ISOMETRIC EMBEDDING OF RIEMANNIAN MANIFOLDS

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Introduction

Ever since Riemann introduces the concept of Riemann manifold, and abstract manifold with a metric structure, we want to ask if an abstract Riemann manifold is a simply a submanifold of some Euclidean space with its induced metric. This is isometric embedding question.

In this project, we use the book Isometric Embedding of Riemann Manifold into Euclidean Spaces written by Qin Han and Jiaxing Hong, and basically introduce that every smooth n-dimensional Riemannian manifold admits a global smooth isometric embedding in the Euclidean space $\mathbb{R}^N$, $N = max\{s_n + 2n, s_n + n + 5\}$, with $s_n = n(n + 1)/2$. This is first proved by John Nash in 1956 with larger N. Here we use the proof by Gunther, which simplifies Nash's original proof.

1. Fundamental Theorems

Given a smooth Riemannian manifold (M^n, g), we are interested in finding a smooth map $u : M^n \to \mathbb{R}^q$, for some positive integer q, such that

$$dudu = g. \tag{1}$$

with $u = (u^1, ..., u^q)$, this is equivalent to

$$(du^1)^2 + ... + (du^q)^2 = g. \tag{2}$$

We then call the map u an isometric imbedding or immersion according to whether u is an imbedding or an immersion. There is also a local version of the above problem in which only a sufficiently small neighbourhood of some specific point on the manifold is to be isometrically embedded in $\mathbb{R}^q$.

Now let us examine (2) closely. Suppose in some local coordinate system the metric g is given by

$$g = \Sigma^n_{i,j=1} g_{ij} dx_i dx_j \quad in \quad B_1 \subset \mathbb{R}^n \tag{3}$$

For the local isometric embedding, we need to find

$$\Sigma^q_{k=1} \partial_i u^k \partial_j u^k = g_{ij}, \quad 1 \le i \le j \le n, \quad in \quad B_1 \tag{4}$$

There are $s_n = \frac{n(n+1)}{2}$ equations in (4), and s_n is called the Janet dimension. In general, the dimension q of the target space should be bigger than or equal to s_n, i.e. $q \ge s_n$. Throughout this project, s_n will always denote the janet dimension and the dimension q

of the target space always satisfied $q \geq s_n$. There're three sections in this project. We discuss the local isometric embedding of analytic Riemannian manifolds in the first section and that of smooth Riemannian manifolds in the second section. In the last section, we discuss the global isometric embedding to smooth Riemann manifolds. The main results are the following.

Theorem 1. Any analytic n-dimensional Riemannian manifold admits on analytic local isometric embedding in $\mathbb{R}^{s_n}$.

Theorem 2. Any smooth n-dimensional Riemannian manifold admits a smooth local isometric embedding in $\mathbb{R}^{s_n+n}$.

Theorem 3. Any smooth n-dimensional Riemann compact Riemannian manifold admits a smooth isometric embedding in $\mathbb{R}^q$ for $q = max\{s_n + 2n, s_n + n + 5\}$.

We prove Theorem 1-3 by solving (1). For Theorems 1 and 2, it suffices to solve the local version (4).

2. Local Isomeric Embedding of Analytic Metric

In this section, we discuss the local isometric embedding of analytic Riemannian manifolds and prove Theorem 1 by solving (4). The proof is based on the Cauchy-Kowalevski Theorem. We rewrite the equation to apply Cauchy-Kowalevski Theorem. We first introduce a concept, which will be useful throughout this project. Let M^n be a C^2 n-dimensional manifold and let $u : M^n \to \mathbb{R}^q$ be a C^2 map. For any given point $p \in M^n$, we define the osculating space $T_p^2(u)$ by

$$T_p^2(u) = span\{\partial_i u(p), \partial_{ij} u(p); i, j = 1...n\}. \tag{5}$$

Such a definition is independent of coordinates.

Definition 1.1. The map u is free at the point $p \in M^n$ if $\dim(T_p^2(u))$=$s_n + n$, or $\partial_i u(p), \partial_{ij} u(p)$, $i, j = 1...n$, are linearly independent as vectors in $\mathbb{R}^q$. Moreover, u is a free map if u is free at each point in M^n.

It's easy to see that if $u : M^n \to \mathbb{R}^q$ is free, then $q \geq s_n + n$ and u must be an immersion. Finally, if $\phi : M^n \to N^n$ is a C^2 diffeomorphism and $u : N^n \to \mathbb{R}^q$ is free, then the composition $u\phi$ is also free. The map

$$(x_1, ..., x_n) \in \mathbb{R}^n \to (x_1, ..., x_n, \frac{1}{2}x_1^2, x_1 x_2, ..., \frac{1}{2}x_n^2) \in \mathbb{R}^{s_n+n} \tag{6}$$

gives the simplest example of a free map from $\mathbb{R}^n$ to $\mathbb{R}^{s_n+n}$. From (6) and the local charts of manifolds, it is easy to see that every C^2 differential manifold M^n has a local free map into $\mathbb{R}^{s_n+n}$. The differential system (4) is highly degenerate. We will transform it to an equivalent differential system which is easier to analyze. We first express the metric in a special form, adopting the notation $x = (x', x_n) = (x_1, ..., x_n)$.

Lemma 1.2. Let (M^n, g) be a smooth n-dimensional Riemannian manifold. Then for anp $p \in M^n$, there exists a local coordinate system $(x_1, ..., x_n)$ in a neighbourhood $N(p)$ of p where g is of the form

$$g = \Sigma_{k,l=1}^{n-1} g_{kl}(x', x_n) dx_k dx_l + dx_n^2, \tag{7}$$

with

$$g_{kl}(0) = \delta_{kl}, \partial_n g_{kl}(0) = 0 \quad for \quad any k, l = 1, ..., n-1 \tag{8}$$

Proof. We start with a normal coordinate system $(x_1, ..., x_n)$ centered at p, and let $M^{n-1} = \{x_n = 0\}$ and e be the unit normal field slong M^{n-1} in M^n. For any $q \in M^{n-1}$, consider the geodesic $c = c(t)$ in M^n with the initial conditions $c(0) = q$ and $c'(0) = e(q)$. Then $(x_1, ..., x_{n-1}, t)$ forms a local coordinate system in a neighborhood of p. First, we note $g(\partial_t, \partial_t) = 1$ since each t- curve is an are-length parametrized geodesic. Next, we have for any $k = 1, ..., n-1$

$$\partial_t g(\partial_t, \partial_k) = g(\nabla_t \partial_t, \partial_k) + g(\partial_t, \nabla_t \partial_k) = g(\partial_t, \nabla_k \partial_t) = \frac{1}{2}\partial_k g(\partial_t, \partial_t) = 0$$

Hence $g(\partial_t, \partial_k) = 0$ since it is zero at $t = 0$. Therefore, the metric g is of the form (7) in the coordinate $(x_1, ..., x_{n-1}, t)$. To prove (8), we have for any $k, l = 1, ..., n-1$

(9)
$$\partial_t g(\partial_k, \partial_l) = g(\nabla_t \partial_k, \partial_l) + g(\partial_k, \nabla_t \partial_l) = g(\nabla_k \partial_t, \partial_l) + g(\partial_k, \nabla_l \partial_t) = \partial_k g(\partial_t, \partial_l) + \partial_l g(\partial_k, \partial_t) - g(\partial_t, \nabla_k \partial$$

Since $(x_1, ..., x_{n-1}, t)$ are normal coordinates at p, we have $\nabla_k \partial_l(0) = 0$ for all k and l. Hence we have $\partial_t g(\partial_k, \partial_l) = 0$ at the origin.

Remark 1.3. If $n = 2$, we may simply take M^1 as a geodesic parameterized by the arc-length parameter x_1. Then g is of the form

$$g = g_{11}(x_1, x_2)dx_1^2 + dx_2^2,$$

where

$$g_{11}(x_1, 0) = 1, \partial_2 g_{11}(x_1, 0) = 0.$$

Suppose g is a smooth metric given by (7) in $\mathbb{B}^n$. In order to construct a smooth isometric imbedding of g in $\mathbb{R}^q$, we need to find a smooth map $u : \mathbb{B}^n \to \mathbb{R}^q$ satisfying

$$\partial_k u \partial_n u = 0 \tag{10}$$

$$\partial_n u \partial_n u = 1 \tag{11}$$

$$\partial_k u \partial_l u = g_{kl}, \tag{12}$$

in $\mathbb{B}^n$ for any $k, l = 1, ..., n-1$. Before proceeding, we derive some identities. By a straightforward calculation, we have

$$\partial_n(\partial_k u \partial_l u) = \partial_{kn} u \partial_l u + \partial_k u \partial_{ln} u = \partial_k(\partial_l u \partial_n u) + \partial_l(\partial_k u \partial_n u) - 2\partial_{kl} u \partial_n u, \tag{13}$$

and

$$\partial_{nn}(\partial_k u \partial_l u) = 2\partial_{kn} u \partial_{ln} u + \partial_{knn} u \partial_l u + \partial_k u \partial_{lnn} u = 2\partial_{kn} u \partial_{ln} u - 2\partial_{kl} u \partial_{nn} u + \partial_k(\partial_l u \partial_{nn} u) + \partial_l(\partial_k u \partial_{nn} u) \tag{14}$$

