by term as

$$\Big| \int_{\partial B_\varepsilon} E \partial_r u \Big| \leq |\phi(\varepsilon)| \cdot M \cdot |S^{n-1}| \varepsilon^{n-1} \to 0 \qquad \text{as} \quad \varepsilon \to 0, \tag{30}$$

where M is an upper bound on $|\nabla u| \geq |\partial_r u|$, and

$$\begin{aligned} \int_{\partial B_\varepsilon} u \partial_r E &= \phi'(\varepsilon) |S^{n-1}| \varepsilon^{n-1} \big(u(0) + O(\varepsilon) \big) \\ &= a \varepsilon^{1-n} |S^{n-1}| \varepsilon^{n-1} u(0) + O(\varepsilon) \to a |S^{n-1}| u(0) \qquad \text{as} \quad \varepsilon \to 0, \end{aligned} \tag{31}$$

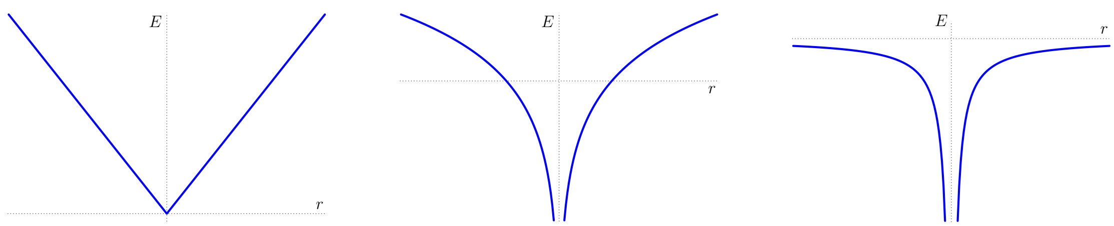
because $\partial_r E(x) = \phi'(\varepsilon) = a \varepsilon^{1-n}$ and $|u(x) - u(0)| \leq M\varepsilon$ for $x \in \partial B_\varepsilon$. Thus we conclude that

$$\int_\Omega E \Delta u = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} E \Delta u = a |S^{n-1}| u(0), \tag{32}$$

and so E with $a = \frac{1}{|S^{n-1}|}$ fulfills Definition 4, i.e.,

$$E(x) = \begin{cases} \frac{1}{(2-n)|S^{n-1}|} |x|^{2-n} & \text{for } n \neq 2, \\ \frac{1}{2\pi} \log |x| & \text{for } n = 2, \end{cases} \tag{33}$$

is a fundamental solution of the Laplace operator in $\mathbb{R}^n$.



(A) In 1 dimension, E is unbounded as $|x| \to \infty$, and continuous at $x = 0$.

(B) In 2 dimensions, E is unbounded both as $|x| \to \infty$, and as $x \to 0$.

(C) In 3 and higher dimensions, E is unbounded as $x \to 0$, but tends to 0 as $|x| \to \infty$.

FIGURE 3. Behaviour of E in different dimensions.

Exercise 6. By explicit computation, show that E given by (33) is harmonic in $\mathbb{R}^n \setminus \{0\}$.

Remark 7 (An alternative derivation)**.** In light of Coulomb's law (2) (or (15)), a physical interpretation of $E(x) = -\frac{1}{4\pi|x|}$ that appears in (17) would be that it is the electric potential produced by a point charge of quantity $q = -1/(4\pi C)$, located at the origin. The density ρ of such a point charge satisfies $\rho(x) = 0$ for $x \neq 0$, and yet it must have the property $\int \rho = q$. Obviously, ρ is not a function, since the condition $\rho(x) = 0$ for $x \neq 0$ already implies $\int \rho = 0$, whatever the value $\rho(0)$ is. However, we can realize it as $\rho = q\delta$, with δ being the *Dirac measure*, defined by

$$\int_A \delta(x) \mathrm{d}x = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{if } 0 \notin A, \end{cases} \tag{34}$$

for any set $A \subset \mathbb{R}^n$, where $\int_A \delta(x)\mathrm{d}x$ is an abuse of notation for $\delta(A)$, the δ-measure of the set A. Now, assuming that (16) is true even when ρ is a measure, we are led to the formal equality $\Delta E = \delta$. This equality is definitely true in $\mathbb{R}^3 \setminus \{0\}$, since $\Delta E(x) = 0$ for $x \neq 0$. At $x = 0$ though, both ΔE and δ are not defined, hence the equality $\Delta E = \delta$ must be understood only in an "integrated" sense, as in (34). Consequently, we need to ascribe a meaning to $\int_A \Delta E$ when $0 \in A$. This can be achieved as follows. Let $A \ni 0$ be an open set with $\mathscr{C}^1$ boundary, and let $\bar{B}_\varepsilon \subset A$, with $B_\varepsilon = B_\varepsilon(0)$ and $\varepsilon > 0$. Then a formal application of Green's zeroeth identity (10) yields

$$\int_A \Delta E = \int_{\partial A} \partial_\nu E = \int_{\partial B_\varepsilon} \partial_\nu E, \tag{35}$$

where the second equality follows from

$$0 = \int_{A\setminus B_\varepsilon} \Delta E = \int_{\partial A} \partial_\nu E - \int_{\partial B_\varepsilon} \partial_\nu E, \tag{36}$$

because E is harmonic in $\mathbb{R}^3 \setminus \{0\}$. In particular, the integral $\int_{\partial B_\varepsilon} \partial_\nu E$ does not depend on $\varepsilon > 0$, so that we can use it as the *definition* of $\int_A \Delta E$ for any open set $A \ni 0$. Finally, an explicit calculation gives

$$\int_A \Delta E = \int_{\partial B_\varepsilon} \partial_\nu E = 4\pi\varepsilon^2 \Big(\frac{\partial}{\partial r} \frac{1}{4\pi r}\Big)\Big|_{r=\varepsilon} = 1, \tag{37}$$

confirming that ΔE coincides with δ on open sets.

Our next goal is to find functions satisfying $\Delta E = \delta$ in general dimensions. Thus we look for a function $E \in \mathscr{C}^2(\mathbb{R}^n \setminus \{0\})$ satisfying $\Delta E = 0$ in $\mathbb{R}^n \setminus \{0\}$ and

$$\int_{\partial B_r} \partial_\nu E = 1 \qquad \text{for} \quad r > 0. \tag{38}$$

As noted before, we are using the equality (35) as the definition of ΔE, since the pointwise value of ΔE might not be defined at the origin. Now we make the ansatz $E(x) = \phi(|x|)$ for $x \in \mathbb{R}^n \setminus \{0\}$, with a twice differentiable function $\phi : (0, \infty) \to \mathbb{R}$. Then for $r > 0$, we require

$$1 = \int_{\partial B_r} \partial_\nu E = |S^{n-1}| r^{n-1} \phi'(r), \tag{39}$$

which immediately gives (33).

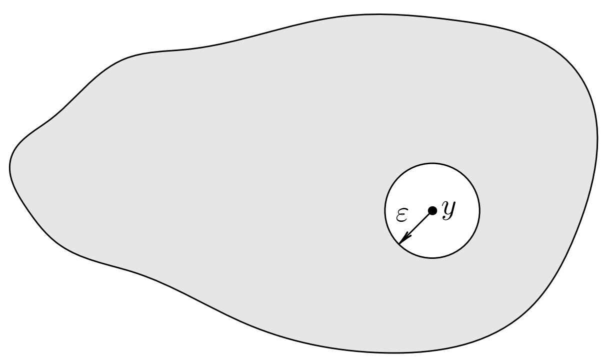


FIGURE 4. This illustration is relevant in two separate places in the text. In the current section (§3), the potential produced by a given charge distribution is split into two parts: the potential produced by the shaded part is harmonic in the ball $B_\varepsilon(y)$, while the potential produced by the ball itself satisfies the Poisson equation at y, cf. (41). In the next section (§4), we apply an identity to the shaded area, which we call Ω_ε, and then take the limit $\varepsilon \to 0$, in order to derive Green's representation formula.

Remark 8 (Poisson's proof)**.** Let us see how Poisson proved (16), that is, the implication

$$u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(z)\mathrm{d}z}{|x-z|} \qquad \Longrightarrow \qquad \Delta u = f. \tag{40}$$

Since the argument is from the early 19-th century, note that it does not necessarily conform to today's standard of rigour. The idea was to split the integral over $\mathbb{R}^3$ in (40) into two parts, one over the small ball $B_\varepsilon(y) = \{z \in \mathbb{R}^3 : |y - z| < \varepsilon\}$, and one over the complement $\mathbb{R}^3 \setminus B_\varepsilon(y)$, see Figure 4. What we have in mind is the situation $x \approx y$. Then the integral over the complement is harmonic in $B_\varepsilon(y)$ by Laplace's result (5), and the other integral can be formally manipulated as

$$\Delta_x u(x) = \frac{1}{4\pi} \int_{B_\varepsilon(y)} (f(x) - f(z))\, \Delta_x \frac{1}{|x-z|} \mathrm{d}z - \frac{f(x)}{4\pi} \int_{B_\varepsilon(y)} \Delta_x \frac{1}{|x-z|} \mathrm{d}z, \tag{41}$$

where Δ_x means that the Laplacian acts on the x variable. Note that we are treating $1/|x-z|$ as if it was a smooth function. For any smooth function ϕ, a direct computation reveals

$$\Delta_x \phi(|x-z|) = \phi''(|x-z|) + \frac{(n-1)\phi'(|x-z|)}{|x-z|} = \Delta_z \phi(|x-z|). \tag{42}$$

Taking this into account at a formal level, for the second integral of (41), an application of the (Green's zeroeth) identity (10) gives

$$\int_{B_\varepsilon(y)} \Delta_x \frac{1}{|x-z|} \mathrm{d}z = \int_{B_\varepsilon(y)} \Delta_y \frac{1}{|x-z|} \mathrm{d}z = \int_{\partial B_\varepsilon(y)} \partial_\nu \frac{1}{|x-z|} \mathrm{d}^2 z, \tag{43}$$

where $\partial B_\varepsilon(y)$ is the boundary of the ball $B_\varepsilon(y)$, ∂_ν is the normal derivative at z with respect to the variable z, and $\mathrm{d}^2 z$ denotes the surface area element of $\partial B_\varepsilon(y)$. Then putting $x = y$ into (43), we get

$$\Big(\int_{B_\varepsilon(y)} \Delta_x \frac{1}{|x-z|} \mathrm{d}z \Big)\Big|_{x=y} = 4\pi\varepsilon^2 \big(\frac{\partial}{\partial r} \frac{1}{r} \big)\Big|_{r=\varepsilon} = -4\pi. \tag{44}$$

As for the first integral of (41), the function $1/|x-z|$ is harmonic except at $x = z$, and if $f(x) - f(z)$ vanishes sufficiently fast as $x \to z$, we expect that the integral would come out as zero, finally giving a (very) formal justification of (16). The whole argument can be made rigorous, hence (16) is valid, e.g., provided that f is a $\mathscr{C}^1$ function (The reader might want to take this problem as a challenge). Poisson argued that (16) is valid for *any continuous function* f, which would be wrong (there exist counterexamples).

4. Green's representation formula

The defining property of the fundamental solution E we found in the preceding section is

$$u(y) = \int_{\mathbb{R}^n} E(x-y) \Delta u(x) \mathrm{d}x, \tag{45}$$

for all $y \in \mathbb{R}^n$ and for all compactly supported functions $u \in \mathscr{C}^2(\mathbb{R}^n)$. The fundamental solution is a generalization of the potential produced by a point charge to arbitrary dimensions, as it satisfies $\Delta E = \delta$ under a suitable interpretation.

Since the fundamental solution in a certain sense inverts the action of Δ, we hope to be able to use it as a key to unlock the secrets of the Laplace operator, and in particular, of harmonic functions. As a start, if $u \in \mathscr{C}^2(\mathbb{R}^n)$ is a compactly supported harmonic function, then an application of (45) reveals that $u \equiv 0$. In other words, nontrivial entire harmonic functions cannot be compactly supported. If $u(y) \neq 0$, then Δu in (45) must be nonzero somewhere. An approach to study nontrivial harmonic functions with the help of (45) is as follows. Let $u \in \mathscr{C}^2(B_R)$ be harmonic in B_R. Pick $0 < r < R$, and let $\chi \in \mathscr{C}^2(\mathbb{R}^n)$ satisfy $\chi \equiv 1$ in B_r and $\chi \equiv 0$ in $\mathbb{R}^n \setminus B_{R-\varepsilon}$ for some small $\varepsilon > 0$. Then the product $v = \chi u$ satisfies $v \in \mathscr{C}^2(\mathbb{R}^n)$, $v = u$ in B_r, and $v = 0$ in $\mathbb{R}^n \setminus B_R$. Now we apply (45) to v, and get

$$u(y) = \int_A E(x-y) \Delta v(x) \mathrm{d}x \qquad \text{for } y \in B_r, \tag{46}$$

where $A = B_R \setminus B_r$. Notice that $x \notin B_r$ while $y \in B_r$ in the integrand, and so the integrand $E(x-y)$ is smooth in the variable y. Therefore u must be smooth in B_r, see Example 10 below for details. By applying this argument at each point in an open set Ω, we have

$$u \in \mathscr{C}^2(\Omega) \quad \text{and} \quad \Delta u = 0 \text{ in } \Omega \qquad \Longrightarrow \qquad u \in \mathscr{C}^\infty(\Omega). \tag{47}$$

Results such as this one, that allow one to conclude higher regularity from a lower regularity and a differential equation, are called *regularity theorems*.

