1. Let $Q = (0, 1)^n$ and let $Q_h = (h, 1 - h)^n$. For $h > 0$ small, define the trace map
   \[ \gamma_h : C^1(Q) \to C(\partial Q_h) \] by $\gamma_h v = v|_{\partial Q_h}$.
   a) Prove that $\gamma_h$ can be uniquely extended to a bounded map $\gamma : H^1(Q) \to L^2(\partial Q_h)$.
   b) Make sense of the boundary trace $\gamma_0 u = \lim_{h \to 0} \gamma_h u$ in $L^2(\partial Q)$ for $u \in H^1(Q)$.
   c) Show that $\gamma_0 u = 0$ for $u \in H_0^1(Q)$.
   d) Let $u \in H_0^1(Q)$ and let $u$ be continuous at 0. Show that $u(0) = 0$.

2. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $W^{1,1}_\text{loc}(\Omega)$ be the set of locally integrable functions whose (weak) derivatives are locally integrable (that is, in $L^1_{\text{loc}}(\Omega)$).
   a) Show that if $u, v \in W^{1,1}_\text{loc}(\Omega)$ and $uv, u\partial_i v + v\partial_i u \in L^1_{\text{loc}}(\Omega)$, then $uv \in W^{1,1}_\text{loc}(\Omega)$ and $\partial_i (uv) = u\partial_i v + v\partial_i u$.
   b) Let $\phi : \Omega \to \Omega'$ be a $C^1$-diffeomorphism between $\Omega$ and $\Omega'$. Show that if $u \in W^{1,1}_\text{loc}(\Omega')$ then $v = u \circ \phi \in W^{1,1}_\text{loc}(\Omega)$ and $\partial_i v(x) = \sum_j \partial_i \phi_j(x) (\partial_j u)(\phi(x))$, where $\phi_j$ is the $j$-th component of $\phi$, and $(\partial_j u)(\phi(x))$ is the evaluation of $\partial_j u$ at the point $\phi(x)$.
   c) Let $f \in C^1(\mathbb{R})$ with both $f$ and $f'$ bounded, and let $u \in W^{1,1}_\text{loc}(\Omega)$. Prove that $f \circ u \in W^{1,1}_\text{loc}(\Omega)$ and that $\partial_i (f \circ u) = (f' \circ u) \partial_i u$.
   d) Let $u \in W^{1,1}_\text{loc}(\Omega)$ and let $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ pointwise. Prove that $\partial_i u^+ = \theta(u) \partial_i u$ and $\partial_i u^- = \theta(-u) \partial_i u$ a.e., where $\theta$ is the Heaviside step function.
   In particular, show that $|u| \in W^{1,p}(\Omega)$ if $u \in W^{1,p}(\Omega)$.

3. Let $H$ be a (real) Hilbert space, and let $H'$ be its dual, defined as the space of continuous linear functionals on $H$. Let us denote the inner product of $H$ by $\langle \cdot, \cdot \rangle$. Observe that any $y \in H$ defines an element $f \in H'$ by $f(x) = \langle y, x \rangle$ for $x \in H$. This defines a map $J : H \to H'$. The Riesz representation theorem (for Hilbert spaces)\(^1\) states that $J$ is invertible, that is, any continuous linear functional on $H$ can be realized through the inner product with an element of $H$. We would like to prove this theorem by using a variational method. Let $f \in H'$, and let
   \[ E(x) = \langle x, x \rangle - 2f(x), \quad x \in H, \]
   and consider the problem of finding a minimizer of $E$ over $H$.
   a) Show that a minimizing sequence for $E$ exists and is Cauchy in $H$.
   b) Demonstrate that the limit minimizes $E$ over $H$.

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\(^1\)There is another result called Riesz representation theorem that is about representing linear functionals on a space of continuous functions as measures.

Date: Fall 2013.
c) Denoting by $y \in H$ the minimizer, show that $\langle y, x \rangle = f(x)$ for all $x \in H$.
d) Finally, show that $y$ depends continuously on $f$.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and consider the bilinear form

$$a(u, v) = \int_{\Omega} (a_{ij} \partial_i u \partial_j v + cv),$$

where the repeated indices are summer over, and the coefficients $a_{ij}$ and $c$ are smooth functions on $\overline{\Omega}$, with $a_{ij}$ satisfying the uniform ellipticity condition

$$a_{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2, \quad \xi \in \mathbb{R}^n, \quad x \in \overline{\Omega},$$

for some constant $\lambda > 0$.

a) Show that the mapping $A : H_0^1(\Omega) \to [H_0^1(\Omega)]'$, defined by $\langle Au, v \rangle = a(u, v)$, is bounded, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $[H_0^1(\Omega)]'$ and $H_0^1(\Omega)$.
b) Show that if $c \geq 0$ then

$$\langle Au, u \rangle \geq \alpha \|u\|^2_{H_1}, \quad u \in H_0^1(\Omega),$$

for some constant $\alpha > 0$. Show also that the inequality is still true (with possibly different $\alpha > 0$) if $c$ is slightly negative.
c) Supposing that the condition in b) holds, show that given $f \in L^2(\Omega)$, there exists a unique function $u \in H_0^1(\Omega)$ satisfying $a(u, v) = \int_{\Omega} fv$ for all $v \in H_0^1(\Omega)$.
d) Suppose that $u \in H_0^1(\Omega)$ is sufficiently smooth and satisfies $a(u, v) = \int_{\Omega} fv$ for all $v \in H_0^1(\Omega)$. What differential equation does $u$ satisfy in $\Omega$? Is $u = 0$ on $\partial \Omega$? In the language of variational methods, this is an essential boundary condition because it is incorporated into the space $H_0^1(\Omega)$.

5. Let $a$ be the bilinear form as in the preceding question.

a) Show that the mapping $A : H^1(\Omega) \to [H^1(\Omega)]'$, defined by $\langle Au, v \rangle = a(u, v)$, is bounded, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $[H^1(\Omega)]'$ and $H^1(\Omega)$.
b) Show that if $c > 0$ in $\overline{\Omega}$, then

$$\langle Au, u \rangle \geq \alpha \|u\|^2_{H_1}, \quad u \in H^1(\Omega),$$

for some constant $\alpha > 0$.
c) Supposing that the condition in b) holds, show that given $f \in L^2(\Omega)$, there exists a unique function $u \in H^1(\Omega)$ satisfying $a(u, v) = \int_{\Omega} fv$ for all $v \in H^1(\Omega)$.
d) Suppose that $u \in H^1(\Omega)$ is sufficiently smooth and satisfies $a(u, v) = \int_{\Omega} fv$ for all $v \in H^1(\Omega)$. What differential equation does $u$ satisfy in $\Omega$? What boundary condition does $u$ satisfy? This is a natural boundary condition because it arises from the equation $u$ has to satisfy in the weak sense.

6. In the context of the preceding question, assuming $c \equiv 0$, prove that there exists a function $u \in H^1(\Omega)$ satisfying $a(u, v) = \int_{\Omega} fv$ for all $v \in H^1(\Omega)$ if and only if $f = 0$. Moreover, show that such a function is unique up to addition of a constant.