By differentiating (10)-(12) with respect to x_n, we get for any $k, l = 1, ..., n-1$

$$\partial_k u \partial_{nn} u = 0 \tag{15}$$

$$\partial_n u \partial_{nn} u = 0 \tag{16}$$

$$\partial_{kl} u \partial_{nn} u = \partial_{kn} u \partial_{ln} u - \frac{1}{2}\partial_{nn} g_{kl}. \tag{17}$$

We call (15)-(17) the Janet system. There are s_n equations in this system, and , as we know, we should require that $q \geq s_n$. Now we prescribe Cauchy data for (15)-(17) as follows:

$$u|_{x_n=0} = u_0, \partial_n u|_{x_n=0} = u_1. \tag{18}$$

By (10)-(13), u_0 and u_1 satisfy in $\mathbb{B}^{n-1}$

$$\partial_k u_0 \partial_l u_0 = g_{kl}(, 0), \tag{19}$$

$$\partial_k u_0 u_1 = 0, \tag{20}$$

$$\partial_{kl} u_0 u_1 = -\frac{1}{2}\partial_n g_{kl}(,0), \tag{21}$$

$$u_1 u_1 = 1 \tag{22}$$

We want to show that the Janet system (15)-(22) is equivalent to (10)-(12). It's clear that we can derive (15)-(22) from (10)-(12), just the way we did. Now, we want to derive (10)-(12) from (15)-(22). Suppose u is a C^3 solution to the Cauchy problem of the Janet system (15)-(22). Then (16) together with (22), immediately implies (11). In a similar way, (10) follows from (15) and (20). To prove (12), we first note (19) implies $g_{lk} - \partial_l u \partial_k u = 0$ at $x_n = 0$. Considering the initial condition (20) and (21), we have, by (13),

$$\partial_n(g_{kl} - \partial_k u \partial_l u) = -2\partial_{kl} u_0 u_1 + 2\partial_{kl} u_0 u_1 - \partial_k(u_1 \partial_l u_0) - \partial_l(u_1 \partial_k u_0) = 0$$

at $x_n = 0$. Next by (14),(17),(15), we have

$$\partial_{nn}(g_{kl} - \partial_k u \partial_l u) = -\partial_k(\partial_l u \partial_{nn} u) - \partial_l(\partial_k u \partial_{nn} u) = 0$$

Hence (12) is valid. Therefore, we have proved the equivalence of them.

If $\partial_k u, \partial_{kl} u, \partial_n u$, $1 \le k, l \le n-1$, are linearly independent, we can solve $\partial_{nn} u$ from (15)-(17) to get

$$\partial_{nn} u = F(x, \partial_k u, \partial_n u, \partial_{kl} u, \partial_{kn} u) near x_n = 0, \tag{23}$$

where F is smooth in x and analytic in other arguments and k, l run over $1, ..., n-1$. Moreover, F is analytic in x if g ia an analytic metric. Note that there are s_n equations in (15)-(17). If $q = s_n$, then ∂_{nn} can be solved uniquely, for $q \ge s_n$, solutions may not be unique. If g is an analytic metric, the Cauchy-Kowalevski theorem implies that (23) always admits an analytic solution in a neithborhood of the origin with the given Cauchy data (18). Hence we have the following result.

Lemma 1.4. Let g be an analytic metric of the form (7) in $\mathbb{B}^n \subset \mathbb{R}^n$ and $q \ge s_n$. Suppose that there exist analytic functions $u_0, u_1 : \mathbb{B}^{n-1} \to \mathbb{R}^q$ satisfying (19)-(22) and that $\partial_k u_0, \partial_{kl} u_0, u_1$, $1 \le k, l \le n-1$, are linearly independent in $\mathbb{B}^{n-1}$. Then g, restricted to a neighborhood of the origin $0 \in \mathbb{B}^n$, admits an analytic isometric embedding in $\mathbb{R}^q$.

Sometimes we are interested in free isometric embeddings.

Lemma 1.5. Let g be an analytic metric of the form (7) in $\mathbb{B}^n \subset \mathbb{R}^n$ and $q \geq s_n$. Suppose that there exist analytic functions $u_0, u_1 : \mathbb{B}^{n-1} \to \mathbb{R}^q$ satisfying (19)-(22) and that $\partial_k u_0, \partial_{kl} u_0, u_1, \partial_k u_1$, $1 \leq k, l \leq n-1$, are linearly independent in $\mathbb{B}^{n-1}$. Then g, restricted to a neighborhood of the origin $0 \in \mathbb{B}^n$, admits an analytic isometric embedding in $\mathbb{R}^q$.

Proof. We look for a free isometric embedding $u : \mathbb{B}^n \to \mathbb{R}^q$ of the following form:

$$u(x) = (\tilde{u}(x), \frac{1}{2}x_n^2). \tag{24}$$

By (15)-(17), $\tilde{u}$ satisfies for $1 \leq k, l \leq n-1$

$$\partial_k \tilde{u} \partial_{nn} \tilde{u} = 0 \partial_n \tilde{u} \partial_{nn} \tilde{u} = -x_n \partial_{kl} \tilde{u} \partial_{nn} \tilde{u} = \partial_{kn} \tilde{u} \partial_{ln} \tilde{u} - \frac{1}{2} \partial_{nn} g_{kl} \tag{25}$$

By the Cauchy-Kowalevski theorem, there exists an analytic solution $\tilde{u}$ in a neighborhood of the origin $0 \in \mathbb{B}^n$ with the Cauchy data

$$\tilde{u}|_{x_n=0} = u_0, \partial_n \tilde{u}|_{x_n=0} = u_1.$$

Then by the derivation of (25), the map u solves the Janet system (15)-(17) and satisfied (18)-(22). Hence u is an analytic isometric embedding for g. Now we need to prove only that u is free in a neighborhood of $x_n = 0$. By the assumption, $\partial_k u, \partial_n u, \partial_{kl} u, \partial_{kn} u, 1 \leq k, l \leq n-1$, are liniarly independent at $x_n = 0$. By (24), the last components of all these vectors are 0 at $x_n = 0$ and $\partial_{nn} u = (..., 1)$. Hence $\partial_k u, \partial_n u, \partial_{kl} u, \partial_{kn} u, \partial_{nn} u, l \leq n-1$ are linearly independent for $x_n = 0$.

The following is the main result in this section.

Theorem 1.6. Any n-dimensional analytic Riemannian manifold admits a local analytic isometric embedding in $\mathbb{R}^{s_n}$ and a local free analytic isometric embedding in $\mathbb{R}^{s_n+n}$.

Obviously, Theorem 1 is a part of Theorem 1.6. Proof. We prove Theorem 1.6 by an induction on n. We first consider $n = 2$. Suppose g is given in a local coordinate system by

$$g = g_{11}(x_1, x_2) dx_1^2 + dx_2^2 \quad in \mathbb{B}^2,$$

where g_{11} satisfies $g_{11}(x_1, 0) = 1$ and $\partial_2 g_{11}(x_1, 0) = 0$. This can always be arranged by Remark 1.3. Now let

$$u_0 = (\cos x_1, \sin x_1, 0), u_1 = (0, 0, 1). \tag{26}$$

Then u_0, u_1 satisfy (19)-(22) and $\partial_1 u_0, \partial_{11} u_0, u_1$ are linearly independent in $\mathbb{B}^1$. We may apply Lemma 1.4 to get a local analytic isometric embedding of g in $\mathbb{R}^3$. Now we prove the existence of a local free analytic isometric embedding for $n = 2$. In this case, $s_n + n = 5$. Set

$$u_0^* = (\cos x_1, \sin x_1, 0, 0), u_1^* = (0, 0, \cos x_1, \sin x_1). \tag{27}$$

Obviously, u_0^*, u_1^* still satisfy (19)-(22) and $\partial_1 u_0^*, \partial_{11} u_0^*, u_1^*, \partial_1 u_1^*$ are linearly independent. We may apply Lemma 1.5 to get a free local analytic isometric embedding of g in $\mathbb{R}^5$. Now let us assume Theorem 1.6 holds for $n-1$ for some integer $n \geq 3$. We consider the case n. By Lemma 1.2, we assume that the metric g is of the form

$$g = \tilde{g} + dx_n^2 = \Sigma_{k,l=1}^{n-1} g_{kl}(x', x_n) dx_k dx_l + dx_n^2 \quad in \mathbb{B}^n, \tag{28}$$

with $\partial_n g_{kl}(0) = 0$ for $1 \leq k, l \leq n-1$. Note $s_n = s_{n-1} + n$ and $\tilde{g}(,0)$ is an analytic metric in $\mathbb{B}^{n-1}$. By the induction hypothesis, we may assume there exists a free analytic map $f : \mathbb{B}^{n-1} \to \mathbb{R}^{s_n}$ satisfying $df df = \tilde{g}(,0)$. To prove the existence of a local analytic isometric embedding in $\mathbb{R}^{s_n}$, we consider $u_0 = (f, 0)$ and $\tilde{u_1} = (0, ..., 0, 1) \in \mathbb{R}^{s_n}$. It is easy to see that

$$\partial_k u_0 \tilde{u_1} = 0, \partial_{kl} u_0 \tilde{u_1} = 0, \tilde{u_1}\tilde{u_1} = 1.$$