The representation formula (46) can be derived for a general domain Ω as well, by invoking a cut-off function χ that is nonzero only in an ε-neighbourhood of the boundary $\partial\Omega$. Then a natural question is what happens in the limit $\varepsilon \to 0$. This can be answered by repeating the derivation of (45) with appropriate modifications, as follows. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\mathscr{C}^1$ boundary. Let $y \in \Omega$, and let $\varepsilon > 0$ is so small that $\overline{B_\varepsilon(y)} \subset \Omega$. We apply Green's second identity (12) to the pair $u \in \mathscr{C}^2(\bar{\Omega})$ and $E_y(x) := E(x-y)$, in the region $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon(y)}$, see Figure 4, to infer

$$\int_{\Omega_\varepsilon} u\Delta E_y - E_y \Delta u = \int_{\partial\Omega} (u\partial_\nu E_y - E_y\partial_\nu u) - \int_{\partial B_\varepsilon(y)} (u\partial_r E_y - E_y \partial_r u), \tag{48}$$

where we have taken into account that the boundary of Ω_ε consists of two parts $\partial\Omega$ and $\partial B_\varepsilon(y)$, and ∂_r is the radial derivative centred at y. Since E_y is harmonic except at $x = y$, the term with $u\Delta E_y$ vanishes. Moreover, we know from the preceding section that the integral

$$\int_\Omega E_y \Delta u = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} E_y \Delta u, \tag{49}$$

is an absolutely convergent, and that

$$\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon(y)} (u\partial_r E_y - E_y \partial_r u) = u(y). \tag{50}$$

The result is *Green's representation formula* (or *Green's third identity*)

$$u(y) = \int_\Omega E_y \Delta u + \int_{\partial\Omega} u\partial_\nu E_y - \int_{\partial\Omega} E_y \partial_\nu u, \tag{51}$$

valid for $u \in \mathscr{C}^2(\bar{\Omega})$. Green's formula delivers a way to represent an arbitrary function as the sum of certain special kinds of functions, called *potentials*. We call the first term in the right hand side the *Newtonian potential*, the second term the *double layer potential*, and the last term the *single layer potential*. Note that if $\Delta u = 0$ in Ω, then

$$u(y) = \int_{\partial\Omega} u\partial_\nu E_y - \int_{\partial\Omega} E_y \partial_\nu u, \tag{52}$$

and so in particular, we can conclude that $u \in \mathscr{C}^\infty(\Omega)$, because the single- and double layer potentials are smooth in the interior of the domain. However, (52) is *not* a solution formula for either Dirichlet or Neumann problem, since (52) requires both u and $\partial_\nu u$ on the boundary.

From (51), we see that Δu (or a constant multiple of it) can be understood as the *volume charge density*, and $-\partial_\nu u$ as the *surface charge density*. To clarify a possible meaning of the other term, consider a point charge of quantity $-q$ placed at $x \in \partial\Omega$, and another charge of quantity q placed at $x + l\nu$, where $l \neq 0$ is a scalar. Such a system is called a *dipole*, and the potential produced by it at the point y is

$$u(y) = qE(x + l\nu - y) - qE(x - y) = ql \cdot \frac{E(x + l\nu - y) - E(x-y)}{l}. \tag{53}$$

If we now send $l \to 0$ and $q \to \infty$, such that the *dipole moment* $\mu = ql$ remains constant, then the preceding expression approaches $u(y) = \mu\partial_\nu E_y(x)$, which makes it clear that the second integral in the right hand side of (51) represents the potential produced by dipoles distributed over the surface $\partial\Omega$, and that the role of u in the same integral should be the *surface dipole moment density.* This explains the designation "double layer potential."

As a technical aside, we include here the following simple result on differentiating under the integral sign, which has been used in this section, and will be used many times later.

Theorem 9 (Leibniz rule)**.** *Let X be a compact topological space, and let $f : X \times (a,b) \to \mathbb{R}$ be a (jointly) continuous function. We label the variables of f by $(x,t) \in X \times (a,b)$, and assume that $\frac{\partial f}{\partial t} : X \times (a,b) \to \mathbb{R}$ is also continuous. Let $T : \mathscr{C}(X) \to \mathbb{R}$ be a bounded linear map, i.e.,*

$$|Tu| \leq c\|u\|_{\mathscr{C}(X)}, \qquad u \in \mathscr{C}(X), \tag{54}$$

for some constant c. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Tf(\cdot,t) = T\frac{\partial f}{\partial t}(\cdot,t), \qquad t \in (a,b). \tag{55}$$

Proof. In this proof, we fix $t \in (a,b)$ once and for all. Then by the mean value theorem, for $x \in X$ and $h > 0$, there is some $0 \leq \theta(x,h) \leq h$ such that

$$\frac{f(x,t+h) - f(x,t)}{h} = f_t'(x, t+\theta(x,h)), \tag{56}$$

where we have abbreviated $f_t' = \frac{\partial f}{\partial t}$. For a fixed t, the left hand side is a continuous function of $x \in X$, hence we can apply T to both sides, and conclude that

$$\frac{Tf(\cdot,t+h) - Tf(\cdot,t)}{h} = T\big(\frac{f(\cdot,t+h) - f(\cdot,t)}{h}\big) = Tf_t'(\cdot, t+\theta(\cdot,h)). \tag{57}$$

The lemma would follow from linearity and boundedness of T, upon showing that

$$\sup_{x\in X} |f_t'(x,t+\theta(x,h)) - f_t'(x,t)| \to 0 \qquad \text{as} \quad h \to 0. \tag{58}$$

To see that this is true, let

$$\omega(x,s) = |f_t'(x,t+s) - f_t'(x,t)|, \tag{59}$$

which is a continuous function of $(x,s) \in X \times [0,\varepsilon)$ for some $\varepsilon > 0$, satisfying $\omega(x,0) = 0$ for all $x \in X$. Since $0 \leq \theta(x,h) \leq h$, we have

$$\sup_{x\in X} |f_t'(x,t+\theta(x,h)) - f_t'(x,t)| \leq \sup_{(x,s)\in X\times[0,h]} \omega(x,s) =: w(h). \tag{60}$$

The function w is continuous because ω is continuous, and $w(0) = 0$ because $\omega(\cdot,0) = 0$, thus (58) is established. □

Example 10. Let us show that u from (46) is smooth in B_r. To apply the preceding theorem, we put $X = \bar{A}$, and $T : \mathscr{C}(X) \to \mathbb{R}$ will be the integration over X. Take a smooth curve $\gamma(-\varepsilon,\varepsilon) \to B_r$ in B_r, and set $f(x,t) = E(x-\gamma(t))\Delta v(x)$. Since

$$\frac{\partial f}{\partial t} = -\gamma'(t) \cdot \nabla E(x - \gamma(t)), \tag{61}$$

is continuous in $X \times (-\varepsilon,\varepsilon)$, by the preceding theorem, the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} Tf(\cdot,t) = \frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) \tag{62}$$

exists, and is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t} Tf(\cdot,t) = -\int_A \gamma'(t) \cdot \nabla E(x-\gamma(t))\Delta v(x)\mathrm{d}x. \tag{63}$$

In particular, taking $\gamma(t)$ to be a line parallel to an arbitrary coordinate axis, passing through an arbitrary point in B_r, we conclude that ∇u exists in B_r, and

$$\nabla u(y) = -\int_A \nabla E(x-y)\Delta v(x)\mathrm{d}x. \tag{64}$$

Although this is hardly new, as we already have $u \in \mathscr{C}^2(B_R)$, new information will be produced if we inductively apply the same argument to (64). Thus in the k-th iteration, we deduce that all k-th order partial derivatives of u exists in B_r, and therefore that all $(k-1)$-st order partial derivatives of u are continuous in B_r. Since k is arbitrary, we have $u \in \mathscr{C}^\infty(B_r)$.

5. Mean value property

Let $B = B_r(y)$ and $u \in \mathscr{C}^2(\bar{B})$. Then Green's representation formula (51) yields

$$\begin{aligned} u(y) &= \int_B E_y \Delta u + \int_{\partial B} u \partial_\nu E_y - \int_{\partial B} E_y \partial_\nu u \\ &= \int_B E_y \Delta u + \phi'(r) \int_{\partial B} u - \phi(r) \int_{\partial B} \partial_\nu u, \end{aligned} \tag{65}$$

where ϕ is the function such that $E(x) = \phi(|x|)$. Of course, the reason for this simplification is the spherical symmetry of E. Now in light of (10), that is,

$$\int_{\partial B} \partial_\nu u = \int_B \Delta u, \tag{66}$$

we conclude

$$\begin{aligned} u(y) &= \phi'(r) \int_{\partial B} u + \int_B \big(E_y - \phi(r)\big)\Delta u \\ &= \frac{1}{|\partial B|} \int_{\partial B} u + \int_B \big(E_y - \phi(r)\big)\Delta u, \end{aligned} \tag{67}$$

where we have taken into account that

$$\phi'(r) = \frac{1}{|\partial B|} \int_{\partial B} \partial_\nu E_y = \frac{1}{|\partial B|}, \tag{68}$$

cf., e.g., (39). In particular, if u is harmonic in B, we have

$$u(y) = \frac{1}{|\partial B|} \int_{\partial B} u, \tag{69}$$

which is called the *mean value property* of harmonic functions. Furthermore, we can get a volume averaged version, instead of the surface averaged one, by radial integration. Let us summarize these findings in the following theorem.

Theorem 11 (Gauss 1840)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in \mathscr{C}^2(\Omega)$ with $\Delta u = 0$ in Ω. Then for any ball $\overline{B_r(y)} \subset \Omega$, we have*

$$\int_{\partial B_r(y)} \partial_\nu u = 0, \tag{70}$$

and

$$u(y) = \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u = \frac{1}{|B_r|} \int_{B_r(y)} u. \tag{71}$$

Direct proof. We present here an alternative proof that does not rely on Green's representation formula. Without loss of generality, we set $y = 0$, and for $0 < \rho \leq r$, let

$$g(\rho) = \frac{1}{|\partial B_\rho|} \int_{\partial B_\rho} u(x)\, \mathrm{d}^{n-1}x = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} u(\rho\xi)\, \mathrm{d}^{n-1}\xi. \tag{72}$$

Then we have

$$g'(\rho) = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \xi\cdot\nabla u(\rho\xi)\,\mathrm{d}^{n-1}\xi = \frac{1}{|\partial B_\rho|}\int_{\partial B_\rho} \partial_\nu u(x)\,\mathrm{d}^{n-1}x = 0, \tag{73}$$

by (70). For a justification of the differentiation under the integral sign, we refer to the Leibniz rule (Theorem 9). On the other hand, $g(\rho)\to u(0)$ as $\rho\to 0$ by continuity, hence establishing the first equality of (71). □

Both (70) and (71) can be compared to Cauchy's theorem (or perhaps Cauchy's integral formula) in complex analysis. We will later prove that the mean value property characterizes harmonicity, which would be an analogue of Morera's theorem. For now, let us look at some of its immediate consequences.

Lemma 12 (Harnack inequality). *Let $u\in\mathscr{C}^2(\Omega)$ be a nonnegative function harmonic in Ω. Let $\overline{B_R(y)}\subset\Omega$ with $R>0$, and $x\in B_R(y)$. Then we have*

$$u(x) \leq \left(\frac{R}{R-|x-y|}\right)^n u(y) = \left(\frac{1}{1-k}\right)^n u(y), \tag{74}$$

where $k=|x-y|/R$.

Proof. From the mean value property and the positivity of u, with $r=R-|x-y|$ we have

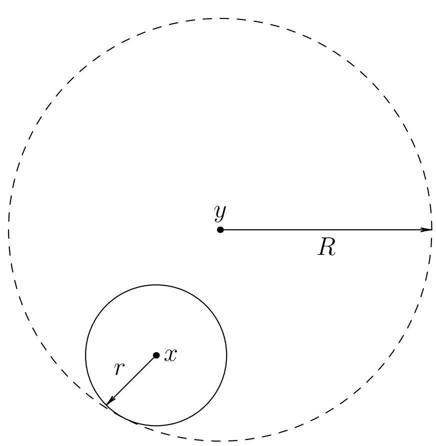
$$u(x) = \frac{1}{|B_r|}\int_{B_r(x)} u \leq \frac{1}{|B_r|}\int_{B_R(y)} u = \frac{|B_R|}{|B_r|}\,u(y), \tag{75}$$

establishing the claim. See Figure 5(a) for an illustration. □

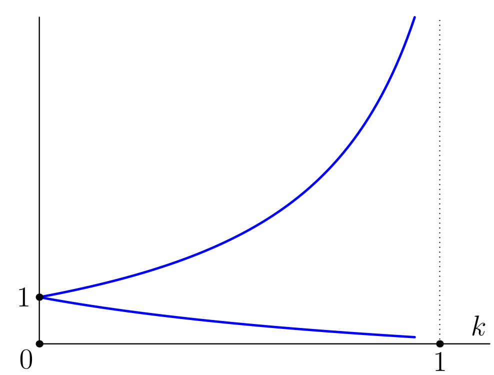
If $y\in B_R(x)$ as well, then $u(y)\leq\frac{1}{(1-k)^n}u(x)$, that is, we have

$$(1-k)^n\,u(y) \leq u(x) \leq \left(\frac{1}{1-k}\right)^n u(y). \tag{76}$$

Note that the inequalities become more stringent as $k=|x-y|/R$ tends to 0, and therefore the function cannot oscillate too much when $x\approx y$, cf. Figure 5(b).



(A) Setting of the proof of Lemma 12.



(B) Behavior of the constants A_k and B_k in the Harnack inequality $A_k u(y)\leq u(x)\leq B_k u(y)$, where $k=|x-y|/R$.

FIGURE 5. The Harnack inequality.

For nonnegative *entire* harmonic functions, we can apply (74) with fixed x,y, and take the limit $R\to\infty$ to get the following result.

Corollary 13 (Liouville's theorem[5]). *Let $u \in \mathscr{C}^2(\mathbb{R}^n)$ be nonnegative and harmonic in $\mathbb{R}^n$. Then u is constant.*

Note that Liouville's theorem still holds if we relax the condition "nonnegative" to "bounded from above or below," since if u is harmonic then so are $u - c$ and $c - u$ for any constant $c \in \mathbb{R}$.

