1. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in C^2(\Omega)$ be a nonconstant harmonic function. Show that $u$ cannot have a local maximum in $\Omega$.

2. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u \in H^1_{\text{loc}}(\Omega)$ satisfy
   \[ \int_{\Omega} \nabla u \cdot \nabla \varphi = 0, \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \]
   Show that $u \in C^\omega(\Omega)$ and that $u$ is harmonic.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and consider the Friedrichs inequality
   \[ \int_{\Omega} |u|^2 \leq C \int_{\Omega} |\nabla u|^2, \]
   that holds for all $u \in H^1_0(\Omega)$, and for some constant $C = C(\Omega) > 0$.
   a) Characterize the best constant $C$ via an eigenvalue problem.
   b) Compute the best constant for the cube $\Omega = (0,1)^n$.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set having the $H^1$ extension property.
   a) Show that the first Neumann eigenvalue of $\Omega$ is $\lambda_1 = 0$, and the dimension of the eigenspace corresponding to this eigenvalue (i.e., the multiplicity of $\lambda_1$) is equal to the number of connected components of $\Omega$.
   b) Show that $\Omega$ has finitely many connected components.