To ensure that u_0 and u_1 satisfy (19)-(22), we take $u_1 = \tilde{u_1} + \delta$, where $\delta : \mathbb{B}^{n-1} \to \mathbb{R}^{s_n}$ is a perturbation of 0 satisfying

$$\partial_k u_0 \delta = 0, \partial_{kl} u_0 \delta = -\frac{1}{2}\partial_n g_{kl}, \tilde{u_1}\delta = -\frac{1}{2}\delta\delta. \tag{29}$$

Note that $\partial_n g_{kl}(0) = 0$. By the implicit function theorem, we can get an analytic solution δ to (29) near $x' = 0$. Obviously, $\partial_k u_0, \partial_{kl} u_0, u_1, 1 \leq k, l \leq n-1$, are linearly independent. We may apply Lemma 1.4 to get a local analytic isometric embedding of g in $\mathbb{R}^{s_n}$. Next we prove the existence of a local free analytic isometric embedding in $\mathbb{R}^{s_n+n}$. Let $u_0^* = (f, 0, ..., 0) : \mathbb{B}^{n-1} \to \mathbb{R}^{s_n+n-1}$ with all the last n components zero, and take $\tilde{u_1^*} = (0, e)$ for some smooth vector $e : \mathbb{B}^{n-1} \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then $u_0^*, \tilde{u_1^*}$ satisfy

$$\partial_k u_0^* \tilde{u_1^*} = 0, \partial_{kl} u_0^* \tilde{u_1^*} = 0, \tilde{u_1^*}\tilde{u_1^*} = 1.$$

We may choose e such that $\partial_k u_0^*, \partial_{kl} u_0^*, \tilde{u_1^*}, \partial_k \tilde{u_1^*}, 1 \leq k, l, \leq n-1$, are linearly independent. Analogously, by the implicit function theorem we can select an analytic map $u_1^* = \tilde{u_1^*} + \delta^*$ where $\delta^* : \mathbb{B}^{n-1} \to \mathbb{R}^{s_n+n-1}$ satisfies

$$\partial_k u_0^* \delta^* = 0, \partial_{kl} u_0^* \delta^* = -\frac{1}{2}\partial_n g_{kl}, \tilde{u_1^*}\delta^* = -\frac{1}{2}\delta^*\delta^*.$$

Therefore, $u_0^*, u_1^* : \mathbb{B}^{n-1} \to \mathbb{R}^{s_n+n-1}$ satisfy (19)-(22), and $\partial_k u_0^*, \partial_{kl} u_0^*, u_1^*, \partial_k u_1^*, 1 \leq k, l \leq n-1$, are linearly independent in $\mathbb{B}^{n-1}$. Now we may apply Lemma 1.5 to get a free local analytic isometric embedding of g in $\mathbb{R}^{s_n+n}$.

3. Local Isometric Embedding of Smooth Metrics

Turning to the smooth case from the analytic case, the seential difficulty we encounter is finding a technique to replace the Cauchy-Kowalevski theorem. As far as the existence of solutions to partial differential equations is concerned, there are essential differences between the analytic case and the smooth case. As is well known, even a linear partial differential equation with smooth coefficients may have no local solutions. The crucial step here is to solve a perturbation problem for the isometric embedding. The nonlinear partial differential equation for such a perturbed problem exhibits a loss of differentiability. By applying the Laplacian operator, rearranging the equation and then applying the inverse of the Laplacian operator, we may transfrom this equation into an equivalent one which is essentially elliptic. Then there is no loss of differentiability and we may solve it by a standard iteration.

We begin our discussion with the following question: If a given metric is isometrically embedded, can we isometrically embed nearby metrics? Let $u \in C^\infty(\mathbb{B}^n, \mathbb{R}^{s_n+n})$ be a smooth map and $h = h_{ij}dx^i dx_j$ be a small smooth quadratic differential form in $\mathbb{B}^n$. We look for a small map $v \in C^\infty(\mathbb{B}^n, \mathbb{R}^{s_n+n})$, such that

$$d(u+v)d(u+v) = dudu + h. \tag{30}$$

To this end, we need to solve the following differential system for v:

$$\partial_i u \partial_j v + \partial_i v \partial_j u + \partial_i v \partial_j v = h_{ij}, 1 \le i, j \le n \tag{31}$$

To proceed, we rewrite (31) as

$$\partial_j(\partial_i uv) + \partial_i(\partial_j uv) - 2\partial_{ij}uv = h_{ij} - \partial_i v \partial_j v, 1 \le i, j \le n \tag{32}$$

In place of (32), we consider a new system

$$\partial_i uv = 0, i = 1, ..., n \quad \partial_{ij}uv = -\frac{1}{2}h_{ij} + \frac{1}{2}\partial_i v \partial_j v, i, j = 1, ..., n. \tag{33}$$

The left-hand side of (33) is a linear algebraic system. If u is free, we can rewrite (33) as

$$v = E(u)F(h, v), \tag{34}$$

where F is given by

$$F(h, v) = (0, ..., 0, ..., -\frac{1}{2}h_{ij} + \frac{1}{2}\partial_i v \partial_j v, ...)^T \in \mathbb{R}^{s_n+n}.$$

When we attempt to solve (34) by an iteration, there is a loss of order one derivative at each step. This is obvious since only v itself appears at the left-hand side, while first order derivatives of v appear at the right-hand side. Hence, we cannot prove the convergence of the sequence generated in the iteration. In order to overcome this difficulty, we rewrite the system (33). As we have observed, the difficulty arises from the nonlinear terms $\partial_i v \partial_j v$. Applying the Laplacian operator yields

$$\Delta(\partial_i v \partial_j v) = \partial_i(\Delta v \partial_j v) + \partial_j(\partial_i v \Delta v) + 2\partial_{il} v \partial_{jl} v - 2\Delta v \partial_{ij} v$$

If we could invert the Laplacian operator Δ, we could then have

$$\partial_i v \partial_j v = \Delta^{-1}\partial_i(\Delta v \partial_j v) + \Delta^{-1} partial_j(\partial_i v \Delta v) + 2\Delta^{-1}(\partial_{il} v \partial_{jl} v - \Delta v \partial_{ij} v) \tag{35}$$

Instead of (33), now (32) becomes the following system

(36)
$$\partial_i uv = -\Delta^{-1}(\Delta v \partial_i v), i = 1, ..., n \quad \partial_{ij} uv = -\frac{1}{2}h_{ij} + \Delta^{-1}(\partial_{il} v \partial_{jl} v - \Delta v \partial_{ij} v), i, j = 1, ..., n.$$

Here is an important observation. The expression for v on the left-hand side of (36) has the same regularity as the expression for v on the right-hand side, by the regularity theory for elliptic equations. To be specific, if v is $C^{m,\alpha}$ for some $m \geq 2$ and $0 < \alpha < 1$, then the right side expressions for v are also $C^{m,\alpha}$. There's no loss of derivatives. Hence, if h_{ij} is small, we may solve (36) by the contraction mapping principle. When we try to make the above procedure rigorous, a series of difficulties arises in inverting Δ to get (35), since the boundary values involved. To remedy this, we try to use cutoff functions. We assume $u + a^2 v$ is a solution of (30), then like the calculation before, we get

$$\partial_i uv = N_i(v, a), i = 1, ..., n \quad \partial_{ij} uv = -\frac{1}{2}h_{ij} + M_{ij}(v, a), i, j = 1, ..., n. \tag{37}$$

where

$$N_i(v, a) = -a\Delta^{-1}(a\Delta v \partial_i v) - a\partial_i a |v|^2, \tag{38}$$

(39)
$$M_{ij}(v, a) = \frac{1}{2}a\Delta^{-1} r_{ij}(v, a) - (a\partial_{ij} a + \partial_i a \partial_j a)|v|^2 - \frac{3}{2}(\partial_i a \Delta^{-1}(a\Delta v \partial_j v) + \partial_j a \Delta^{-1}(a\Delta v \partial_i v)),$$

$$r_{ij}(v, a) = \Delta a \partial_i v \partial_j v - \partial_i a \Delta v \partial_j v - \partial_j a \partial_i v \Delta v + 2\partial_l a \partial_l(\partial_i v \partial_j v) + 2a(\partial_{il} v \partial_{jl} v - \Delta v \partial_{ij} v).$$

Let $v : \bar{\mathbb{B}}^n \to \mathbb{R}^{s_n + n}$ be a solution of (38)-(39) in $\mathbb{B}^n$. Obviously, v satisfies the isometrically embedding. We assume $u : \bar{\mathbb{B}}^n \to \mathbb{R}^{s_n + n}$ is free and write (38)-(39) as

$$v = E(u)F(v, h), \tag{40}$$

where

$$E(u) = ((\partial_i u, \partial_{ij} u)^T)^{-1}$$
$$F(v, h) = (N_i(v, a), -\frac{1}{2}h_{ij} + M_{ij}(v, a))$$

We should point out that $E(u)$ involves first order and second order derivatives of u, not u itself.