Remark 14. If a harmonic function is bounded from below, we can always add a constant to it and make it a nonnegative harmonic function. Therefore the Harnack inequality can be used to turn one sided bounds into two sided bounds. For instance, suppose that $u \in \mathscr{C}^2(\mathbb{R}^n)$ is entire harmonic, and $u(x) \geq -p(|x|)$ for some nondecreasing function $p : [0, \infty) \to [0, \infty)$. Fix an arbitrary $x \in \mathbb{R}^n$, and consider the ball B_{2r} with $r = |x|$. Then $v = u + p(2r)$ is a nonnegative harmonic function in $\bar{B}_{2r}$, and so the Harnack inequality (74) gives

$$v(x) \leq 2^n v(0). \tag{77}$$

In terms of u, this can be written as

$$u(x) \leq 2^n(u(0) + p(2r)) - p(2r) \leq 2^n(u(0) + p(2|x|)). \tag{78}$$

Thus for example, if $u(x) \geq -A(1+|x|^m)$ for some constants A and m, then $u(x) \leq B(1+|x|^m)$ for some constant B.

6. Maximum principles

The mean value property tells us that the value of a harmonic function at the centre of a sphere is equal to the function averaged over the sphere. This leads to the intuition that the central point cannot be a maximum or a minimum. We can in fact eliminate only maximums (or only minimums), with the help of the following more primitive concept.

Definition 15. A continuous function $u \in \mathscr{C}(\Omega)$ is called *subharmonic* in Ω, if for any $y \in \Omega$, there exists $r^* = r^*(y) > 0$ such that

$$u(y) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u, \qquad 0 < r < r^*. \tag{79}$$

We can also define the concept of *superharmonic* functions, by flipping the inequality in the preceding definition. The mean value property then implies that harmonic functions are both subharmonic and superharmonic. In 1 dimension, the concepts of harmonicity, subharmonicity, and superharmonicity reduce to linearity, convexity, and concavity, respectively.

Remark 16. Let $u \in \mathscr{C}^2(\Omega)$ be subharmonic in Ω. Then (67) gives

$$\int_{B_r(y)} (\phi(r) - E_y)\Delta u = -u(y) + \frac{1}{|\partial B_r|} \int_{\partial B_r(y)} u \geq 0, \tag{80}$$

for all $0 < r < r^*$. Since $E_y < \phi(r)$ in B_r, this implies that $\Delta u \geq 0$ in Ω. On the other hand, if $u \in \mathscr{C}^2(\Omega)$ satisfies $\Delta u \geq 0$ in Ω, then it is immediate from (67) that u is subharmonic in Ω.

Theorem 17 (Strong maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in \mathscr{C}(\Omega)$ be subharmonic in Ω. If $u(z) = \sup_{\Omega} u$ for some $z \in \Omega$, then u is constant in the connected component of Ω that contains z.*

Proof. Let $M = \sup_\Omega u$. Then by hypothesis the set $\Sigma = \{y \in \Omega : u(y) = M\}$ is nonempty and closed. Now suppose that $y \in \Sigma$. Then by subharmonicity of u we have

$$\frac{1}{|B_r|} \int_{B_r(y)} M = u(y) \leq \frac{1}{|B_r|} \int_{B_r(y)} u, \tag{81}$$

[5] Sometimes attributed to Picard

for small $r > 0$, giving

$$\int_{B_r(y)} (u(x) - M)\mathrm{d}x \geq 0. \tag{82}$$

This means that $u \equiv M$ in $B_r(y)$, hence Σ is open. □

If Ω is bounded and if u is continuous up to the boundary of Ω, then u has its maximum in $\bar{\Omega}$. Since a maximum in the interior of Ω would imply that u is constant in the connected component of Ω containing the maximum, in any case the maximum value is attained at the boundary of Ω. We emphasize here that the subharmonicity condition is imposed only in the *interior* of Ω, and nothing except continuity of u is assumed at the points of $\partial\Omega$.

Corollary 18 (Weak maximum principle)**.** *Let Ω be a bounded open set, and let $u \in \mathscr{C}(\bar{\Omega})$ be subharmonic in Ω. Then*

$$\sup_{\Omega} u = \max_{\partial\Omega} u, \tag{83}$$

i.e., u achieves its maximum at the boundary.

Remark 19. The boundedness condition on Ω cannot be removed, as seen from the example $u(x, y) = y$ defined in the upper half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

An immediate consequence of the weak maximum principle is a *uniqueness theorem* for the Dirichlet problem. Indeed, by linearity, the question of uniqueness reduces to the uniqueness for the homogeneous problem $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$. Then we can apply the weak maximum principle to u and to $-u$ to infer $u = 0$. Note that we require $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}(\bar{\Omega})$ and that Ω be a bounded open set, which is weaker than the conditions in the uniqueness proof using the identity (14). Maximum principles applied to $-u$ are sometimes called *minimum principles* for u.

If we apply a maximum principle to the difference between two functions, we obtain a *comparison principle*. We state here one version of it.

Corollary 20 (Comparison principle)**.** *Let Ω be a bounded open set, and let u and v be elements of $\mathscr{C}(\bar{\Omega})$. Assume that $u - v$ is subharmonic in Ω and that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

Recall that if $u, v \in \mathscr{C}^2(\bar{\Omega})$, then $u - v$ is subharmonic in Ω if and only if $\Delta u \geq \Delta v$ in Ω. As a simple application of the comparison principle, let us prove the following *a priori* bound.

Corollary 21. *Let Ω be a bounded open set. Then for $u \in \mathscr{C}^2(\Omega) \cap \mathscr{C}(\bar{\Omega})$ we have*

$$\sup_{\Omega} |u| \leq C \sup_{\Omega} |\Delta u| + \sup_{\partial\Omega} |u|, \tag{84}$$

where $C > 0$ is a constant depending only on Ω.

Proof. Suppose that Ω is contained in the strip $\{x \in \mathbb{R}^n : 0 < x_1 < d\}$ with some $d > 0$. Let $v(x) = \alpha - \gamma x_1^2$ with constants α and γ to be determined. We have $\Delta v = -2\gamma$, meaning that the choice $\gamma = \frac{1}{2} \sup_{\Omega} |\Delta u|$ would ensure that $\Delta u \geq \Delta v$ in Ω. Then in order to have $u \leq v$ on $\partial\Omega$, we put $\alpha = \sup_{\partial\Omega} |u| + \gamma d^2$, which gives the bound $u \leq v \leq \alpha$ in Ω. The same function v works also for $-u$. □

Yet another application of the comparison principle gives the following important result known as *Hopf's boundary point lemma.*

Corollary 22. *Let $u \in \mathscr{C}(\bar{B}_r)$ be a function subharmonic in B_r, and suppose that $u \leq 0$ in B_r, with $u(0) < 0$. Then there exists $\gamma > 0$ such that*

$$u(x) \leq \gamma\big(|x| - r\big) \qquad \text{for all } x \in B_r. \tag{85}$$

In particular, if the normal derivative $\partial_\nu u$ exists at some $z \in \partial B_r$ with $u(z) = 0$, then it must be strictly positive: $\partial_\nu u(z) > 0$.

Proof. By continuity, there is a small $\rho > 0$ such that

$$\beta = \max_{\bar{B}_\rho} u < 0. \tag{86}$$

We want to apply the comparison principle in $A = B_r \setminus \bar{B}_\rho$. The function $v(x) = e^{-\alpha r^2} - e^{-\alpha|x|^2}$ satisfies $v = 0$ on ∂B_r, and $v = e^{-\alpha\rho^2}(e^{-\alpha(r^2-\rho^2)} - 1) \geq \beta$ on ∂B_ρ for $\alpha > 0$ sufficiently large. Furthermore, we have

$$\Delta v(x) = 2\alpha e^{-\alpha|x|^2}\big(n - 2\alpha|x|^2\big), \tag{87}$$

so that $\Delta v \leq 0$ in A for $\alpha > 0$ sufficiently large. To conclude, $u - v$ is subharmonic in A and $u - v \leq 0$ on ∂A, implying that $u \leq v$ in A, or $u \leq \max\{v, \beta\}$ in B_r. Finally, since $(-e^{-\alpha t^2})' > 0$ at $t = r > 0$, it is clear that $\max\{v, \beta\}$ can be bounded from above in B_r by a function of the form $\gamma(|x| - r)$ with some $\gamma > 0$. □

Remark 23. In 1 dimension, a notable property of convex functions is that if $u \in \mathscr{C}(\mathbb{R})$ is convex and bounded from above, then $u = \text{const}$. Does the same property hold for subharmonic functions in higher dimensions? It turns out that for $n \geq 3$, there are subharmonic functions in $\mathbb{R}^n$ that are bounded from above. For example the following function will do:

$$u(x) = -\min\big\{1, |x|^{2-n}\big\}. \tag{88}$$

Indeed, it is harmonic in $B = B_1(0)$ and in $\mathbb{R}^n \setminus \bar{B}$, so we need to check the subharmonicity condition only at points $y \in \partial B$. But this is obvious as

$$u(y) = \frac{1}{|B_r|}\int_{B_r(y)} 1 \leq \frac{1}{|B_r|}\int_{B_r(y)} u. \tag{89}$$

Now for $n = 2$, we claim that if $u \in \mathscr{C}(\mathbb{R}^2)$ is subharmonic and bounded from above, then $u = \text{const}$. To establish this, we will show that u attains its global maximum somewhere in $\bar{B}$, and so the strong maximum principle would finish the job. First, by adding a constant to u, without loss of generality, we can assume that

$$\max_{\bar{B}} u = 0. \tag{90}$$

Then we consider the comparison function

$$v(x) = \varepsilon \log|x| \qquad \text{in } A = B_R \setminus \bar{B}, \tag{91}$$

for $R > 0$ large. Note that v is harmonic in A and $u \leq 0 = v$ on ∂B. Furthermore, since u is bounded from above, and $v(x) \to \infty$ as $|x| \to \infty$, for all large R, we have $u \leq v$ on ∂B_R, and so $u \leq v$ in A. As R is arbitrary, this means that

$$u(x) \leq \varepsilon \log|x| \qquad \text{for } |x| \geq 1. \tag{92}$$

Finally, by sending $\varepsilon \to 0$ for each fixed $x \in \mathbb{R}^2 \setminus \bar{B}$, we get

$$u(x) \leq 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \bar{B}, \tag{93}$$

and on account of (90), we conclude that

$$\max_{\mathbb{R}^2} u = \max_{\bar{B}} u, \tag{94}$$

which is what we wanted to show.

7. Green's function

In this and the next sections, we consider the so called *Green's function approach* to the Dirichlet and Neumann problems for the Laplace equation. The treatment is largely heuristic, as we do not include rigorous justifications, but it adds to the insight into the problems under consideration, and in fact allows us to derive explicit formulas to solve those problems for certain special domains. If the reader skips this and the next section, and proceed directly to §9, they would still be able to follow the main thread of this chapter.

Adding (12) to (51), we get the *generalized Green formula*

$$u(y) + \int_\Omega u\Delta\varphi = \int_\Omega \Phi_y \Delta u + \int_{\partial\Omega} u\partial_\nu \Phi_y - \int_{\partial\Omega} \Phi_y \partial_\nu u, \tag{95}$$

for $u, \varphi \in \mathscr{C}^2(\bar{\Omega})$ with $\Phi_y(x) = E(x-y) + \varphi(x)$ and $y \in \Omega$. Recall that Ω is assumed to be a bounded $\mathscr{C}^1$ domain in $\mathbb{R}^n$. Consider the *Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{96}$$

where $f \in \mathscr{C}(\Omega)$ and $g \in \mathscr{C}(\partial\Omega)$ are given functions, respectively called the *source term* and *Dirichlet datum*. Then assuming that u satisfies (96), and applying (95) to it, we observe that

$$u(y) = \int_\Omega \Phi_y f + \int_{\partial\Omega} g\, \partial_\nu \Phi_y, \tag{97}$$

provided that $\Delta\varphi = 0$ in Ω and $\Phi_y = 0$ on $\partial\Omega$. The latter conditions are equivalent to

$$\begin{cases} \Delta \varphi = 0 & \text{in } \Omega, \\ \varphi = -E_y & \text{on } \partial\Omega. \end{cases} \tag{98}$$

So this approach potentially reduces the general Dirichlet problem (96) to a set of special Dirichlet problems (98). Note that we have to solve one special problem for each $y \in \Omega$. For this reason, it is preferable to denote φ in (98) by φ_y, so in particular, $\Phi_y(x) = E_y(x) + \varphi_y(x)$. The function $(y, x) \mapsto \Phi_y(x)$ is called *Green's function* for the Dirichlet problem (96), and $(y, x) \mapsto \varphi_y(x)$ is called the corresponding *correction function*.

We remark that formally, the problem (98) is equivalent to

$$\begin{cases} \Delta \Phi_y = \Delta E_y \equiv \delta_y & \text{in } \Omega, \\ \Phi_y = 0 & \text{on } \partial\Omega, \end{cases} \tag{99}$$

where $\delta_y(x) = \delta(x-y)$ is the Dirac measure concentrated at y.

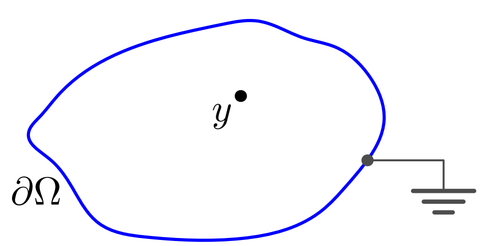


Figure 6. In 3 dimensions, Green's function Φ_y for a domain Ω can be thought of as the potential produced by a point charge at y, when the conducting surface $\partial\Omega$ is grounded, cf. (99). Any deviation of the potential from 0 on the surface $\partial\Omega$ would induce electrostatic force that pushes the charges in the right direction within the conductor or between the conductor and the Earth, until the potential is 0 throughout $\partial\Omega$.

Now, in order to justify the whole thing, we need to address the following questions.

(i) Does Φ_y exists, i.e., is the problem (98) solvable?
(ii) Supposing that Φ_y exists, does the function u defined by (97) solve the problem (96)?

