1. Let Ω be an open subset of \( \mathbb{R}^n \).
   (a) Show that if \( u \in C^2(\Omega) \) is harmonic in \( \Omega \) then
   \[
   \int_{\partial B} \partial_\nu u = 0,
   \]
   for any ball \( B \) whose closure is contained in \( \Omega \). Here \( \partial_\nu \) is the normal derivative.
   (b) Suppose that \( u \in C^1(\Omega) \) and that for each \( y \in \Omega \) there exists \( r^* > 0 \) such that
   \[
   \int_{\partial B_r} \partial_\nu u = 0,
   \]
   for all \( 0 < r < r^* \). Show that \( u \) is harmonic in \( \Omega \) (Böcher 1905).

2. We say \( u \in C(\Omega) \) is subharmonic in \( \Omega \) if for each \( y \in \Omega \) there exists \( r^* > 0 \) such that
   \[
   u(y) \leq \frac{1}{|B_r|} \int_{B_r(y)} u, \quad \forall r \in (0, r^*).\]
   Prove the following statements.
   (a) A function \( u \in C^2(\Omega) \) is subharmonic in \( \Omega \) iff \( \Delta u \geq 0 \) in \( \Omega \).
   (b) A function \( u \in C(\Omega) \) is subharmonic in \( \Omega \) iff for any closed ball \( B \subset \Omega \) and any
   harmonic function \( v \) in a neighbourhood of \( B \), \( u \leq v \) on \( \partial B \) implies \( u \leq v \) in \( B \).
   (c) A function subharmonic in \( \mathbb{R}^2 \) and bounded from above must be constant. Is this
   statement true in \( \mathbb{R}^n \) for \( n \geq 3 \)?

3. Prove that the function \( u \) given by the Poisson formula for the Dirichlet problem on
   a ball, say, \( B_r \), is harmonic in \( B_r \) for boundary data \( g \in L^1(\partial B_r) \), and takes correct
   boundary values wherever \( g \) is continuous.

4. (a) Find the region of \( \mathbb{R}^2 \) in which the power series \( \sum_n x_1^n x_2^n \) is absolutely convergent.
   (b) The domain of convergence of a power series is the interior of the region in which the
   series converges absolutely. Exhibit a two-variable real power series whose domain
   of convergence is the unit disk \( \mathbb{D} = \{ x \in \mathbb{R}^2 : |x| < 1 \} \).
   (c) Show that if \( \Omega \subset \mathbb{R}^2 \) is a domain of convergence of some power series centred
   at 0, then \( \Omega \) is reflection symmetric with respect to the coordinate axes, and
   \( \{(\log x_1, \log x_2) : x \in \Omega, x_1 > 0, x_2 > 0\} \) is a convex domain.

5. (Poincaré 1887) In this exercise, we will implement Poincaré’s method of sweeping out
   (méthode de balayage) to solve the Dirichlet problem. Let \( \Omega \) be a bounded domain in
   \( \mathbb{R}^n \), and let \( g \in C(\Omega) \). Suppose that \( u_0 \in C(\overline{\Omega}) \) is a function subharmonic in \( \Omega \) and

\text{Date: Fall 2013.}
$u_0 = g$ on $\partial \Omega$. The idea is to iteratively improve the initial approximation $u_0$ towards a harmonic function by solving the Dirichlet problem on a suitable sequence of balls.

(a) Show that there exist countably many open balls $B_k$ such that $\Omega = \bigcup_k B_k$.
(b) Consider the sequence $B_1, B_2, B_1, B_2, B_3, B_1, \ldots$, so that each $B_k$ is occurring infinitely many times, and let us reuse the notation $B_k$ to denote the $k$-th member of this sequence. Then we define the functions $u_1, u_2, \ldots \in C(\Omega)$ by the following recursive procedure: For $k = 1, 2, \ldots$, put $u_k = u_{k-1}$ in $\Omega \setminus B_k$, and let $u_k$ be the solution of $\Delta u_k = 0$ in $B_k$ with the boundary condition $u_{k-1}|_{\partial B_k}$. Prove that $u_k \to u$ locally uniformly in $\Omega$, for some $u \in C^\infty(\Omega)$ that is harmonic in $\Omega$.

(c) Show that if there exists $v \in C(\overline{\Omega})$ satisfying $\Delta v = 0$ in $\Omega$ and $v = g$ on $\partial \Omega$, then indeed $u = v$, where $u$ is the function we constructed in (b). So if there exists a solution, then our method would produce the same solution. However, we want to demonstrate existence without any prior assumption on existence.

(d) Prove that if there exists a barrier at $z \in \partial \Omega$, then $u(x) \to g(z)$ as $\Omega \ni x \to z$, where $u$ is the function we constructed in (b). Recall that a function $\varphi \in C(\overline{\Omega})$ is called a barrier for $\Omega$ at $z \in \partial \Omega$ if

- $\varphi$ is subharmonic in $\Omega$,
- $\varphi(z) = 0$,
- $\varphi < 0$ in $\overline{\Omega} \setminus \{z\}$.

We call the boundary point $z \in \partial \Omega$ regular if there is a barrier for $\Omega$ at $z \in \partial \Omega$.

(e) Assuming that all boundary points are regular, this procedure reduces the Dirichlet problem into the problem of constructing a subharmonic function $u_0$ with $u_0|_{\partial \Omega} = g$. Instead of constructing such $u_0$ for the given $g$ directly, let us approximate $g$ by functions for which such a construction is simpler. Show that if $\{v_j\} \subset C(\overline{\Omega})$ is a sequence with $\Delta v_j = 0$ in $\Omega$ and $v_j \to g$ uniformly on $\partial \Omega$, then there exists a function $u \in C(\overline{\Omega})$ satisfying $\Delta u = 0$ in $\Omega$ and $u = g$ on $\partial \Omega$.

(f) Show that any polynomial can be written as the difference of two subharmonic functions in $\Omega$. Hence it suffices to extend $g$ into a continuous function on $\overline{\Omega}$, and approximate the resulting function by polynomials (explain why). State what standard results we need in order to realize this.

6. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$.

(a) Show that if the Dirichlet problem in $\Omega$ is solvable for any boundary condition $g \in C(\partial \Omega)$, then each boundary point $z \in \partial \Omega$ admits a barrier.

(b) Why is regularity of a boundary point a local property? In other words, if $z \in \partial \Omega$ is regular, and if $\Omega'$ is a domain that coincides with $\Omega$ in a neighborhood of $z$ (hence in particular $z \in \partial \Omega'$), then is $z$ also regular as a point on $\partial \Omega'$?

7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in C^2(\Omega)$ satisfy

$$E_s(u) := \int_{\Omega} (|\nabla u|^2 + |u|^2) < \infty.$$ 

Prove the followings.

(a) If $\Delta u = u$ in $\Omega$, then $E_s(u + v) > E_s(u)$ for all nontrivial $v \in \mathcal{D}(\Omega)$.

(b) Conversely, if $E_s(u + v) \geq E_s(u)$ for all $v \in \mathcal{D}(\Omega)$, then $\Delta u = u$ in $\Omega$. 