Theorem 2.1. Let $u \in C^{2,\alpha}(\bar{\mathbb{B}}^n, \mathbb{R}^{s_n+n})$ be a free map and h be a $C^{2,\alpha}$ quadratic differential form in $\bar{\mathbb{B}}^n$, $0 < \alpha < 1$. For any $a \in C_c^\infty(\mathbb{B}^n)$, there's a positive constant θ_* depending only on $|\alpha|_{4,\alpha}$, such that if

$$\theta(u,h) \equiv |E(u)|_{2,\alpha}|E(u)F(0,h)|_{2,\alpha} \leq \theta_*, \tag{41}$$

there exists a map $v \in C^{2,\alpha}(\bar{\mathbb{B}}^n, \mathbb{R}^{s_n+n})$ satisfying (38)-(39) and

$$|v|_{2,\alpha} \leq C|E(u)F(0,h)|_{2,\alpha}, \tag{42}$$

where C is a positive constant independent of u and v. Moreover, $v \in C^{m,\alpha}(\mathbb{B}^n)$ or C^∞ if $u, h \in C^{m,\alpha}(\mathbb{B}^n)$ or C^∞, for some $m \geq 3$.

Proof. We need to solve (38)-(39), or (40), for v. Letting $v \equiv \mu w$, we rewrite (40) as

$$w = \mu E(u)F(w,0) + \frac{1}{\mu}E(u)F(0,h). \tag{43}$$

Set

$$\Sigma = \{w \in C^{2,\alpha}(\bar{\mathbb{B}}^n, \mathbb{R}^{s_n+n}); |w|_{2,\alpha} \leq 1\},$$

and

$$Tw = \mu E(u)F(w,0) + \frac{1}{\mu}E(u)F(0,h)$$

In the following, we take

$$\mu = (\frac{|E(u)F(0,h)|_{2,\alpha}}{|E(u)|_{2,\alpha}})^{\frac{1}{2}}.$$

We claim that $T : \Sigma \to \Sigma$ is a contraction mapping if $\theta = \theta(u,h)$ is small. First, we have for any $w \in \Sigma$

(44)

$$|Tw|_{2,\alpha} \leq \mu|E(u)|_{2,\alpha}|F(w,0)|_{2,\alpha} + \frac{1}{\mu|E(u)F(0,h)|_{2,\alpha}} = \sqrt{\theta}(|F(w,0)|_{2,\alpha}+1) = \sqrt{\theta}C_1$$

where C_1 is a positive constant depending only on $|a|_{4,\alpha}$, and θ is as in (41). Hence, T maps Σ into Σ if $\theta \leq \frac{1}{C_1^2}$. Next, we note that $F(w,0)$ is quadratic in w. Then we have for $w_1, w_2 \in \Sigma$

$$|Tw_1 - Tw_2|_{2,\alpha} \leq \mu|E(u)|_{2,\alpha}|F(w_1,0) - F(w_2,0)|_{2,\alpha} \leq \sqrt{\theta}C_2|w_1 - w_2|_{2,\alpha}, \tag{45}$$

where C_2 is a positive constant depending only on $|a|_{4,\alpha}$. Therefore, T is a contraction if $\theta \le \frac{1}{C_1^2+C_2^2}$. By the contraction mapping principle, we have a fixed point of T in Σ. To prove (42), we rewrite (43) as

$$v = \mu^2 E(u)F(w,0) + E(u)F(0,h)$$

Then (42) follows easily from the definition of μ and the boundedness of $F(w,0)$ in $C^{2,\alpha}$-norm. The higher regularity is a consequence of regularity result for elliptic differential equations. By setting $w = E(u)W$ and applying Δ to (43), we obtain

$$\Delta W - \mu\Delta(F(w,0)) = \frac{1}{\mu}\Delta F(0,h). \tag{46}$$

Note that $\Delta(F(w,0))$ can be considered as a bilinear form in the derivatives of w up to the second order, with coefficients given by derivatives of a up to the second order. Then the linearized operator associated with (46) is given by

$$\Delta\tilde{W} + \Sigma_{|\beta|=2}\mu A^\beta(w)E(u)\partial_\beta\tilde{W} + lower order terms, \tag{47}$$

where $A^\beta(w)$ is a matrix whose entries can be viewed as linear combinations of derivatives of w up to the second order, with coefficients involving derivatives of a up to the second order. in view of $|w|_{2,\alpha} \le 1$ and the expressions of $N_i(w,a)$ and $M_{ij}(w,a)$, we have

$$\Sigma_{|\beta|=2}|\mu A^\beta(w)E(u)| \le \mu C_3|E(u)|_{2,\alpha} \le C_3\sqrt{\theta} \le \frac{1}{4},$$

if

$$\theta \le \theta_* = \frac{1}{C_1^2 + C_2^2 + 16C_3^2}. \tag{48}$$

If (48) holds, then (47), and therefore (46), are elliptic differential systems. Then, the (interior) regularity of W an dthus that of w follow from the regularity theory for non-linear elliptic systems.

Remark 2.2. For a fixed free map u as in Theorem 2.1, we may conclude $u + a^w v$ is freeif $|h_{2,\alpha}|$ is small depending also on u. This can be seen easily from (42).

Remark 2.3. For any smooth function a in $\mathbb{B}^n$ with compact support, let $v \in C^{2,\alpha}(\bar{\mathbb{B}}^n, \mathbb{R}^{s_n+n})$ be the solution as in Theorem 2.1. Then, v satisfies isometric embedding.

Now we state the main result of this section.

Theorem 2.4. Any smooth n-dimensional Riemannian manifold admits a smooth local free isometric embedding in $\mathbb{R}^{s_n+n}$.

Note that Theorem 2.4 implies Theorem 2 easily.

Proof. We prove Theorem 2.4 by an induction on n. We start with the assumption that it holds for $n-1, n \geq 3$, since the proof for $n=2$ can be formulated easily from the following argument. We divide the proof into two steps. Step 1. Suppose the metric g is of the form (28) in $\mathbb{B}^{n-1} \times (-1,1)$. We claim that, for any $m \in \mathbb{Z}^+$, we may find a free map $u_m : \mathbb{B}^{n-1} \times (-1,1) \to \mathbb{R}^{s_n+n}$ such that

$$g = du_m du_m + x_n^m h^{(m)}(x) in \mathbb{B}^{n-1} \times (-1,1) \tag{49}$$

for some smooth quadratic differential form $h^{(m)}$. In an argument similar to the proof for the second part of Theorem 1.6, we can find smooth maps $f_0, f_1 : \mathbb{B}^{n-1} \to \mathbb{R}^{s_n+n}$ such that f_0 and f_1 satisfy (19)-(22) and $\partial_k f_0, \partial_{kl} f_0, f_1, \partial_k f_1, 1 \leq k, l \leq n-1$, are linearly independent. Here we used the induction hypothesis. If $n=2$, f_0, f_1 were constructed in the proof of Theorem 1.6. Now we prove inductively that for any integer $m \geq 1$ there exists a smooth map $u_m : \mathbb{B}^{n-1} \times (-1,1) \to \mathbb{R}^{s_n+n}$ such that the following hold for $k, l = 1, ..., n-1$:

$$g_{kl} = \partial_k u_m \partial_l u_m + O(x_n^{m+1}),$$

$$g_{kn} = \partial_k u_m \partial_n u_m + O(x_n^m),$$

$$g_{nn} = \partial_n u_m \partial_n u_m + O(x_n^m).$$

By taking $u_m = (w_m, \frac{1}{2}x_n^2)$, we need to find a smooth map $w_m : \mathbb{B}^{n-1} \times (-1,1) \to \mathbb{R}^{s_n+n-1}$ for which the following hold for $k, l = 1, ..., n-1$:

(50)

$$g_{kl} = \partial_k w_m \partial_l w_m + O(x_n^{m+1}), g_{kn} = \partial_k w_m \partial_n w_m + O(x_n^m), \tilde{g}_{nn} = \partial_n w_m \partial_n w_m + O(x_n^m).$$

where $\tilde{g}_{nn} = g_{nn} - x_n^2$. We prove (50) by induction on m. For $m=1$, we set $w_1(x', x_n) = f_0(x') + f_1(x') x_n$. It's easy to check that w_1 satisfies (50) by (19)-(22). Suppose (50) holds for some $m \geq 1$. We write (50) as

(51)

$$g_{kl} = \partial_k w_m \partial_l w_m + g_{kl}^{(m)} x_n^{m+1} + O(x_n^{m+2}), g_{kn} = \partial_k w_m \partial_n w_m + g_{kn}^{(m)} x_n^m + O(x_n^{m+1}), \tilde{g}_{nn} = \partial_n w_m \partial_n w_m + g_{nn}^{(m)}$$

where $g_{ij}^{(m)}$ are smooth functions in $\mathbb{B}^{n-1}$. We set

$$w_{m+1}(x', x_n) = w_m(x', x_n) + f_{m+1}(x') x_n^{m+1}.$$

From a straight forward calculation, we require

$$\partial_k f_0 f_{m+1} = \frac{1}{m+1} g_{kn}^{(m)}, \partial_{kl} f_0 f_{m+1} = -\frac{1}{2} g_{kl}^{(m)} - \frac{1}{2(m+1)} \partial_k g_{ln}^{(m)} - \frac{1}{2(m+1)} \partial_l g_{kn}^{(m)}, f_1 f_{m+1} = \frac{1}{m+1} g_{nn}^{(m)}.$$

This is an algebraic linear system for f_{m+1}. We may solve for f_{m+1} since the coefficients are linearly independent. Therefore, w_{m+1} satisfies (50). In the following, we simply write u instead of u_m for a fixed m to be determined. Obviously, u is free, at least in a neighborhood of the origin.