In general, the first question is essentially as difficult as solving the general problem (96), but when Ω is simple, e.g., a ball or a half space, we can solve (99) explicitly, and hence we will have an integral formula for the solution of (96). On the other hand, there is a heuristic argument on the solvability of (99), which we quote straight from the source (see Figure 6):

> To convince ourselves that there does exist such a function as we have supposed Φ_y to be; conceive the surface to be a perfect conductor put in communication with the earth, and a unit of positive electricity to be concentrated in the point y, then the total potential function arising from y and from the electricity it will induce upon the surface, will be the required value of Φ_y (Green 1828).

We will see in the next chapter that this intuition is correct if the boundary of Ω is nice in a certain sense. For domains with highly nonsmooth boundaries, the problem (98) is not always solvable.

As for Question (ii), it can be answered without much difficulty when the domain boundary is sufficiently regular, but we will not go into details here as the next sections do not depend on it. The main use we have for Green's function is to give a derivation of Poisson's formula, whose validity will be verified independently. Moreover, solvability of (96) will be treated by more direct methods.

Remark 24. Let $G(y, x) = \Phi_y(x)$ be Green's function considered as a locally integrable function on $\Omega \times \Omega$. Then assuming that G exists, we have

- G is unique and $G < 0$.
- $G(y, x) > E(x - y)$ if $n \geq 3$ and $G(y, x) > E(x - y) - \frac{1}{2\pi} \log \operatorname{dist}(y, \partial\Omega)$ if $n = 2$.
- $G(x, y) = G(y, x)$ for $x \neq y$.

Uniqueness follows from the uniqueness theorem for (98). Since $\varphi|_{\partial\Omega} = -E_y|_{\partial\Omega} > 0$ for $n \geq 3$, the maximum principle says that $\varphi > 0$, which implies $G(y, x) > E(x - y)$. For $n = 2$, we have $E_y|_{\partial\Omega} < \frac{1}{2\pi} \log \operatorname{dist}(y, \partial\Omega)$, so $G(y, x) > E(x - y) - \frac{1}{2\pi} \log \operatorname{dist}(y, \partial\Omega)$. The negativity $G < 0$ is because Φ_y is harmonic in Ω except at y, and satisfies $\Phi_y|_{\partial\Omega} = 0$ and $\Phi_y(x) \to -\infty$ as $x \to y$. As for the symmetry $G(x, y) = G(y, x)$, let $\Omega_\varepsilon = \Omega \setminus (B_\varepsilon(y) \cup B_\varepsilon(z))$ where $\varepsilon > 0$ is sufficiently small, and invoke Green's second identity (12) to write

$$\begin{aligned} \int_{\Omega_\varepsilon} \Phi_y \Delta \Phi_z - \Phi_z \Delta \Phi_y &= \int_{\partial\Omega} (\Phi_y \partial_\nu \Phi_z - \Phi_z \partial_\nu \Phi_y) \\ &\quad + \int_{\partial B_\varepsilon(y)} (\Phi_z \partial_r \Phi_y - \Phi_y \partial_r \Phi_z) + \int_{\partial B_\varepsilon(z)} (\Phi_z \partial_r \Phi_y - \Phi_y \partial_r \Phi_z). \end{aligned} \tag{100}$$

Note that in the last two integrals, r denotes the radial variable centred at y and z, respectively. Since $\Delta\Phi_y = \Delta\Phi_z = 0$ in Ω_ε and $\Phi_y = \Phi_z = 0$ on $\partial\Omega$, the entire left hand side as well as the first integral in the right hand side vanish. Then taking into account that

$$\int_{\partial B_\varepsilon(y)} \Phi_y \partial_r \Phi_z \to 0, \qquad \int_{\partial B_\varepsilon(y)} \Phi_z \partial_r \Phi_y \to \Phi_z(y), \tag{101}$$

as $\varepsilon \to 0$, and analogous limits for the remaining terms, we get

$$0 = \int_{\partial B_\varepsilon(y)} (\Phi_z \partial_r \Phi_y - \Phi_y \partial_r \Phi_z) + \int_{\partial B_\varepsilon(z)} (\Phi_z \partial_r \Phi_y - \Phi_y \partial_r \Phi_z) \to \Phi_z(y) - \Phi_y(z), \tag{102}$$

as $\varepsilon \to 0$, which establishes that $\Phi_z(y) = \Phi_y(z)$.

Example 25. Let us find Green's function for $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$, the "upper half space". The defining condition (99) for Green's function arose from discussions about bounded domains, but the equations (99) make perfect sense even in unbounded domains. Let $y \in \Omega$. Then we have $\Delta E_y = \delta_y$, but we have to correct it by a function φ harmonic in Ω so as to have $(E_y+\varphi)|_{x_n=0} = 0$. It is obvious that $\varphi(x) = -E_{-y}(x) = -E(x+y)$ works, since $|x-y| = |x+y|$ if $x_n = 0$, and E_{-y} is harmonic in Ω. Thus Green's function is $G_y(x) = E(x-y) - E(x+y)$. A physical interpretation is that when a point charge of quantity q is at $y \in \Omega$, near the infinite conducting plane $\{x_n = 0\}$, the charges on the plane will arrange themselves so that the electrostatic field in Ω would be as if there is an additional charge of quantity $-q$ at $y^* = -y$. This additional, "imaginary" charge can be thought of as the *mirror image* of q with respect to the plane $\{x_n = 0\}$, cf. Figure 7(a).

The formula (97) involves $\partial_\nu G_y$, so let us compute it. For $n \geq 3$, we have

$$\partial_n |x-y|^{2-n} = (2-n)|x-y|^{-n}(x_n - y_n), \tag{103}$$

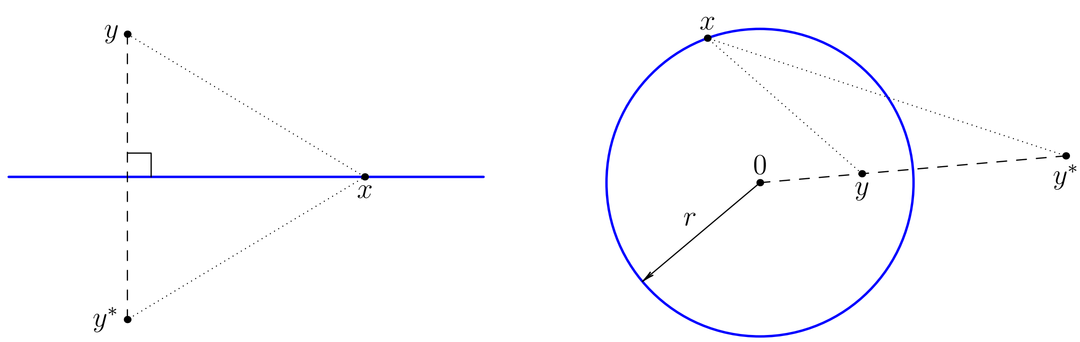
and so

$$\partial_\nu G_y(x) = -\partial_n G_y(x) = \frac{2y_n}{|S^{n-1}| \cdot |x-y|^n}, \tag{104}$$

which is called the *Poisson kernel* for the half space. For $n = 2$ we have

$$\partial_2 \log|x-y| = |x-y|^{-2}(x_2 - y_2), \qquad \text{hence} \qquad \partial_\nu G_y(x) = \frac{2y_2}{2\pi|x-y|^2}, \tag{105}$$

meaning that (104) holds for all $n \geq 2$.



(A) The effect of an infinite conducting plane near a point charge q at y is the same as that of an additional point charge of quantity $-q$ at the mirror image y^* of y with respect to the plane.

(B) Given a point charge at y, we want to place another point charge at a suitably chosen point y^*, so that the combined potential at the surface ∂B_r is constant. It turns out that the spherical inversion $y^* = \frac{r^2}{|y|^2} y$ works.

FIGURE 7. Green's function construction by the method of images.

Example 26. Let us now find Green's function for the ball B_r. Fix $y \in B_r$, and we look for the Green function in the form

$$G_y(x) = E(x-y) - qE(x-y^*) + c, \tag{106}$$

with $y^* = \lambda y$, where q, c and λ are real numbers possibly depending on y. For now, we assume $y \neq 0$. If λ is so large that y^* is outside B_r, then the second term is harmonic in B_r, so all we need to do is to ensure that $G_y = 0$ on ∂B_r. We see that

$$|x-\lambda y|^2 = |x|^2 + \lambda^2|y|^2 - 2\lambda x \cdot y = r^2 + \lambda^2|y|^2 - 2\lambda x \cdot y, \tag{107}$$

is a constant multiple of $|x-y|^2 = r^2 + |y|^2 - 2x\cdot y$ for all $x \in \partial B_r$, if $\lambda = 1$ or $\lambda|y|^2 = r^2$. Since we want $\lambda > 1$, the latter is clearly our choice, with which we then have

$$|x-y^*| = \sqrt{\lambda}|x-y| = \frac{r}{|y|}|x-y|, \qquad (x\in\partial B_r). \tag{108}$$

This implies

$$E(x-y^*) = \big(\frac{|y|}{r}\big)^{n-2}E(x-y), \qquad (x\in\partial B_r), \tag{109}$$

for $n \geq 3$, and

$$E(x-y^*) = E(x-y) + \frac{1}{2\pi}\log\big(\frac{r}{|y|}\big), \qquad (x\in\partial B_r), \tag{110}$$

for $n = 2$. Then from (106) it is easy to figure out the values of q and c that ensures $G_y = 0$ on ∂B_r, resulting in

$$G_y(x) = E(x-y) - \big(\frac{r}{|y|}\big)^{n-2}E(x-y^*) + \begin{cases} 0 & \text{for } n\geq 3, \\ \frac{1}{2\pi}\log\big(\frac{r}{|y|}\big) & \text{for } n = 2, \end{cases} \tag{111}$$

where $y^* = \frac{r^2}{|y|^2}\,y$, cf. Figure 7(b). This definition can be extended to $y = 0$ by continuity. Namely, as $y \to 0$, we have $|x-y^*| \sim \frac{r^2}{|y|}$, so $G_y(x) \to E(x) - \frac{1}{(2-n)|S^{n-1}|r^{n-2}}$ for $n \geq 3$, and $G_y(x) \to E(x) - \frac{1}{2\pi}\log r$ for $n = 2$.

Assuming $n \geq 3$, for $a \in B_r$ and $x \in \partial B_r$, we have

$$\partial_\nu|x-a|^{2-n} = (2-n)|x-a|^{-n}\frac{|x|^2 - a\cdot x}{|x|} = \frac{(2-n)(r^2 - a\cdot x)}{r|x-a|^n}, \tag{112}$$

and hence

$$\partial_\nu E(x-y) = \frac{r^2 - y\cdot x}{r|S^{n-1}|\cdot|x-y|^n}, \tag{113}$$

and

$$\partial_\nu E(x-y^*) = \frac{r^2 - y^*\cdot x}{r|S^{n-1}|\cdot|x-y^*|^n} = \frac{|y|^{n-2}(|y|^2 - y\cdot x)}{r^{n-1}|S^{n-1}|\cdot|x-y|^n}. \tag{114}$$

Then substituting those into (111), we get the *Poisson kernel*

$$\Pi(y,x) := \partial_\nu G_y(x) = \frac{r^2 - |y|^2}{r|S^{n-1}|\cdot|x-y|^n}. \tag{115}$$

and *Poisson's formula*

$$u(y) = \int_{\partial B_r}\Pi(y,x)g(x)\mathrm{d}^{n-1}x, \tag{116}$$

the latter being true if $u \in \mathscr{C}^2(\Omega)\cap\mathscr{C}^1(\bar{\Omega})$ satisfies $\Delta u = 0$ in B_r and $u = g$ on ∂B_r. In particular, putting $u \equiv 1$ immediately gives

$$\int_{\partial B_r}\Pi(y,x)\mathrm{d}^{n-1}x = 1, \qquad y\in B_r. \tag{117}$$

For $n = 2$, the formula (112) must be replaced by

$$\frac{1}{2}\partial_\nu\log|x-a|^2 = \frac{1}{|x-a|^2}\cdot\frac{|x|^2 - a\cdot x}{|x|} = \frac{r^2 - a\cdot x}{r|x-a|^2}, \tag{118}$$

which implies that (113) and (114) are valid for $n = 2$, and hence that the Poisson kernel for $n = 2$ is given by the same formula (115) with $n = 2$.