Step 2. Set $x = \epsilon y$ and $u_\epsilon(y) = u(\epsilon y)$. In the y-coordinate, we may write (49) as

$$g = du_\epsilon du_\epsilon + \tilde{h},$$

where $\tilde{h}(y) = \epsilon^m y_n^m h^{(m)}(\epsilon y)$. Now we seek a solution $\tilde{u}$ with the form

$$\tilde{u}(y) = u_\epsilon(y) + a^2(y)v(y) = u(\epsilon y) + a^2(y)v(y)$$

where $a \in C_0^\infty(\mathbb{B}^n)$ with $a = 1$ on $\mathbb{B}^n_{3/4}$. Then (38)-(39) takes the following form:

$$\epsilon^2 \partial_i u(\epsilon y) v = \epsilon N_i(v, a), \tag{52}$$

$$\epsilon^2 \partial_{ij} u(\epsilon y) v = -\frac{1}{2}\tilde{h}(y) + M_{ij}(v, a). \tag{53}$$

In this case, we have

$$|E(u_\epsilon)|_{2,\alpha} |E(u_\epsilon) F(0, \tilde{h})|_{2,\alpha} \leq C_m \epsilon^{m-4},$$

for some constant C_m independent of ϵ. Therefore, if $m = 5$ and ϵ is small, the requirement fulfilled. By Theorem 2.1, there exists a smooth solution $v : \mathbb{B}^n \to \mathbb{R}^{s_n+n}$ to (52)-(53). By Remark 2.2 and 2.3, we get $dwdw = g$.

4. Global Isometric Embedding of Smooth metrics

In this section, we discuss the global isometric embedding of smooth Riemannian manifolds in Euclidean spaces. We divide our discussion into two parts. First, we prove that any n-dimensional smooth Riemannian manifold admits a global smooth isometric embedding in some Euclidean space. Second, we seek to determine the lowest dimension of the target space.

The discussion of global isometric embedding consists of two steps. First, for any Riemannian metric g on a compact manifold M^n, we find an embedding $u_0 : M^n \to \mathbb{R}^q$ such that $g - du_0 du_0$ is also a metric. Second, we modify u_0 to get u to satisfy $dudu = f$. During this process, we need to be sure that u remains an embedding. The technique developed in the previous section plays an important role. The starting point of the second step is to localize the problem so that we need to modify u_0 in each local chart is very similar to that in the proof of Theorem 2.4 of the previous section. Both steps are quite easy if we are content with finding an isometric embedding in some Euclidean space. However, the second step becomes extremely complicated when we attempt to find the lowest target dimension.

First of all, we localize the global problem.

Lemma 3.1. Let (M^n, g) be a compact smooth n-dimensional Riemannian manifold. Then there's a (finite) open covering $\{U_j\}$ and smooth functions $\phi_j \in C_c^\infty(U_j)$ and $\zeta_j \in C^\infty(U_j)$ such that

$$g = \Sigma\phi_j^2 d\zeta_j^2. \tag{54}$$

Proof. For any $p \in M^n$, there is a local chart (V_p, η_p), where $\eta_p : V_p \to \mathbb{B}^n$ is a diffeomorphism. By Theorem 2.4, there exists a neighborhood N of $\eta_p(p)$ in $\mathbb{B}^n$ such that

$$(\eta_p^{-1})^* g = \Sigma_{j=1}^{s_n+n} dx_j^2 \quad inN,$$

for some smooth function $x_1, ..., x_{s_n+n}$ on N. Hence we have

$$g = \eta_p^*(\eta_p^{-1})^* g = \Sigma_{j=1}^{s_n+n} d(x_j\eta_p)^2 \quad inU_p,$$

where $U_p = \eta_p^{-1}(N) \subset V_p$ is a neighborhood of p. Since M_n is compact, we can complete the proof by using the partitions of unity.

Remark 3.2. Let (M^n, g) be a precompact smooth n-dimensional Riemannian manifold. Then (54) still holds. Moreover, (54) can be written in the form

$$g = \Sigma\phi_j^4 d\zeta_j^2.$$

This is useful for later discussions.

Lemma 3.3. Let (M^n, g) be a compact smooth Riemannian manifold. Then there exists a smooth free embedding $u : M^n \to \mathbb{R}^{s_n+2n}$ such that $g - dudu$ is a smooth Riemannian metric in M^n.

Proof. It suffices to prove that there is a smooth free embedding of M^n into $\mathbb{R}^{s_n+n}$. In fact, if u is such a map, by the compactness of M^n, it follows that $g - c^2dudu$ is positive definite on M^n if c is sufficiently small.

The proof consists of two parts. First, we prove that there is a smooth free embedding from M^n into $\mathbb{R}^q$ for some q. Second, we prove that the existence of a smooth free embedding from M^n into $\mathbb{R}^q$ for $q > s_n + 2n$ implies the existence of such a map into $\mathbb{R}^{q-1}$.

By Whitney's embedding theorem, there exists a smooth embedding $\phi : M^n \to \mathbb{R}^{2n}$. Set $\sigma : \mathbb{R}^{2n} \to \mathbb{R}^{s_n+2n}$ by

$$\sigma(x_1, ..., x_{2n}) = \{x_1, ..., x_{2n}, x_1^2, x_1x_2, ..., x_{2n}^2\}.$$

Obviously σ is free, and hence $\sigma\phi$ is a smooth free embedding of M^n into $\mathbb{R}^q$ with $q = s_{2n} + 2n$.

Next, we assume ϕ is a smooth free embedding into $\mathbb{R}^q$ with $q > s_n + 2n$. We denote by $\Sigma(\phi(p))$ the unit sphere of the osculating space $T_p^2(\phi) \subset T_*(\mathbb{R}^q)$ and by Σ the smooth submanifold of $M^n \times \mathbb{R}^q$ consisting of $\{(p, \Sigma(\phi(p))); p \in M^n\}$. Denote by P the smooth map : $(p, e) \in \Sigma \to e \in \mathbb{S}^{q-1}$. Since $\dim\{\Sigma\} = s_n + 2n - 1 < q - 1$, it follows that $P\{\Sigma\}$ is of measure zero in $\mathbb{S}^{q-1}$. Hence $\mathbb{S}^{q-1} \setminus P\{\Sigma\}$ is not empty, and we may take an $e_* \in \mathbb{S}^{q-1} \setminus P\{\Sigma\}$. We denote by P_{e_*} the projection of $\mathbb{R}^q$ onto the subspace perpendicular to e_*. Then $P_{e_*}\phi$ is a smooth free embedding of M^n into $\mathbb{R}^{q-1}$.

Now, we prove that any smooth compact Riemannian manifold admits a smooth free isometric embedding in a Euclidean space. The dimension of the target space could be large. We include this result to illustrate how to get the global isometric embedding from the local version.

Proposition 3.4. Any smooth n-dimensional compact Riemannian manifold admits a smooth free isometric embedding in $\mathbb{R}^q$ for some integer q.

Proof. Let (m^n, g) be a smooth n-dimensional compact Riemannian manifold. By Lemma 3.3, there exists a smooth free embedding $u : M^n \to \mathbb{R}^{q_1}$ such that $g - dudu$ is positive definite on M^n, where $q_1 = s_n + 2n$. By Lemma 3.1, there is a finite collection of open sets $\{U_l\}$ of M^n with $\phi_l \in C_c^\infty(U_l)$, $\eta_l \in C^\infty(U_l)$ such that

$$g - dudu = \Sigma_{l=1}^L \phi_l^2 d\eta_l^2,$$

for some positive integer L. Set $u_\epsilon : M^n \to \mathbb{R}^{2L}$ by

$$u_\epsilon = \epsilon(\phi_1 \cos\frac{\eta_1}{\epsilon}, \phi_1 \sin\frac{\eta_1}{\epsilon}, ..., \phi_L \cos\frac{\eta_L}{\epsilon}, \phi_L \sin\frac{\eta_L}{\epsilon}).$$

We then have

$$g = du_\epsilon du_\epsilon + dudu - \epsilon^2(d\phi_1^2 + ... + d\phi_L^2).$$

Now we claim there is a smooth free embedding $w : M^n \to \mathbb{R}^{q_1}$ such that

$$dwdw = dudu - \epsilon^2 \Sigma_{l=1}^L d\phi_l^2. \tag{55}$$

If (55) holds, then $(w, u_\epsilon) : M^n \to \mathbb{R}^{q_1+2L}$ is a smooth free isometric embedding of g.

It suffices to prove (55). To present the idea clearly, we discuss only the case $L = 1$. In the sequel, we omit the subscript l and consider the equation

$$dwdw = dudu - \epsilon^2 d\phi^2$$

in the unit ball $\mathbb{R}^n$, with $\phi \in C_c^\infty(\mathbb{R}^n)$. Assume $w = u + a^2 v$ for some $a \in C_c^\infty(\mathbb{R}^n)$ with $a = 1$ on suppϕ. Then according to the proof in previous section, if ϵ is small enough, we can find such v. Thus we can find w. Proof done.

The dimension q in Proposition3.4 depends on the manifold M^n itself, and not just n. In the rest of the section, we try to find the best q. We first state the main result.

Theorem 3.5. Let (M^n, g) be a smooth n-dimensional Riemannian manifold and let $u_0 : M^n \to \mathbb{R}^q$ be a smooth free map such that $g - du_0du_0$ is positive definite, with $q \geq s_n + n + 5$. Then there is a free map $u \in C^\infty(M^n, \mathbb{R}^q)$ such that $g = dudu$. Furthermore, if u_0 is an embedding, so is u.