8. Neumann's function

Green's function approach can be extended to the *Neumann problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \partial_\nu u = g & \text{on } \partial\Omega, \end{cases} \tag{119}$$

where $f \in \mathscr{C}(\Omega)$ and $g \in \mathscr{C}(\partial\Omega)$ are given functions. Let us note that in order for a solution u to exist, the data must satisfy the consistency condition

$$\int_\Omega f = \int_{\partial\Omega} g, \tag{120}$$

that follows from (10), and that the solution u is determined by (119) only up to a constant. Recall from (95) the generalized Green formula

$$u(y) = \int_\Omega \Phi_y \Delta u - \int_{\partial\Omega} \Phi_y \partial_\nu u + \int_{\partial\Omega} u \partial_\nu \Phi_y - \int_\Omega u \Delta\varphi, \tag{121}$$

which is valid for $u, \varphi \in \mathscr{C}^2(\bar{\Omega})$ with $\Phi_y(x) = E(x-y) + \varphi(x)$ and $y \in \Omega$. Imitating what we did for the Dirichlet case, for any solution u of (119), from (121) we get

$$u(y) = \int_\Omega \Phi_y f - \int_{\partial\Omega} \Phi_y g, \tag{122}$$

provided that $\Delta\varphi = 0$ in Ω and $\partial_\nu \Phi_y = 0$ on $\partial\Omega$. The analogue of (98) is

$$\begin{cases} \Delta\varphi_y = 0 & \text{in } \Omega, \\ \partial_\nu \varphi_y = -\partial_\nu E_y & \text{on } \partial\Omega, \end{cases} \tag{123}$$

where $\Phi_y(x) = E_y(x) + \varphi_y(x)$. This problem is *not* solvable, since (10) requires

$$\int_{\partial\Omega} \partial_\nu \varphi_y = \int_\Omega \Delta\varphi_y = 0, \tag{124}$$

but (51) with $u \equiv 1$ implies

$$\int_{\partial\Omega} \partial_\nu E_y = 1. \tag{125}$$

Thus in order to have a chance at solvability, we have to replace the problem (123) by

$$\begin{cases} \Delta\varphi_y = \rho_y & \text{in } \Omega, \\ \partial_\nu \varphi_y = \chi_y & \text{on } \partial\Omega, \end{cases} \tag{126}$$

where $\rho_y \in \mathscr{C}(\Omega)$ and $\chi_y \in \mathscr{C}(\partial\Omega)$, $(y \in \Omega)$, are suitably chosen functions satisfying

$$\int_\Omega \rho_y = \int_{\partial\Omega} \chi_y \qquad (y \in \Omega). \tag{127}$$

Assuming that such φ_y exists for each $y \in \Omega$, we put $N(y,x) = N_y(x) = E_y(x) + \varphi_y(x)$, which is called *Neumann's function* or *Green's function of the second kind*. It is easy to see that the analogue of (99) is

$$\begin{cases} \Delta N_y = \delta_y + \rho_y & \text{in } \Omega, \\ \partial_\nu N_y = \partial_\nu E_y + \chi_y & \text{on } \partial\Omega, \end{cases} \tag{128}$$

and by (121), the analogue of (97) is

$$u(y) = \int_\Omega N_y f - \int_{\partial\Omega} N_y g + \int_{\partial\Omega} (\partial_\nu E_y + \chi_y) u - \int_\Omega \rho_y u. \tag{129}$$

If we think of the latter formula as a solution formula for u in terms of f and g, then it requires us to know beforehand the quantity $\int_{\partial\Omega} (\partial_\nu E_y + \chi_y) u - \int_\Omega \rho_y u$ for each $y \in \Omega$. As the solution of the Neumann problem is determined up to a constant, a general solution formula must involve *one and only one* additive constant that can be chosen at will. The formula (129) would have such a property if $\rho_y = \rho$, i.e., one function ρ for all y, and if $\chi_y = \chi - \partial_\nu E_y$ for some $\chi \in \mathscr{C}(\partial\Omega)$. The compatibility condition (127) now reads

$$\int_{\partial\Omega} \chi = 1 + \int_\Omega \rho, \tag{130}$$

and (129) becomes

$$u(y) = \int_\Omega N_y f - \int_{\partial\Omega} N_y g + \int_{\partial\Omega} \chi u - \int_\Omega \rho u. \tag{131}$$

Two standard choices for ρ and χ are

$$\text{(i)} \quad \rho \equiv 0, \quad \chi \equiv \frac{1}{|\partial\Omega|}, \qquad \text{and} \qquad \text{(ii)} \quad \rho \equiv -\frac{1}{|\Omega|}, \quad \chi \equiv 0. \tag{132}$$

As with the Dirichlet case, we have not addressed the solvability of (126), and if (131) solves the Neumann problem (119). Assuming that everything is fine, (131) offers a solution formula in which we can freely specify the quantity $\int_{\partial\Omega} \chi u - \int_\Omega \rho u$.

Remark 27. We can study the symmetry of N by using the method from Remark 24. First, let us rewrite (100) in the current context as

$$\begin{aligned} \int_{\Omega_\varepsilon} N_y \Delta N_z - N_z \Delta N_y = \int_{\partial\Omega} &(N_y \partial_\nu N_z - N_z \partial_\nu N_y) \\ &+ \int_{\partial B_\varepsilon(y)} (N_z \partial_r N_y - N_y \partial_r N_z) + \int_{\partial B_\varepsilon(z)} (N_z \partial_r N_y - N_y \partial_r N_z). \end{aligned} \tag{133}$$

Since $\Delta N_y = \Delta N_z = \rho$ in Ω_ε and $\partial_\nu N_y = \partial_\nu N_z = \chi$ on $\partial\Omega$, we get

$$\int_{\Omega_\varepsilon} N_y \Delta N_z - N_z \Delta N_y = \int_{\Omega_\varepsilon} (N_y - N_z)\rho, \tag{134}$$

and

$$\int_{\partial\Omega} (N_y \partial_\nu N_z - N_z \partial_\nu N_y) = \int_{\partial\Omega} (N_y - N_z)\chi. \tag{135}$$

Furthermore, we have

$$\int_{\partial B_\varepsilon(y)} N_y \partial_r N_z \to 0, \qquad \int_{\partial B_\varepsilon(y)} N_z \partial_r N_y \to N_z(y), \tag{136}$$

as $\varepsilon \to 0$, and analogous limits for the remaining terms, yielding

$$\int_{\Omega_\varepsilon} (N_y - N_z)\rho - \int_{\partial\Omega} (N_y - N_z)\chi \to N_z(y) - N_y(z), \tag{137}$$

as $\varepsilon \to 0$. After taking the limit, this can be rewritten as

$$N_y(z) + \psi(y) = N_z(y) + \psi(z), \tag{138}$$

where

$$\psi(y) = \int_\Omega N_y \rho + \int_{\partial\Omega} N_y \chi. \tag{139}$$

Since N_y in (128) is defined only up to a constant, the new function

$$N'_y(x) = N_y(x) + \psi(y), \tag{140}$$

still satisfies (128) with $\rho_y = \rho$ and $\chi_y = \chi$, and it is symmetric:

$$N'_y(x) = N'_x(y). \tag{141}$$

To conclude, the possibility to add an arbitrary constant to the Neumann function for each y allows us to require the Neumann function be symmetric. Once we impose this requirement, the Neumann function is fixed up to a global constant.

Example 28. Let us find Neumann's function for $\Omega = \{x \in \mathbb{R}^n : x_n > 0\}$, the upper half space. Looking at (132)(ii), since $|\Omega| = \infty$, we take $\rho \equiv 0$ and $\chi \equiv 0$. Even though all our derivations so far have been for bounded domains, we proceed to finding Neumann's function N_y satisfying (128), which, in the current setting, becomes

$$\begin{cases} \Delta N_y = \delta_y & \text{in } \Omega, \\ \partial_\nu N_y = 0 & \text{on } \partial\Omega. \end{cases} \tag{142}$$

By symmetry, it is obvious that $N_y(x) = E(x-y) + E(x+y)$ works. In particular, since $E(x-y) = E(x+y)$ for $x \in \partial\Omega = \mathbb{R}^{n-1} \times \{0\}$, for $f \equiv 0$, the formula (131) specializes to

$$u(y) = -2 \int_{\partial\Omega} E(x-y) g(x) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}, \tag{143}$$

which may be called the half-space Poisson formula for the Neumann problem.

Example 29. Let us now find Neumann's function for the unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$. We follow the approach (132)(i), and set $\rho \equiv 0$ and $\chi \equiv \frac{1}{2\pi}$. Then (128) becomes

$$\begin{cases} \Delta N_y = \delta_y & \text{in } \mathbb{D}, \\ \partial_\nu N_y = \frac{1}{2\pi} & \text{on } \partial\mathbb{D}. \end{cases} \tag{144}$$

Taking inspiration from Example 26, we look for Neumann's function in the form

$$N_y(x) = \frac{1}{2\pi} \log|x-y| + \frac{q}{2\pi} \log(|x-y^*| \cdot |y|), \tag{145}$$

where $y^* = \frac{1}{|y|^2} y$, and q is a constant, possibly depending on y. The factor $|y|$ in the second term is there to ensure that $N_y(x)$ at $y = 0$ can be defined by contiuity, as $N_0(x) = \frac{1}{2\pi} \log|x|$. Since the second term (as a function of x) is harmonic in $\mathbb{D}$, the condition $\Delta N_y = \delta_y$ is satisfied. Recall from Example 26 that

$$\partial_\nu \log|x-y| = \frac{1 - y \cdot x}{|x-y|^2}, \qquad \text{and} \qquad \partial_\nu \log|x-y^*| = \frac{|y|^2 - y \cdot x}{|x-y|^2}. \tag{146}$$

Hence if we put $q = 1$, for $x \in \partial\mathbb{D}$, we get

$$\partial_\nu N_y(x) = \frac{1 - 2y \cdot x + |y|^2}{2\pi |x-y|^2} = \frac{|x-y|^2}{2\pi |x-y|^2} = \frac{1}{2\pi}, \tag{147}$$

which confirms that $N_y(x) = \frac{1}{2\pi} \log(|x-y| \cdot |x-y^*| \cdot |y|)$ is Neumann's function for the unit disk. In particular, taking into account the fact that $|x-y^*| = |x-y|/|y|$ for $x \in \partial\mathbb{D}$, and setting $f \equiv 0$ and $\int_{\partial\mathbb{D}} u = 0$, the formula (131) specializes to

$$u(y) = -\frac{1}{\pi} \int_{\partial\mathbb{D}} g(x) \log|x-y| \,\mathrm{d}x, \tag{148}$$

which may be called the Poisson formula for the Neumann problem.

Exercise 30. With $\Omega \subset \mathbb{R}^n$ a bounded domain, the *Robin problem* for the Poisson equation is

$$\Delta u = f, \quad \text{in } \Omega, \qquad \partial_\nu u + ku = g, \quad \text{on } \partial\Omega,$$

where f and g are functions defined on Ω and $\partial\Omega$, respectively, and $k > 0$ is a constant. Devise an approach analogous to Green's functions for the Robin problem. The resulting functions are called *Robin's functions*, or *Green's functions of the third kind*. Make some preliminary observations on the behaviour of these functions. Give explicit formulas for the Robin function(s) and the solution of the Robin problem in the case $n = 1$.

9. Poisson's formula

In this section we will solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B_r, \\ u = g & \text{on } \partial B_r, \end{cases} \tag{149}$$

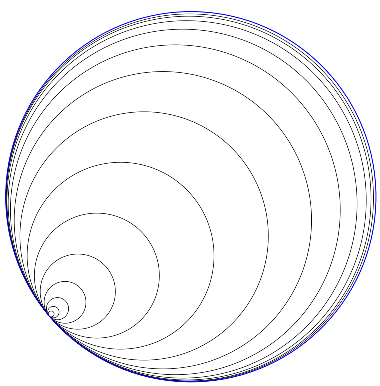
in the ball $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, where $g \in \mathscr{C}(\partial B_r)$ is a given function. It will give us a powerful tool to study harmonic functions, and will be a stepping stone to solving the Dirichlet problem in general domains. By using Green's function approach, we have derived a good *candidate* solution, the *Poisson formula*

$$u(y) = \int_{\partial B_r} \Pi(y, x) g(x) \mathrm{d}^{n-1} x, \qquad (y \in B_r), \tag{150}$$

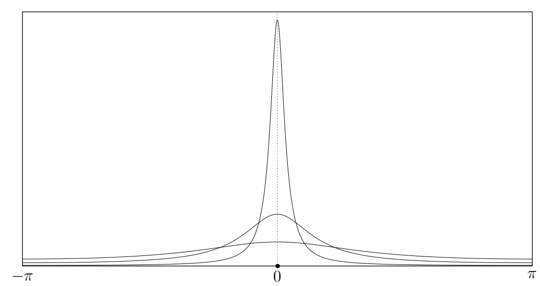
where

$$\Pi(y, x) = \frac{r^2 - |y|^2}{r|S^{n-1}| \cdot |x - y|^n}, \tag{151}$$

is called the *Poisson kernel*. Note that thus far we have not established the fact that the function defined by (150) indeed satisfies (149). We shall establish that fact now.



(A) Contour lines of $\Pi(y, x)$ as a function of y, with x fixed.



(B) Graphs of $\Pi(re^{i\theta}, 1)$ as a function of θ, in complex notation, for the values $r = 0.3$, 0.6, 0.9.

Figure 8. The Poisson kernel.

The following lemma summarizes some of the key properties of the Poisson kernel.

Lemma 31. *For any fixed $x \in \partial B_r$, the Poisson kernel $\Pi(y, x)$ is infinitely differentiable in y and $\Delta_y \Pi(y, x) = 0$ for $y \in B_r$, where Δ_y denotes the Laplace operator with respect to y. Moreover, we have*

$$\frac{\partial}{\partial y_i} \Pi(0, x) = \frac{n x_i}{r^{n+1} |S^{n-1}|}, \qquad \text{for} \quad x \in \partial B_r. \tag{152}$$

Proof. The smoothness of $\Pi(y, x)$ in y is immediate. An explicit calculation gives

$$\frac{\partial}{\partial y_i} \frac{1}{|x - y|^n} = \frac{n(x_i - y_i)}{|x - y|^{n+2}}, \tag{153}$$

and

$$\frac{\partial}{\partial y_i} \frac{|y|^2}{|x - y|^n} = \frac{2y_i |x - y|^2 + n(x_i - y_i)|y|^2}{|x - y|^{n+2}}, \tag{154}$$

which imply (152). Furthermore, we have

$$\frac{\partial^2}{\partial y_i^2}\frac{1}{|x-y|^n} = \frac{n(n+2)(x_i-y_i)^2}{|x-y|^{n+4}} - \frac{n}{|x-y|^{n+2}}, \tag{155}$$

and

$$\frac{\partial^2}{\partial y_i^2}\frac{|y|^2}{|x-y|^n} = \frac{n(n+2)|y|^2(x_i-y_i)^2}{|x-y|^{n+4}} - \frac{n|y|^2+4ny_i(x_i-y_i)}{|x-y|^{n+2}} + \frac{2}{|x-y|^n}, \tag{156}$$

leading to

$$\Delta_y\frac{r^2-|y|^2}{|x-y|^n} = \frac{2n(r^2+|y|^2-x\cdot y)}{|x-y|^{n+2}} - \frac{2n}{|x-y|^n} = 0, \qquad \text{for} \quad |x|=r. \tag{157}$$

This completes the proof. □

To study differentiability properties of u given by the Poisson formula (150), we need to be able to differentiate under the integral sign. The simple Leibniz rule (Theorem 9) will be sufficient for our purposes.

Theorem 32 (Schwarz 1872)**.** *a) Let $g \in \mathscr{C}(\partial B_r)$ and let u be given by the Poisson formula*

$$u(y) = \int_{\partial B_r} \Pi(y,x)g(x)\mathrm{d}^{n-1}x, \qquad (y \in B_r). \tag{158}$$

Then $u \in \mathscr{C}^\infty(B_r)$, $\Delta u = 0$ in B_r, and $u(y) \to g(x)$ as $B_r \ni y \to x \in \partial B_r$.

b) Let $u \in \mathscr{C}^2(B_r) \cap \mathscr{C}(\bar{B}_r)$ be harmonic in B_r. Then

$$u(y) = \int_{\partial B_r} \Pi(y,x)u(x)\mathrm{d}^{n-1}x, \qquad \textit{for} \quad y \in B_r. \tag{159}$$

In particular, $u \in \mathscr{C}^\infty(B_r)$.

Proof. a) By Lemma 31, for any fixed $x \in \partial B_r$, the Poisson kernel $\Pi(y,x)$ is infinitely differentiable in y and $\Delta_y\Pi(y,x) = 0$ for $y \in B_r$. Furthermore, observe that all partial derivatives $\frac{\partial^{k_1+\ldots+k_n}}{\partial^{k_1}y_1\cdots\partial^{k_n}y_n}\Pi(y,x)$ are continuous functions of $(y,x) \in B_r \times \partial B_r$, thus we can apply the Leibniz rule (Theorem 9) to the Poisson integral (158) repeatedly, and conclude that $u \in \mathscr{C}^\infty(B_r)$, and that

$$\Delta u(y) = \Delta\int_{\partial B_r} \Pi(y,x)g(x)\mathrm{d}^{n-1}x = \int_{\partial B_r} g(x)\Delta_y\Pi(y,x)\mathrm{d}^{n-1}x = 0, \tag{160}$$

for $y \in B_r$.

By the preceding result, the function

$$v(y) = \int_{\partial B_r} \Pi(y,x)\mathrm{d}^{n-1}x, \qquad (y \in B_r) \tag{161}$$

is harmonic in B_r. Moreover, by symmetry, v is radial, meaning that $v(y) = \phi(|y|)$ for some function $\phi \in \mathscr{C}^\infty((0,r)) \cap \mathscr{C}([0,r))$. From §3, we know that the only radial harmonic functions in a ball (in particular, continuous at the origin) are constants. Hence $v = \text{const}$, and computing the integral in (161) at $y = 0$ reveals that

$$\int_{\partial B_r} \Pi(y,x)\mathrm{d}^{n-1}x = 1, \qquad y \in B_r. \tag{162}$$

Now let $\hat{x} \in \partial B_r$. Then by (162) we have

$$g(\hat{x}) = \int_{\partial B_r} \Pi(y,x)g(\hat{x})\mathrm{d}^{n-1}x, \tag{163}$$

and so

$$u(y) - g(\hat{x}) = \int_{\partial B_r} \Pi(y, x)(g(x) - g(\hat{x}))\mathrm{d}^{n-1}x. \tag{164}$$

If $|x - \hat{x}| > \delta > 0$ and $|y - \hat{x}| < \frac{\delta}{2}$, then $|x - y| > \frac{\delta}{2}$, so the function $x \mapsto \Pi(y, x)$ converges uniformly in $\partial B_r \setminus B_\delta(\hat{x})$ to 0 as $y \to \hat{x}$. For x close to $\hat{x}$, the continuity of g is enough to counteract the singularity of $x \mapsto \Pi(y, x)$ at $x = \hat{x}$, because this singularity is integrable uniformly in y as seen from (162). To formalize the argument, let $\delta > 0$ be a constant to be adjusted later. Then by using the fact that $\Pi(y, x)$ is positive, we have

$$\begin{aligned}
|u(y) - g(\hat{x})| &\leq \int_{\partial B_r} \Pi(y, x)|g(x) - g(\hat{x})|\mathrm{d}^{n-1}x \\
&\leq \sup_{x \in \partial B_r \cap B_\delta(\hat{x})} |g(x) - g(\hat{x})| \int_{\partial B_r \cap B_\delta(\hat{x})} \Pi(y, x)\mathrm{d}^{n-1}x \\
&\quad + \sup_{x \in \partial B_r \setminus B_\delta(\hat{x})} \Pi(y, x) \int_{\partial B_r \setminus B_\delta(\hat{x})} |g(x) - g(\hat{x})|\mathrm{d}^{n-1}x \\
&\leq \sup_{x \in \partial B_r \cap B_\delta(\hat{x})} |g(x) - g(\hat{x})| + 2\|g\|_{L^\infty} |\partial B_r| \sup_{x \in \partial B_r \setminus B_\delta(\hat{x})} \Pi(y, x).
\end{aligned} \tag{165}$$

For any given $\varepsilon > 0$, we can pick $\delta > 0$ so small that the first term is smaller than ε. Then we choose y so close to $\hat{x}$ that the second term is smaller than ε.

b) We construct a function $v \in \mathscr{C}^\infty(B_r) \cap \mathscr{C}(\bar{B}_r)$ by the Poisson formula with the boundary values given by $v = u$ on ∂B_r. Then by part a) of this theorem, we have $\Delta v = 0$ in B_r and $v = u$ on ∂B_r. Now by uniqueness of the Dirichlet problem (or by the maximum principle) we infer $u \equiv v$ in B_r. □

Remark 33. Let $u \geq 0$ be a harmonic function in some region that contains $\bar{B}_r$. Then from an application of the Poisson formula we infer the following *Harnack inequality*

$$u(y) \leq \frac{r^2 - |y|^2}{r|S^{n-1}| \cdot (r - |y|)^n} \int_{\partial B_r} u(x)\mathrm{d}^{n-1}x = \frac{r^{n-2}(r^2 - |y|^2)}{(r - |y|)^n} u(0) = \frac{1 - k^2}{(1 - k)^n} u(0), \tag{166}$$

where $k = |y|/r$. A lower bound on $u(y)$ can also be obtained, leading to

$$\left(\frac{1}{1 + k}\right)^{n-2} \frac{1 - k}{1 + k}\, u(0) = \frac{1 - k^2}{(1 + k)^n} u(0) \leq u(y) \leq \left(\frac{1}{1 - k}\right)^{n-2} \frac{1 + k}{1 - k}\, u(0), \tag{167}$$

which are a slight quantitative improvement over (74).

One should not be deceived by the fact that Poisson's formula solves a seemingly simple problem. It is a very powerful tool in the study of harmonic functions.

Theorem 34 (Removable singularity)**.** *Let Ω be an open set, and let $z \in \Omega$. Assume that $u \in \mathscr{C}^2(\Omega \setminus \{z\})$ is harmonic in $\Omega \setminus \{z\}$, and satisfies $u(x) = o(E(x - z))$ as $x \to z$. Then $u(z)$ can be defined so that $u \in \mathscr{C}^2(\Omega)$ and $\Delta u = 0$ in Ω.*

Proof. Without loss of generality, let us assume $z = 0$ and $\bar{B}_r \subset \Omega$ with some $r > 0$. Let $v \in \mathscr{C}^2(B_r)$ satisfy $\Delta v = 0$ in B_r and $v = u$ on ∂B_r. Of course, if u has a harmonic extension to B_r then it must be equal to v. For this to work, we need to show that $u = v$ in $B_r \setminus \{0\}$. By the maximum principle, we have $|v| \leq M_r$ in B_r, where $M_r = \sup_{x \in \partial B_r} |u(x)|$. Let $w = u - v$ and $\delta > 0$. Then we have $\Delta w = 0$ in $B_r \setminus B_\delta$ and $w = 0$ on ∂B_r. We can say that $|w| \leq |v| + |u| \leq M_r + M_\delta$ on ∂B_δ. At this point if we apply the maximum principle to w, we would only get $|w| \leq M_r + M_\delta$, which is not what we are after. Let us look at the case $n \geq 3$ first. We define the function $\phi_\delta(x) = (M_r + M_\delta)\delta^{n-2}/|x|^{n-2}$ for comparison purposes. We see that $\Delta \phi_\delta = 0$ in $B_r \setminus B_\delta$, $\phi_\delta \geq 0$ on ∂B_r and $\phi_\delta = M_r + M_\delta$ on ∂B_δ, i.e., $\pm w \leq \phi_\delta$ on the boundary of $B_r \setminus B_\delta$. Then the comparison principle yields $|w(x)| \leq \phi_\delta(x)$ for $x \in B_r \setminus B_\delta$.

Finally, for any fixed $x \in B_r$, sending $\delta \to 0$ and taking into account that $M_\delta = o(\delta^{2-n})$, we infer $|w(x)| = 0$. Figure 9(a) illustrates the phenomenon $\phi_\delta(x) \to 0$ as $\delta \to 0$ for any fixed $x \in B_r$, by taking $M_\delta = \text{const}$ for simplicity. In the case $n = 2$, we can use the comparison function $\phi_\delta(x) = (M_r + M_\delta)\log(r/|x|)/\log(r/\delta)$. □

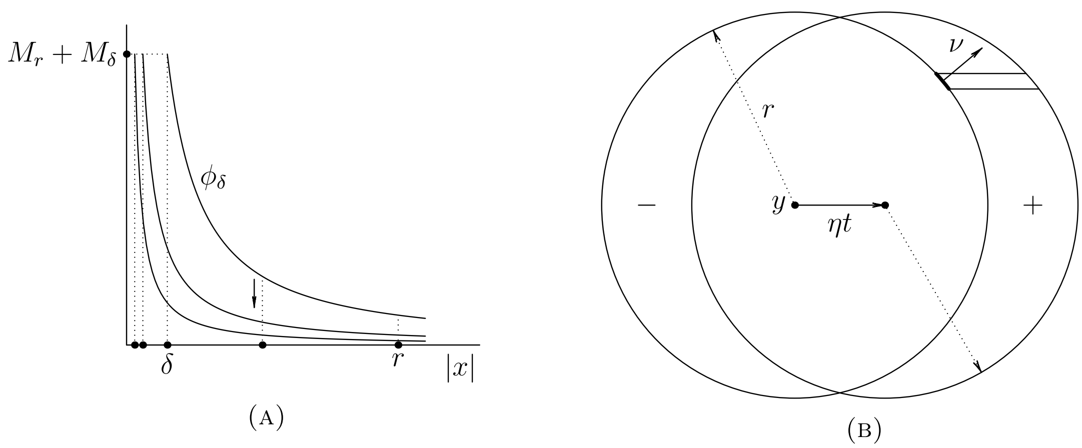


Figure 9. Illustrations for the proofs of Theorem 34 and Theorem 35.

10. Converse to the mean value property

In this section, we prove that the mean value property implies smoothness and harmonicity. We present two proofs, a slick one that is based on the Poisson formula, and one that is direct and elementary. The mean value property says that the values of harmonic functions are averages over balls, which, at an intuitive level, implies that harmonic functions cannot be too rough. The direct proof below makes this intuition precise.

Theorem 35 (Koebe 1906)**.** *Let $u \in \mathscr{C}(\Omega)$ be a function satisfying the mean value property for every ball whose closure is contained in Ω. Then $u \in \mathscr{C}^\infty(\Omega)$ and $\Delta u = 0$. Moreover, for $\eta \in S^{n-1}$ and $\overline{B_r(y)} \subset \Omega$, we have*

$$\partial_\eta u(y) = \frac{1}{|B_r|}\int_{\partial B_r(y)} u\,\eta\cdot\nu. \tag{168}$$

Proof. Let B be an arbitrary ball such that $\bar{B} \subset \Omega$, and by employing the Poisson formula, let us construct a function $v \in \mathscr{C}^\infty(B) \cap \mathscr{C}(\bar{B})$ satisfying $\Delta v = 0$ in B and $v = 0$ on ∂B. Since $u - v$ satisfies the mean value property in B, and $u - v = 0$ on ∂B, by the maximum principle we infer that $u \equiv v$ in B. Hence u is smooth and harmonic. Then the formula (168) follows from (152). □

Direct proof. We will prove (168) first. Let $y \in \Omega$, and let $\overline{B_r(y)} \subset \Omega$. Then for all small t, we have $\overline{B_r(y+\eta t)} \subset \Omega$. The mean value property gives

$$u(y+\eta t) - u(y) = \frac{1}{|B_r|}\int_{B_r(y)} (u(x+\eta t) - u(x))\mathrm{d}^n x, \tag{169}$$

see Figure 9(b). Define $\partial^\pm B_r(y) = \{x \in \partial B_r(y) : (x-y)\cdot\eta \gtrless 0\}$, i.e., $\partial^+ B_r(y)$ is the positive half of $\partial B_r(y)$ with respect to the direction η, and $\partial^- B_r(y)$ is the negative half. Then the above integral can be decomposed as

$$\begin{aligned}\frac{1}{|B_r|}\int_{B_r(y)}(u(x+\eta t)-u(x))\mathrm{d}^n x = \frac{1}{|B_r|}\int_{\partial^+ B_r(y)}\int_0^t u(x+\eta s)\,\eta\cdot\nu \mathrm{d}s\mathrm{d}^{n-1}x \\ - \frac{1}{|B_r|}\int_{\partial^- B_r(y)}\int_0^t u(x+\eta s)(-\eta\cdot\nu)\mathrm{d}s\mathrm{d}^{n-1}x,\end{aligned} \tag{170}$$

where ν is the outer unit normal to $\partial B_r(y)$, and the notation $\mathrm{d}^{n-1}x$ is meant to make it clear that the x-integration is over an $n-1$ dimensional surface. We can recombine the integrals and use uniform continuity to get

$$\begin{aligned}u(y+\eta t)-u(y) &= \frac{1}{|B_r|}\int_{\partial B_r(y)}\int_0^t u(x+\eta s)\,\eta\cdot\nu \mathrm{d}s\mathrm{d}^{n-1}x \\ &= \frac{t}{|B_r|}\int_{\partial B_r(y)} u(x)\,\eta\cdot\nu \mathrm{d}^{n-1}x + o(|t|),\end{aligned} \tag{171}$$

which proves (168).