Theorem 3.5 holds for both compact and noncompact Riemannian manifolds.

Corollary 3.6. Any smooth n-dimensional compact Riemannian manifold admits a smooth free isometric embedding in $\mathbb{R}^q$ for $q = max\{s_n + 2n, s_n + n + 5\}$.

Corollary 3.7. Any smooth 2-dimensional compact Riemannian manifold admits a smooth free isometric embedding in $\mathbb{R}^{10}$.

Before proving Theorem 3.5, we need to introduce several lemmas.

Lemma 3.8. Suppose $e_1, ..., e_m$ are m vectors in $\mathbb{R}^q$, with $q > m$. Then $e_1, ..., e_m$ are linearly independent if and only if

$$det(e_ie_j)_{m\times m} \neq 0$$

It's just some linear algebra stuff, we omit the proof.

Lemma 3.9. Let $e_1, ..., e_{n+5} \in C^\infty(\bar{\mathbb{B}}^n, \mathbb{R}^q)$ be linearly independent for any $x \in \bar{\mathbb{B}}^n$, with $q \geq n+5$, and let P_x be the orthogonal projection

$$P_x : \mathbb{R}^q \to span\{e_1(x), ..., e_{n+5}(x)\}.$$

Then there exist $u, v \in C^\infty(\bar{\mathbb{B}}^n, \mathbb{R}^q) \cap span\{e_1(x), ..., e_{n+5}(x)\}$ such that, for any $x \in \bar{\mathbb{B}}^n$, $s \in \mathbb{R}^1$ and $(a, b) \in \mathbb{S}^1$,

$$u(x), v(x), e_i(x) + sP_x(a\partial_u + b\partial_i v), 1 \leq i \leq n,$$

are linearly independent.

Proof. Consider the unit ball $\mathbb{B}^n$ as a subset of

$$\mathbb{T}^n = \{(x_1, ..., x_n) \in \mathbb{R}^n; -\pi \leq x_i \leq \pi\}$$

Set

$$f_1(x) = \sin x_1, f_k(x) = \Pi_{i=1}^{k-1}(\cos x_i + 2)\sin x_k, k = 2, ..., n, f_{n+1}(x) = \Pi_{i=1}^{k-1}(\cos x_i + 2)\cos x_n.$$

Define $f = (f_1, ..., f_{n+1})$. Obviously, $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a smooth and periodic. Moreover, $\partial_1 f(x), ..., \partial_n f(x)$ are linearly independent for any $x \in \mathbb{T}^n$, and

$$\partial_i f_l = 0 \quad forn \geq i \geq l+1 \geq 2. \tag{56}$$

Set

$$u(x) = \Sigma_{l=1}^{n+1} \epsilon f_l(\frac{x}{\epsilon^2}) e_{l+1}(x) + e_{n+4} \tag{57}$$

$$v(x) = \Sigma_{l=1}^{n+1} \epsilon f_l(\frac{x}{\epsilon^2}) e_{l+2}(x) + e_{n+5} \tag{58}$$

where ϵ is a small parameter to be specified. A differentiation of (57)-(58) yields

$$\partial_i u(x) = \frac{1}{\epsilon} \Sigma_{l=1}^{n+1} \partial_i f_l(\frac{x}{\epsilon^2}) e_{l+1}(x) + O(1) \tag{59}$$

$$\partial_i v(x) = \frac{1}{\epsilon} \Sigma_{l=1}^{n+1} \partial_i f_l(\frac{x}{\epsilon^2}) e_{l+2}(x) + O(1) \tag{60}$$

Combine (59)-(60)gives

$$a\partial_i u + b\partial_v = \frac{1}{\epsilon} \Sigma_{l=1}^{n+1} (ae_{l+1} + be_{l+2}) + O(1)$$

For any $(a, b) \in \mathbb{S}^1$ and any $x \in \bar{\mathbb{B}}^n$, define

$$w_l(x) = ae_{l+1}(x) + be_{l+2}(x), l = 1, ..., n+1, w_l(x) = e_{l+2}(x), l = n+2, n+3 \tag{61}$$

Then, $w_1(x), ..., w_{n+3}(x)$ are linearly independent for any $x \in \bar{\mathbb{B}}^n$. To prove this, we consider constants $c_1, ..., c_{n+3}$ such that $\Sigma_{l=1}^{n+3} c_l w_l(x) = 0$ at some point $x \in \bar{\mathbb{B}}^n$. This implies $c_{n+2} = c_{n+3} = 0$ and

$$c_1 a e_2(x) + (c_1 b + c_2 a) e_3(x) + ... + (c_n b + c_{n+1} a) e_{n+2}(x) + c_{n+1} b e_{n+3}(x) = 0$$

Thus, we get $c_l = 0$, $l = 1, ..., n+1$, since $a^2 + b^2 \neq 0$.

Similarly, we define for $x \in \bar{\mathbb{B}}^n$ and $y \in \mathbb{T}^n$

$$\tilde{w_i}(y, x) = \Sigma_{l=1}^{n+1} \partial_i f_l(y) w_l(x), i = 1, ..., n, \tilde{w_i}(y, x) = w_{i+1}, i = n+1, n+2.$$

Then, $\tilde{w_1}(y, x), ..., \tilde{w_{n+2}}(y, x)$ are linearly independent for any $x \in \bar{\mathbb{B}}^n$ and any $y \in \mathbb{T}^n$. To prove this, we consider constants $\mu_1, ..., \mu_{n+3}$ such that

$$\Sigma_{i=1}^n \mu_i \Sigma_{l=1}^{n+1} \partial_i f_l(y) w_l(x) + \Sigma_{j=1}^2 \mu_{n+j} w_{n+j+1}(x) = 0$$

first we get $\mu_{n+1} = \mu_{n+2} = 0$, and

$$\Sigma_{i=1}^n \mu_i \partial_i f_l(y) = 0, for \quad any \quad l = 1, ..., n+1$$

by the linear independence of $\partial_i f_l(y)$, we get $\mu_1 = ... = \mu_n = 0$. Therefore, Lemma 3.8 yields to

$$det(\tilde{w_i}\tilde{w_j})_{(n+2)\times(n+2)} \geq \theta, \tag{62}$$

for some positive constant θ independent of $x \in \bar{\mathbb{B}}^n$, $y \in \mathbb{T}^n$ and $(a, b) \in \mathbb{S}^1$.

In the following, we prove that $u(x), v(x)$ defined above satisfy the requirements of this lemma. For each $s \in \mathbb{R}^1$ and $(a, b) \in \mathbb{S}^1$, consider $(n+2)$ vectors

$$V_i = d_i(x) + sP_x(a\partial_i u + b\partial_v), i = 1, ..., n \tag{63}$$

$$V_{n+1} = u = e_{n+4} + O(\epsilon), \tag{64}$$

$$V_{n+2} = v = e_{n+5} + O(\epsilon) \tag{65}$$

We may write

$$V_i = e_i + \frac{s}{\epsilon}\tilde{w_i}(\frac{x}{\epsilon^2}, x) + sO(1), i = 1, ..., n,$$

we consider two cases $|s| \geq \epsilon^{1-(1/2n)}$ and $|s| < \epsilon^{1-(1/2n)}$. For the first case, we have

$$V_i = \frac{s}{\epsilon}[\tilde{w_i}(\frac{x}{\epsilon^2}, x) + O(\epsilon^{\frac{1}{2n}})], i = 1, ..., n, V_{n+1} = \tilde{w}_{n+1} + O(\epsilon), V_{n+2} = \tilde{w}_{n+2} + O(\epsilon)$$

In view of (62), it follows that

$$det(V_iV_j) = \frac{s^{2n}}{\epsilon^{2n}}\{det(\tilde{w_i}\tilde{w_j}) + O(\epsilon^{\frac{1}{2n}})\} \geq \frac{1}{\epsilon}\{\theta - C\epsilon^{\frac{1}{2n}}\} \geq \frac{\theta}{2\epsilon}$$

if $\epsilon \in (0, \epsilon_0]$ for some universal constant ϵ_0. By Lemma 3.8, $\{V_1, ..., V_{n+2}\}$ is linearly independent as $\epsilon \in (0, \epsilon_0]$ and $|s| \geq \epsilon^{1-(1/2n)}$.