Since u is continuous, the integral over $B_r(y)$ depends on y continuously, hence $\partial_\eta u \in \mathscr{C}(\Omega)$, implying that $u \in \mathscr{C}^1(\Omega)$. Moreover, from the divergence theorem, we get

$$\partial_\eta u(y) = \frac{1}{|B_r|}\int_{\partial B_r(y)} u\,\eta\cdot\nu = \frac{1}{|\partial B_r|}\int_{\partial B_r(y)}\partial_\eta u, \tag{172}$$

i.e., $\partial_\eta u$ satisfies the mean value property. Then the smoothness of u follows by induction.

As Δu is a linear combination of derivatives of u, it satisfies the mean value property. Applying the divergence theorem to this fact then reveals

$$\Delta u(y) = \frac{1}{|B_r|}\int_{B_r(y)}\nabla\cdot\nabla u = \frac{1}{|B_r|}\int_{\partial B_r(y)}\partial_\nu u. \tag{173}$$

The mean value property can be written as

$$u(y) = \frac{1}{|\partial B_r|}\int_{\partial B_r(y)} u = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} u(y+\xi r)\mathrm{d}^{n-1}\xi. \tag{174}$$

Then

$$0 = \frac{\mathrm{d}}{\mathrm{d}r}\int_{S^{n-1}} u(y+\xi r)\mathrm{d}^{n-1}\xi = \int_{S^{n-1}}\partial_r u(y+\xi r)\mathrm{d}^{n-1}\xi = \frac{1}{r^{n-1}}\int_{\partial B_r(y)}\partial_\nu u, \tag{175}$$

which completes the proof. □

11. Derivative estimates

Suppose that u is a harmonic function in Ω. Then from (168) we infer

$$|\partial_\eta u(y)| \leq \frac{1}{|B_r|}\int_{\partial B_r(y)}|u| \leq \frac{|\partial B_r|}{|B_r|}\sup_{\partial B_r(y)}|u| = \frac{n}{r}\sup_{\partial B_r(y)}|u|, \tag{176}$$

for $\eta \in S^{n-1}$ and $\overline{B_r(y)} \subset \Omega$. This means that a harmonic function tends to be flat towards the middle of the domain on which it is harmonic.

If $u \geq 0$ in (176), we can use the mean value property to get

$$|\partial_\eta u(y)| \leq \frac{1}{|B_r|}\int_{\partial B_r(y)} u = \frac{|\partial B_r|}{|B_r|}u(y) = \frac{n}{r}u(y), \tag{177}$$

which is called a *differential Harnack inequality*. Liouville's theorem follows immediately: If u is nonnegative and entire harmonic, then at each point $y \in \mathbb{R}^n$, taking $r \to \infty$ in (177) implies that $\partial_\eta u(y) = 0$.

Remark 36. The differential Harnack inequality (177) can be integrated to get a Harnack inequality. Suppose that γ is a differentiable curve parameterized by arc length, with endpoints $x = \gamma(0)$ and $y = \gamma(\ell)$, such that each point on γ is at the distance greater than R from the boundary of Ω. Assume that $u > 0$ in Ω. Then we have

$$\frac{\mathrm{d} \log u(\gamma(t))}{\mathrm{d}t} = \frac{\gamma'(t) \cdot \nabla u(\gamma(t))}{u(\gamma(t))}, \qquad 0 \leq t \leq \ell. \tag{178}$$

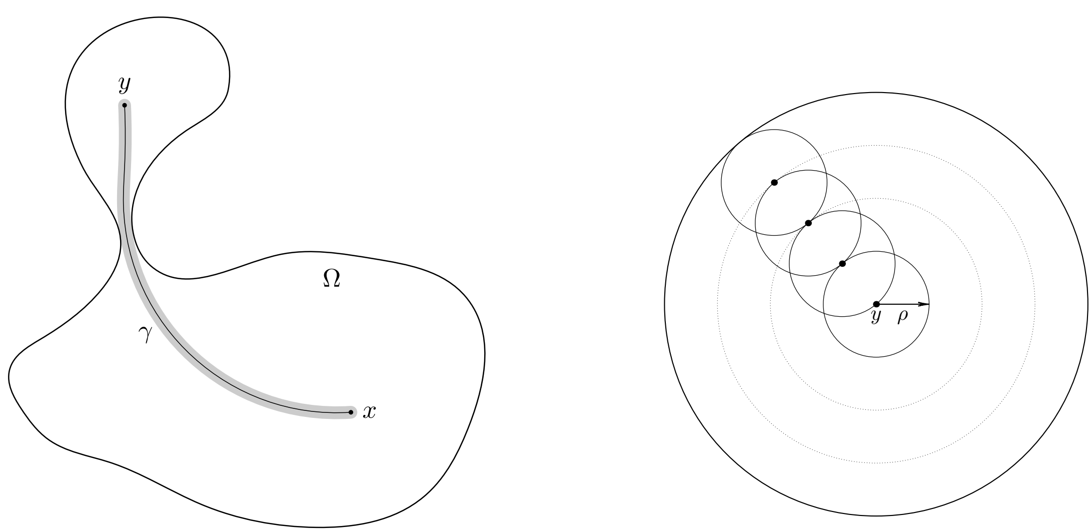
Integrating this, we get

$$|\log u(x) - \log u(y)| \leq \int_0^\ell \Big|\frac{\gamma'(t) \cdot \nabla u(\gamma(t))}{u(\gamma(t))}\Big| \mathrm{d}t \leq \frac{n\ell}{R}, \tag{179}$$

which implies

$$e^{-n\ell/R} \leq \frac{u(x)}{u(y)} \leq e^{n\ell/R}. \tag{180}$$

The essence of Harnack inequalities is the fact that the ratio $u(x)/u(y)$ cannot be too large if, in a certain sense, the domain boundary does not obstruct the "connection" between x and y. For example, if the curve γ has to go through a bottleneck in order to connect x and y, then it would make R smaller, resulting in a weaker control in (180), cf. Figure 10.



(A) The quality of the constants in the Harnack inequality (180) depends on the length of γ and the width of the shaded neighbourhood.

(B) For the proof of Theorem 37.

FIGURE 10. Illustrations for Remark 36 and for the proof of Theorem 37.

We can repeatedly apply (176) to derive estimates on higher derivatives. In the following theorem, we use the convenient notation $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for higher order partial derivative operators, and $|\alpha| = \alpha_1 + \ldots + \alpha_n$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is called a *multi-index.*

Theorem 37. *Let u be harmonic in Ω, and let $\overline{B_r(y)} \subset \Omega$. Then*

$$|\partial^\alpha u(y)| \leq |\alpha|! \big(\frac{ne}{r}\big)^{|\alpha|} \sup_{B_r(y)} |u|. \tag{181}$$

Proof. Let $\rho = \frac{r}{|\alpha|}$ and let β be a multi-index with $|\beta| = |\alpha| - 1$. Then since all derivatives of a harmonic function are also harmonic, from (176) we have

$$|\partial^\alpha u(y)| \leq \frac{n}{\rho} \sup_{\partial B_\rho(y)} |\partial^\beta u|. \tag{182}$$

We can estimate the derivative $\partial^\beta u$ appearing in the right hand side by the same procedure, decreasing the order of derivatives again by one. We continue this process until we get no derivatives in the right hand side, and get

$$|\partial^\alpha u(y)| \leq \big(\frac{n}{\rho}\big)^{|\alpha|} \sup_{B_r(y)} |u| = \big(\frac{n|\alpha|}{r}\big)^{|\alpha|} \sup_{B_r(y)} |u|. \tag{183}$$

The estimate (181) follows from here upon using the elementary inequality $k^k \leq k! e^k$, which can be seen for instance from the convergent series $e^k = 1 + k + \ldots + \frac{k^k}{k!} + \ldots$. □

12. Analyticity

In this section, we will show that harmonic functions are analytic. Let us first clarify the notion of analyticity. In $\mathbb{R}^n$, a *power series* is an expression of the form

$$f(x) = \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} a_{\alpha_1,\ldots,\alpha_n} (x_1 - y_1)^{\alpha_1} \cdots (x_n - y_n)^{\alpha_n}, \tag{184}$$

with the coefficients $a_{\alpha_1,\ldots,\alpha_n} \in \mathbb{R}$, and the centre $y \in \mathbb{R}^n$. Introducing the multi-index notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$, this series can also be written as

$$f(x) = \sum_\alpha a_\alpha (x - y)^\alpha. \tag{185}$$

If the preceding series converges for some x, then obviously there is a constant $M < \infty$, such that $|a_\alpha||x_1 - y_1|^{\alpha_1} \cdots |x_n - y_n|^{\alpha_n} \leq M$ for all α. In particular, if this series converges in a neighbourhood of y, then there are constants $M < \infty$ and $r > 0$, such that $|a_\alpha| \leq M r^{-|\alpha|}$ for all α. On the other hand, if $r \in \mathbb{R}^n$ and $M < \infty$ satisfy $|a_\alpha| r_1^{\alpha_1} \cdots r_n^{\alpha_n} \leq M$ for all α, then the series converges absolutely and uniformly for all $x \in \mathbb{R}^n$ satisfying $|x_i - y_i| < r_i$ for each i. For our purposes, the take away message here is that if (185) converges in a neighbourhood of y, then the convergence is absolute and uniform in a (possibly smaller) neighbourhood of y.

Exercise 38. Prove the statements in the previous paragraph.

Definition 39. Let Ω be an open subset of $\mathbb{R}^n$. A real-valued function $f : \Omega \to \mathbb{R}$ is called *(real) analytic at* $y \in \Omega$ if it is developable into a power series around y, i.e, if there are coefficients $a_\alpha \in \mathbb{R}$, $(\alpha \in \mathbb{N}_0^n)$, such that the power series (185) converges in a neighbourhood of y. Moreover, f is said to be *(real) analytic in* Ω if it is analytic at each $y \in \Omega$. The set of analytic functions in Ω is denoted by $C^\omega(\Omega)$.

In parallel to the single variable case, one can show that if f is analytic at y, then the series (185) is its multivariate *Taylor series*, i.e., the coefficients are given by

$$a_\alpha = \frac{\partial^\alpha f(y)}{\alpha!} = \frac{\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f(y)}{\alpha_1! \cdots \alpha_n!}, \tag{186}$$

where we have introduced the convention $\alpha! = \alpha_1! \cdots \alpha_n!$. In other words, f is analytic at $y \in \Omega$ if and only if

$$f(x) = \sum_\alpha \frac{\partial^\alpha f(y)}{\alpha!} (x - y)^\alpha, \tag{187}$$

holds in a neighbourhood of y. Note that this requires not only that the series in the right hand side converges, but also that the limit value is equal to the function in the left hand side.

Example 40. Convergence of the series in the right hand side of (187) is not enough to guarantee analyticity. A classical counterexample is

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases} \tag{188}$$

This function is in $\mathscr{C}^\infty(\mathbb{R})$, and $f^{(n)}(0) = 0$ for all n. Hence the Taylor series (187) of f at $y = 0$ is identically 0, but f itself is clearly a nontrivial function.

The reason behind the failure of analyticity in the aforementioned example can be attributed to the fact that the derivatives $f^{(n)}(x)$ grow too fast with n in *any neighbourhood* of 0. The following lemma establishes a sufficient condition for analyticity based on growth estimates of derivatives. In fact, we will see later in the course that this condition is also necessary.

Lemma 41. *A function f is real analytic in Ω if for any point $y \in \Omega$ there exist a ball $B = B_r(y)$ with $r > 0$ and $\bar{B} \subset \Omega$, and constants $\delta > 0$ and $M < \infty$ such that*

$$\max_{|\alpha|=m} \sup_{x \in B} |\partial^\alpha f(x)| \leq M \frac{m!}{\delta^m} \qquad \textit{for all } m \in \mathbb{N}. \tag{189}$$

Proof. Let $y \in \Omega$ be an arbitrary point, and assume that (189) is satisfied for some $B = B_r(y)$ as hypothesized in the statement of the lemma. Our goal is now by using the estimates (189) to show that the Taylor series (187) converges in a neighbourhood of y. Without loss of generality, let us assume that $y = 0$. Given $z \in B$, consider the function $g(t) = f(zt)$. Taylor's theorem tells us

$$f(z) = g(1) = \sum_{k=0}^{m-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(m)}(s)}{m!}, \tag{190}$$

where $0 \leq s \leq 1$. Let us compute the derivatives of g. We have

$$\begin{aligned} g'(t) &= (z_1\partial_1 + \ldots + z_n\partial_n) f(zt), \\ g''(t) &= (z_1\partial_1 + \ldots + z_n\partial_n)^2 f(zt), \ldots \\ g^{(k)}(t) &= (z_1\partial_1 + \ldots + z_n\partial_n)^k f(zt) \\ &= \sum_{\alpha_1 \ldots + \alpha_n = k} \frac{k!}{\alpha_1! \ldots \alpha_n!} z_1^{\alpha_1} \ldots z_n^{\alpha_n} \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f(zt) \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} z^\alpha \partial^\alpha f(zt), \end{aligned} \tag{191}$$

by the multinomial theorem, so

$$f(z) = \sum_{|\alpha|<m} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha + \underbrace{\frac{(z_1\partial_1 + \ldots + z_n\partial_n)^m f(sz)}{m!}}_{R_m}, \tag{192}$$

with $0 \leq s \leq 1$. We can estimate the remainder term by

$$|R_m| \leq M\delta^{-m} n^m |z|^m = M\big(\frac{n|z|}{\delta}\big)^m, \tag{193}$$

which tends to 0 if $|z| < \frac{\delta}{n}$. This establishes the lemma. □

Now we can state and prove our main theorem.

Theorem 42. *Let $u \in \mathscr{C}^2(\Omega)$ be harmonic in an open set Ω. Then $u \in \mathscr{C}^\omega(\Omega)$.*

Proof. Let $y \in \Omega$, and choose $\rho > 0$ such that $\bar{B}_\rho(y) \subset \Omega$. Then for all $x \in B_r$ wih $0 < r < \rho$, by Theorem 37 we have

$$|\partial^\alpha u(x)| \leq |\alpha|!\big(\frac{ne}{\rho - r}\big)^{|\alpha|} \sup_{B_\rho(y)} |u|. \tag{194}$$

This gives the desired estimate in Lemma 41, with $\delta = \frac{\rho - r}{ne}$ and $M = \sup\limits_{B_\rho(y)} |u|$, to establish that u is real analytic in Ω. □

Before closing this section, we include here a basic result known as the identity theorem for multivariate analytic functions.

Theorem 43 (Identity theorem)**.** *Let $f \in C^\omega(\Omega)$ with Ω a connected open set in $\mathbb{R}^n$, and with some $y \in \Omega$, let $\partial^\alpha f(y) = 0$ for all α. Then $f \equiv 0$ in Ω. In particular, the same conclusion holds if f vanishes in some open subset of Ω.*

Proof. Each $\Sigma_\alpha = \{x \in \Omega : \partial^\alpha f(x) = 0\}$ is relatively closed in Ω, so the intersection $\Sigma = \bigcap_\alpha \Sigma_\alpha$ is also relatively closed. On the other hand, Σ is open, because $x \in \Sigma$ implies that $f \equiv 0$ in a neighbourhood of x by a Taylor series argument. As Σ is nonempty, we have $\Sigma = \Omega$. □

Corollary 44. *If u is harmonic in a domain Ω and $u = 0$ in an open subset of Ω, then $u \equiv 0$.*

13. Multipole expansions

Analyticity of f at y is equivalent to $T_m f \to f$ as $m \to \infty$ in a neighbourhood of y, where

$$T_m f(x) = \sum_{|\alpha| \leq m} \frac{\partial^\alpha f(y)}{\alpha!} (x - y)^\alpha, \tag{195}$$

are the *Taylor polynomials* of f at y. In light of the fact that harmonic functions are analytic, a natural question arises: Are the Taylor polynomials of a harmonic function also harmonic? We claim that the answer is yes.

Obviously, $T_0 f$ and $T_1 f$ are harmonic. With the intent of applying induction, if $T_{m-1} f$ is harmonic, then harmonicity of $T_m f$ is equivalent to harmonicity of $T_m f - T_{m-1} f$. An obvious quality of the latter is that it is a *homogenous polynomial*, i.e., it is of the form

$$h_m(x) = \sum_{|\alpha| = m} a_\alpha x^\alpha, \tag{196}$$

which can also be characterized by the property

$$h_m(\lambda x) = \lambda^m h_m(x) \qquad \text{for any } \lambda \in \mathbb{R}. \tag{197}$$

It is clear that Δh_m is homogeneous of degree $m - 2$, and so

$$\Delta h_m(x) = |x|^{m-2} \Delta h_m\Big(\frac{x}{|x|}\Big). \tag{198}$$

Now consider the special case $m = 2$. Supposing that f is harmonic, and without loss of generality, letting $y = 0$, we have

$$0 = \Delta f(x) = \Delta h_2(x) + e_2(x) = \Delta h_2\Big(\frac{x}{|x|}\Big) + e_2(x), \tag{199}$$

where $e_2(x) = O(x)$ as $x \to 0$. If $\Delta h_2(x) \neq 0$ for some $x \neq 0$, then by homogeneity, $\Delta h_2(\omega) \neq 0$ for $\omega = \frac{x}{|x|} \in S^{n-1}$. Since $e_2(x)$ can be made arbitrarily small by taking $|x|$ small, we conclude that $\Delta h_2 = 0$ on S^{n-1}, and hence $\Delta h_2 \equiv 0$. This argument works for general m, and shows in the end that if f is harmonic in a neighbourhood of 0, then the series

$$f(x) = \sum_{m \geq 0} |x|^m h_m\Big(\frac{x}{|x|}\Big), \tag{200}$$

converges in a neighbourhood of 0, where h_m is a *homogeneous, harmonic polynomial* of degree m, for each m.

14. Problems and exercises

1. Prove that the space of entire harmonic functions in $\mathbb{R}^n$ $(n \geq 2)$ is infinite dimensional.

2. Show that real and imaginary parts of a holomorphic function satisfy the Laplace equation.

3. Compute the electrostatic potential produced by a uniformly charged circular wire.

4. (a) Consider a 3-dimensional solid ball B of radius a centred at the origin, with a spherically symmetric charge density $\rho(x) = g(|x|) \geq 0$, for some continuous function g of a single variable. Show that the electrostatic potential generated by B at the point $x \in \mathbb{R}^3$ with $|x| > a$, is the same as if its charge were all concentrated at the origin, that is, $u(x) = CQ/|x|$, where Q is the total charge of B.

(b) Consider now a spherical shell S, defined by $a \leq |x| \leq b$, with spherically symmetric, continuous charge density function. Show that this shell exerts *no* electrostatic force on a point charge q located at the point $(c, 0, 0)$ *inside* S (that is, $|c| < a$).

(c) In both cases (a) and (b), find the electrostatic potential u and compute Δu everywhere, that is, inside and outside of the body.

5. Let $f : B_R \setminus \{0\} \to \mathbb{R}$ and let there be a constant $M > 0$ such that

$$\int_{B_R \setminus B_\varepsilon} |f| \leq M \qquad \text{for any } \varepsilon > 0, \tag{201}$$

where the integral is understood in the Riemann sense. Then the *improper Riemann integral* of f over B_R is defined to be

$$\int_{B_R} f = \lim_{\varepsilon \to 0} \int_{B_R \setminus B_\varepsilon} f, \tag{202}$$

and we say that f is *absolutely integrable*.

(a) Let $\{U_\varepsilon\}$ be a family of open sets in $\mathbb{R}^n$, satisfying $U_\varepsilon \subset U_\delta$ for $\varepsilon < \delta$ and $\bigcap_\varepsilon U_\varepsilon = \{0\}$. Show that if f is absolutely integrable, then

$$\lim_{\varepsilon \to 0} \int_{B_R \setminus U_\varepsilon} f = \lim_{\varepsilon \to 0} \int_{B_R \setminus B_\varepsilon} f, \tag{203}$$

meaning that the improper Riemann integral does not depend on the family $\{U_\varepsilon\}$.

(b) Show that if f is absolutely integrable, then the Lebesgue integral of f over B_R exists, and is equal to the improper Riemann integral (202).

6. Let $\phi : (0, 1] \to \mathbb{R}$ be a continuous nonnegative function satisfying

$$\int_\varepsilon^1 \phi(r) r^{n-1} \mathrm{d}r \leq M,$$

for any $\varepsilon > 0$, with a fixed constant M. Define $f(x) = \phi(|x|)$ for $x \in B_1$ where B_1 is the open unit ball centred at the origin. Prove in complete detail that $f \in L^1(B_1)$.

7. Let Ω and u be as in Green's representation formula (51), and in addition, let $\Delta u = 0$ in Ω. Then by using Green's formula, prove that $u \in \mathscr{C}^\infty(\Omega)$ with

$$\sup_{y \in K} |\partial^\alpha u(y)| \leq C(\sup_\Omega |u| + \sup_\Omega |\nabla u|), \tag{204}$$

for all multi-indices α and compact sets $K \subset \Omega$, with the constant C possibly depending on α and K.

8. Let $u \in \mathscr{C}^2(\mathbb{R}^n)$ be a function satisfying $\Delta u = 0$ in $\mathbb{R}^n \setminus B_r$, where B_r is the open ball of radius $r > 0$ centred at the origin. Show that

$$\frac{1}{|\partial B_r|}\int_{\partial B_r} u = \frac{r^{2-n}}{(2-n)|S^{n-1}|}\int_{B_r} \Delta u \equiv E(r)\int_{B_r} \Delta u,$$

provided $n \geq 3$ and

$$\frac{1}{|\partial B_\rho|}\int_{\partial B_\rho} u \to 0 \qquad \text{as} \qquad \rho \to \infty.$$

9. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in C^2(\Omega)$ be a nonconstant harmonic function. Show that u cannot have any *local* maximum in Ω. (Note that the strong maximum principle as stated rules out only *global* maximums.)

10. Let $u \in \mathscr{C}^2(\Omega)$ be harmonic in a bounded domain Ω. By using the Harnack inequality show that unless u is constant, it cannot achieve its extremums in Ω.

11. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the boundary value problem

$$\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 1 \quad \text{on } \partial\Omega. \tag{205}$$

Prove the following.

(a) Any solution of (205) in $C^2(\Omega) \cap C(\bar{\Omega})$ with $f(u) = u^m$ where $m \in \mathbb{N}$ is odd, must satisfy $0 \leq u \leq 1$ in $\bar{\Omega}$, and is unique.
(b) The only solution of (205) in $C^2(\Omega) \cap C(\bar{\Omega})$ with $f(u) = u - u^{-1}$ is $u \equiv 1$.

12. Prove the following statements.

(a) A function $u \in C(\Omega)$ is subharmonic in Ω iff for any closed ball $B \subset \Omega$ and any harmonic function v in a neighbourhood of B, $u \leq v$ on ∂B implies $u \leq v$ in B.
(b) If u is harmonic in Ω, then $|\nabla u|^2$ is subharmonic in Ω.
(c) Let $u \in \mathscr{C}(\mathbb{R}^n)$ be subharmonic in $\mathbb{R}^n$ where $n \leq 2$, and suppose that $u(x) = o(|x|)$ or $u(x) = o(\log|x|)$, depending on whether $n = 1$ or $n = 2$. Show that u is constant.

13. Let $\Omega \subset \mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n : x_n > 0\}$ be a domain, and let $\Sigma = \{x \in \partial\Omega : x_n = 0\}$ be a nonempty open subset of the hyperplane $\partial\mathbb{R}^n_+ \equiv \{x_n = 0\}$. Prove the following.

(a) Suppose that $u \in C^2(\Omega) \cap C(\Omega \cup \Sigma)$ is harmonic in Ω and $u = 0$ on Σ. Denote by x^* the reflection $(x_1, \ldots, x_{n-1}, -x_n)$ of $x = (x_1, \ldots, x_{n-1}, x_n)$, and let

$$\tilde{\Omega} = \Omega \cup \Sigma \cup \{x^* : x \in \Omega\}.$$

Then the function $\tilde{u} \in C(\tilde{\Omega})$ defined by $\tilde{u} = u$ in $\Omega \cup \Sigma$ and $\tilde{u}(x^*) = -u(x)$ for $x \in \Omega$ is harmonic in $\tilde{\Omega}$. This result is known as the *Schwarz reflection principle*.
(b) The Cauchy problem for the Laplace equation $\Delta u = 0$ with the Cauchy data $u = 0$ and $\partial_n u = g$ on the hyperplane $\{x_n = 0\}$ has no solution in any neighbourhood of $0 \in \mathbb{R}^n$, if g is not analytic at $0 \in \mathbb{R}^{n-1}$.
(c) A bounded harmonic function in $\mathbb{R}^n_+$ with $u = 0$ on $\{x_n = 0\}$ is identically zero.

14. Let u be an entire harmonic function in $\mathbb{R}^n$. Prove the following.

(a) If $u \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$ then $u \equiv 0$.
(b) Any tangent hyperplane to the graph of u intersects the graph more than once.
(c) If u satisfies $u(x) \geq -C(1 + |x|)^m$ for some constants C and $m \in \mathbb{N}$, then u is a polynomial of degree less or equal to m.

15. Let $g \in \mathscr{C}(\mathbb{R}^{n-1})$ be a bounded function, and let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be given by

$$u(y) = \int_{\partial\mathbb{R}^n_+} \Pi(y, x) g(x) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}, \qquad (y \in \mathbb{R}^n_+), \tag{206}$$

where $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$, and

$$\Pi(y, x) = \frac{2y_n}{|S^{n-1}| \cdot |x - y|^n}, \tag{207}$$

is the half-space Poisson kernel. Show that $u \in \mathscr{C}^\infty(\mathbb{R}^n_+)$, $\Delta u = 0$ in $\mathbb{R}^n_+$, and $u(y) \to g(x)$ as $\mathbb{R}^n_+ \ni y \to x \in \partial\mathbb{R}^n_+$.

16. Let $g \in \mathscr{C}(\mathbb{R}^{n-1})$ be a compactly supported function, and let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be given by

$$u(y) = 2 \int_{\partial\mathbb{R}^n_+} E(x - y) g(x) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}, \qquad (y \in \mathbb{R}^n_+), \tag{208}$$

where $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$. Show that $u \in \mathscr{C}^\omega(\mathbb{R}^n_+)$, $\Delta u = 0$ in $\mathbb{R}^n_+$, and $\partial_n u(y) \to g(x)$ as $\mathbb{R}^n_+ \ni y \to x \in \partial\mathbb{R}^n_+$.

17. Let $g \in \mathscr{C}(\partial\mathbb{D})$, and let $u : \mathbb{D} \to \mathbb{R}$ be given by

$$u(y) = -\frac{1}{\pi} \int_{\partial\mathbb{D}} g(x) \log|x - y| \, \mathrm{d}x, \tag{209}$$

Show that u is harmonic in $\mathbb{D}$, and continuous in $\bar{\mathbb{D}}$, with $\int_{\partial\mathbb{D}} u = 0$. Moreover, prove that $\partial_r u(y) \to g(x)$ as $\mathbb{D} \ni y \to x \in \partial\mathbb{D}$, where ∂_r is the radial derivative.

18. Let Ω be a domain, and let $\Sigma = \partial\Omega \cap B$ be a smooth and nonempty portion of the boundary, where B is an open ball. Let $u \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma)$ satisfy $\Delta u = 0$ in Ω and $u = \partial_\nu u = 0$ on Σ. Show that u is identically zero in Ω.

19 (Bôcher 1905)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in C^1(\Omega)$ and that for each $y \in \Omega$ there exists $r^* = r^*(y) > 0$ such that

$$\int_{\partial B_r} \partial_\nu u = 0,$$

for all $0 < r < r^*$, where ∂_ν is the normal derivative. Show that u is harmonic in Ω.

20. Let u be a harmonic function, and define

$$q(r) = \int_{\partial B_r} u^2, \qquad \text{for} \quad r > 0.$$

Prove that

a) q is monotone and convex.
b) q is log-convex, i.e., $\log q(r)$ is a convex function of $\log r$.