For the second case $|s| < \epsilon^{1-(1/2n)}$, we assume for some $x \in \bar{\mathbb{B}}^n$

$$\Sigma_{i=1}^{n+2} c_i V_i(x) = 0.$$

Set

$$sup|c_i| = A$$

We may write

$$c_1 + O(A\epsilon^{1-\frac{1}{2n}}) = 0,$$
$$c_2 + \frac{s}{\epsilon} c_1 a \partial_1 f_1(\frac{x}{\epsilon^2}) + O(A\epsilon^{1-\frac{1}{2n}}) = 0,$$
$$c_i + \frac{s}{\epsilon} \Sigma_{l=1}^{i-1} c_l [a \partial_l f_{i-1}(\frac{x}{\epsilon^2}) + b \partial_l f_{i-2}(\frac{x}{\epsilon^2})] + O(A\epsilon^{1-\frac{1}{2n}}) = 0,$$
$$c_{n+1} + O(A\epsilon^{1-\frac{1}{2n}}) = 0,$$
$$c_{n+2} + O(A\epsilon^{1-\frac{1}{2n}}) = 0,$$

Consequently, there hold for some constant C independent of ϵ

$$|c_1| \leq CA\epsilon^{1-\frac{1}{2n}}, |c_2| \leq CA\epsilon^{1-\frac{1}{2n}}, |c_i| \leq CA\epsilon^{1-\frac{i}{2n}}, i = 3, ..., n, |c_{n+1}|, |c_{n+2}| \leq CA\epsilon^{1-\frac{1}{2n}}.$$

Then we have $|c_i| \leq CA\sqrt{\epsilon}$ for all i, and hence

$$A = sup|c_i| \leq CA\sqrt{\epsilon} \leq \frac{1}{2}A,$$

for any $\epsilon \in (0, \epsilon_0]$ with $C\sqrt{\epsilon_0} \leq \frac{1}{2}$. Therefore, $A = 0$ and $V_1, ..., V_{n+2}$ is linearly independent as $|s| < \epsilon^{1-(1/2n)}$ and $\epsilon \in (0, \epsilon_0]$.

Remark 3.10. With the aid fo the normalizing procedure, we may assume $|u(x)| = |v(x)| = 1$ and $u(x)v(x) = 0$, since this procedure does not affect linear independence.

Lemma 3.11. There exist two smooth functions β_1, β_2 on $\mathbb{S}^1$ such that

$$\beta_1'(t)\beta_2''(t) - \beta_1''(t)\beta_2'(t) \neq 0, \tag{66}$$

$$\int_{\mathbb{S}^1} \beta_i'(t)\beta_j(t)dt = 0, i, j = 1, 2, \tag{67}$$

$$\int_{\mathbb{S}^1} \beta_i'(t)\rho(t)dt = 0, i = 1, 2. \tag{68}$$

where

$$\rho(t) = \sqrt{\beta_1'(t)^2 + \beta_2'(t)^2} \tag{69}$$

is positive on $\mathbb{S}^1$.

Proof. Set $\beta_1(t) = \cos t - \frac{1}{2}\cos 2t + b$, $\beta_2(t) = \sin t + \sin 2t + a$. Then, we have

$$\beta_1'(t)\beta_2''(t) - \beta_1''(t)\beta_2'(t) = 3(\sin 2t \sin t - 1) < 0$$

It's easy to check that $\int_{\mathbb{S}^1} \beta_i'(t)\beta_j(t)dt = 0$. It is easy to see that ρ is positive on $\mathbb{S}^1$. Thus a special choice of constants a and b implies (68).

Lemma 3.12. Suppose $u_0 \in C^\infty(\bar{\mathbb{B}}^n, \mathbb{R}^q)$ is a free map, $q \geq s_n + n + 5$. Then there exists a $v_1(t,x) \in C^\infty(\mathbb{S}^1 \times \bar{\mathbb{B}}^n, \mathbb{R}^q))$ such that the following hold for $u_1 = \phi v_1$, for any $\phi \in C_c^\infty(\mathbb{B}^n)$:

$$\partial_k u_0 u_1 = 0, \partial_k u_0 \partial_t u_1 = 0, 1 \leq k \leq n, \partial_{ij} u_0 u_1 = 0, 2 \leq i,j \leq n, \tag{70}$$

(71)
$$\partial_i u_0, \partial_{ij} u_0, 2 \leq i,j \leq n, \partial_1 u_0 + s\partial_t u_1, \partial_{11} u_0 + 2s\partial_{t1} u_1, \partial_{1i} u_0 + s\partial_{ti} u_1, 2 \leq i \leq n, \partial_t v_1, \partial_{tt} v_1$$

are linearly independent for any $(t,x) \in \mathbb{S}^1 \times \bar{\mathbb{B}}^n$ and $s \in \mathbb{R}$,

and

$$|\partial_t u_1|^2 = \phi^4(x)\rho^2(t) \tag{72}$$

for any $(t,x) \in \mathbb{S}^1 \times \bar{\mathbb{B}}^n$, where $\rho(t)$ is a positive function on $\mathbb{S}^1$.

Proof. Since u_0 is free and $q \geq s_n + n + 5$, there exist $f_1, ..., f_5 \in C^\infty(\bar{\mathbb{B}}^n, \mathbb{R}^q)$ such that

$$\partial_k u_0, \partial_{kl} u_0, 1 \leq k,l \leq n, f_1, ..., f_5 \tag{73}$$

are linearly independent in $\bar{\mathbb{B}}^n$. Denote by L_x the space spanned by $\partial_k u_0, 1 \leq k \leq n$ and $\partial_{ij} u_0, 2 \leq i,j \leq n$, and the projection operator by

$$P_x : \mathbb{R}^q \to L_x^\perp. \tag{74}$$

Here $L_x^\perp$ is contained in the linear space spanned by functions in (73). Set

$$e_i(x) = P_x \partial_{1i} u_0(x), i = 1, ..., n, e_i(x) = P_x f_{i-n}(x), i = n+1, ..., n+5. \tag{75}$$

Obviously, $e_1, ..., e_{n+5}$ are linearly independent. By Lemma 3.9, there exist $u, v \in C^\infty(\bar{\mathbb{B}}^n, \mathbb{R}^q) \cap$ $span\{e_1, ..., e_{n+5}\}$ such that

$$u(x), v(x), e_i(x) + P_x(a\partial_i u(x) + b\partial_i v(x)), i = 1, ..., n \tag{76}$$

are linearly independent for any $x \in \bar{\mathbb{B}}^n$, $a, b \in \mathbb{R}^1$, $a^2 + b^2 \neq 0$.

By Remark 3.10, we may assume $|u(x)| = |v(x)| = 1$ and $u(x)v(x) = 0$. With β_1, β_2 constructed in Lemma 3.11, we set

$$v_1(t,x) = \beta_1(t)u(x) + \beta_2(t)v(x) \tag{77}$$

Obviously, (72) holds since $u_1 = \phi^2 v_1$ satisfies

$$|\partial_t u_1(t,x)|^2 = \phi^4(\beta_1'(t)^2 + \beta_2'(t)^2) = \phi^4\rho^2(t)$$

Since $u(x), v(x)$ are in $L_x^\perp$, (70) holds too. By (66), we know that $\partial_t v_1, \partial_{tt} v_1$ are linearly independent and that

$$span\{\partial_t v_1, \partial_{tt} v_1\} = span\{u, v\} \tag{78}$$

Note $P_x\{\partial_{1i}u_0 + s\partial_{ti}v_1\} = e_i(x) + sP_x\{\beta_1'(t)\partial_i u(x) + \beta_2'(t)\partial_i v(x)\}$, thus (71) follows from (76).

Now we may assume that

$$\partial_t h_1 = -(\rho\partial_1 u_0 + \partial_t u_1)\partial_1 u_1, \partial_t h_i = -\rho\partial_1 u_1 \partial_i u_0 - (\rho\partial_1 u_0 + \partial_t u_1)\partial_i u_1, i = 2, ..., n \tag{79}$$

The key step in proving Theorem 3.5 is the following result.

Theorem 3.13. Suppose $u_0 \in C^\infty(\bar{\mathbb{B}}^n, \mathbb{R}^q)$ is a free map and $\phi \in C_c^\infty(\mathbb{B}^n)$ is a smooth function, with $q \geq s_n + n + 5$. Then there exists a compact set $C \subset \mathbb{B}^n$ with $supp\phi \subset C$ such that the following holds: For any $k \in \mathbb{Z}^+$, there is a positive constant ϵ_k such that for any $\epsilon \in (0, \epsilon_0]$ there exists a smooth free map $u_{\epsilon k} : \bar{\mathbb{B}}^n \to \mathbb{R}^q)$ such that

$$du_{\epsilon k} du_{\epsilon k} = du_0 du_0 + \phi^4 dx_1^2 + \epsilon^{k+1} g_{\epsilon k}, \tag{80}$$

where

$$u_{\epsilon k} - u_0 \in C_c^\infty(\bar{\mathbb{B}}^n, \mathbb{R}^q), supp(g_{\epsilon k}) \subset C \tag{81}$$

$$|u_{\epsilon k} - u_0| \leq c\epsilon \tag{82}$$

for some constant c independent of ϵ.

Remark 3.14. As we see in the proof, the function $u_{\epsilon k}$ is of the form

$$u_{\epsilon k} = \tilde{u}_{\epsilon k}(\epsilon, \frac{x_1}{\epsilon}, x).$$

for some $\tilde{u}_{\epsilon k} \in C^\infty((0, \epsilon k] \times \mathbb{S}^1 \times \bar{\mathbb{B}}^n)$. Here $\tilde{u}_{\epsilon k}(, t,)$ is considered to be a function in $t \in \mathbb{S}^1$, or a 2π-periodic function in $t \in \mathbb{R}$. A similar form holds for $g_{\epsilon k}$.

Proof. We prove by an induction on k that $u_{\epsilon k}$ can be taken in the form

$$u_{\epsilon k} = u_0(x) + \epsilon u_1(t,x) + \epsilon^2 w_2(\epsilon,t,x) + ... + \epsilon^k w_k(\epsilon,t,x) \quad with \quad t = \gamma(\frac{x_1}{\epsilon}) \tag{83}$$

where u_1 is as in Lemma 3.12 and $w_l(\epsilon,t,x) = u_l(t,x) + \epsilon v_l(t,x)$ will be determined, $l = 2,...,k$. $\int_0^\gamma \rho(s)ds = t$ defines γ.

We begin with $k = 2$ an look for a function $u_{\epsilon 2}$ of the following form:

$$u_{\epsilon 2} = u_0(x) + \epsilon u_1(t,x) + \epsilon^2 w_2(\epsilon,t,x)$$

for some smooth function $w_2(\epsilon,t,x)$, yet to be determined. Then, we have

$$\partial_1 u_{\epsilon 2} = (\partial_1 u_0 + \frac{1}{\rho}\partial_t u_1) + \epsilon(\partial_1 u_1 + \frac{1}{\rho}\partial_t w_2) + \epsilon^2 \partial_1 w_2, \partial_i u_{\epsilon 2} = \partial_i u_0 + \epsilon \partial_i u_1 + \epsilon^2 \partial_i w_2$$

By (70) and the definition of h, yields

$$\partial_1 u_{\epsilon 2}\partial_1 u_{\epsilon 2} = |\partial_1 u_0|^2 + \phi^4 + \frac{2\epsilon}{\rho}(-\partial_t h_1 + (\partial_1 u_0 + \frac{1}{\rho}\partial_t u_1)\partial_t w_2)$$

$$+\epsilon^2|\partial_1 u_1 + \frac{1}{\rho}\partial_t w_2|^2 + 2\epsilon^2(\partial_1 u_0 + \frac{1}{\rho}\partial_t u_1)\partial_1 w_2 + O(\epsilon^3)$$

$$\partial_1 u_{\epsilon 2}\partial_i u_{\epsilon 2} = \partial_1 u_0 \partial_i u_0 + \frac{\epsilon}{\rho}(-\partial_t h_i + \partial_i u_0 \partial_t w_2)$$

$$+\epsilon^2(\partial_i u_0 \partial_1 w_2 + (\partial_1 u_1 + \frac{1}{\rho}\partial_t w_2)\partial_i u_1 + (\partial_1 u_0 + \frac{1}{\rho}\partial_t u_1)) + O(\epsilon^3)$$

$$\partial_i u_{\epsilon 2}\partial_j u_{\epsilon 2} = \partial_i u_0 \partial_j u_0 + \epsilon^2(\partial_i u_1 \partial_j u_1 + \partial_i u_0 \partial_j w_2 + \partial_j u_0 \partial_i w_2) + O(\epsilon^3)$$

We claim all the above holds if w_2 satisfies

$$(\partial_1 u_0 + \frac{1}{\rho}\partial_t u_1 + \epsilon \partial_1 u_1) w_2 = h_! + O(\epsilon^2)$$

$$(\partial_i u_0 + \epsilon \partial_i u_1) w_2 = h_i + O(\epsilon^2)$$

$$(\partial_{11} u_0 + \frac{2}{\rho}\partial_{t1} u_1) w_2 = \frac{1}{2}(|\partial_1 u_1|^2 + 2\partial_1 h_1) + O(\epsilon)$$

$$(\partial_{1i} u_0 + \frac{1}{\rho}\partial_{ti} u_1) w_2 = \frac{1}{2}(\partial_1 u_1 \partial_i u_1 + \partial_i h_1 + \partial_1 h_i) + O(\epsilon)$$

$$\partial_{ij} u_0 w_2 = \frac{1}{2}(\partial_i u_1 \partial_j u_1 + \partial_i h_j + \partial_j h_i) + O(\epsilon)$$

$$\partial_t u_1 w_2 = 0, \partial_{tt} u_1 w_2 = \frac{\epsilon}{2}|\partial_t w_2|^2 + O(\epsilon^2)$$

Note that it's an algebraic system for w_2, while the above involves derivatives of w_2. To prove the claim, we rewrite it so that no differentiation is applied directly to w_2. We do this for ∂_t first and then for ∂_1 and ∂_i for $i = 2,...,n$. We may omit this calculation, as it's clear and not hard.

To solve the algebraic system of w_2, we consider

$$w_2(\epsilon, t, x) = u_2(t, x) + \epsilon v_2(t, x)$$

By substituting w_2 in the algebraic system and comparing the order of ϵ, we require

$$(\partial_1 u_0 + \frac{1}{\rho}\partial_t u_1)w_2 = h_1, \partial_i u_0 u_2 = h_i \tag{84}$$

$$(\partial_{11} u_0 + \frac{2}{\rho}\partial_{t1} u_1)u_2 = \frac{1}{2}(|\partial_1 u_1|^2 + 2\partial_1 h_1), (\partial_{1i} u_0 + \frac{1}{\rho})u_2 = \frac{1}{2}(\partial_1 u_1 \partial_i u_1 + \partial_i h_1 + \partial_1 h_i) \tag{85}$$

$$\partial_{ij} u_0 u_2 = \frac{1}{2}(\partial_i u_1 \partial_j u_1 + \partial_i h_j + \partial_j h_i), \partial_t v_1 u_2 = 0, \partial_{tt} v_1 u_2 = 0. \tag{86}$$

and

$$(\partial_1 u_0 + \frac{1}{\rho}\partial_t u_1)v_2 = -\partial_1 u_1 u_2 \tag{87}$$

$$\partial_i u_0 v_2 = -\partial_i u_1 u_2, \partial_t v_1 v_2 = 0, \partial_{tt} v_1 v_2 = \frac{|\partial_t u_2|^2}{2\phi^2}. \tag{88}$$

According to Lemma 3.12, the coefficients of u_2 is linearly independent and of full rank in $\mathbb{S}^1 \times \bar{\mathbb{B}}^n$. Thus we may find a $u_2 \in C^\infty(\mathbb{S}^1 \times \bar{\mathbb{B}}^n)$ satisfying the above algebraic system. Besides, $supp u_2 \subset \mathbb{S}^1 \times supp \phi$. So we can find $v_2 \in C_c^\infty(\mathbb{S}^1 \times \bar{\mathbb{B}}^n)$. And thus we have proved the case for $k = 2$.

Now, we assume Theorem 3.13 holds for $k \geq 2$ and then prove it for $k+1$. Suppose

$$u_{\epsilon(k+1)}(x) = u_{\epsilon k}(x) + \epsilon^{k+1} w_{k+1}(\epsilon, t, x), with \quad t = \gamma(\frac{x_1}{\epsilon})$$

for some smooth function $w_{k+1}(\epsilon, t, x)$ to be determined. Like what we do in $k = 2$ case, we find some conditions for $w_{k+1}(\epsilon, t, x)$, and derive a equivalent algebraic system. Then we let $w_{k+1}(\epsilon, t, x) = u_{k+1}(t, x) + \epsilon v_{k+1}(t, x)$ and again get a algebraic system for u_{k+1}, v_{k+1}. And happily, we find the coefficient is just as we stated in Lemma 3.12. So we can solve these u_{k+1}, v_{k+1}, which leads to the statement of the Theorem.

Now we are able to prove Theorem 3.5.

Proof of Theorem 3.5. By Lemma 3.3 and Remark 3.2, we may assume that the metric g is given by

$$g = du_0 du_0 + \Sigma_{l \geq 1} h^{(l)},$$

where $h^{(l)}$ is a smooth symmetric covariant 2-tensor on M such that in a suitable local coordinate system $x : U^{(l)} \to \mathbb{R}^n$

$$h^{(l)} = \phi^4 dx_1^2$$

with $\phi \in C_c^\infty(U^{(l)})$. The support of $h^{(l)}$ is in $U^{(l)}$, and the family $\{U^{(l)}\}$ is a cover of M. To prove Theorem 3.5, it suffices to consider the case where there is only one term in the summation for the expression of g.

In the following, we assume

$$g = du_0 du_0 + \phi^4 dx_1^2,$$

for some $\phi \in C_c^\infty(U^{(l)})$ and some local coordinate system $x : U \to \mathbb{R}^n$. We may identify U with $\mathbb{B}^n$. By Theorem 3.13, we may find a compact set $C \subset \mathbb{B}^n$ with $supp\phi \subset C$, a smooth free map $u_{\epsilon k} : \bar{\mathbb{R}}^n \to \mathbb{R}^q$ and a smooth symmetric covariant 2-tensor $g_{\epsilon k}$ in $\bar{\mathbb{R}}^n$ such that $suppg_{\epsilon k} \subset C$ and

$$du_{\epsilon k} du_{\epsilon k} = du_0 du_0 + \phi^4 dx_1^2 + \epsilon^{k+1} g_{\epsilon k}.$$

Now suppose $u = u_{\epsilon k} + a^2 w$ for some $a \in C_c^\infty(\mathbb{B}^n)$ with $a = 1$ on C. Then $dudu = g$ is equivalent to

$$d(u_{\epsilon k} + a^2 w) d(u_{\epsilon k} + a^2 w) = du_{\epsilon k} du_{\epsilon k} - \epsilon^{k+1} g_{\epsilon k}. \tag{89}$$

Since we have ϵ^{k+1} in front of $g_{\epsilon k}$, as in the section 2, we can find w satisfying the equation. Thus $u = u_{\epsilon k} + a^2 w$ is a free map if ϵ is small enough. This ends the proof of Theorem 3.5.

References

[1] Qing Han, Jia-xing Hong, *Isometric Embedding of Riemannian Manifolds in Euclidean Spaces,* American Mathematical Society, pp. 1-31