FIRST ORDER EQUATIONS

TSOGTGEREL GANTUMUR

ABSTRACT. After reviewing some fundamental results from advanced analysis, we give a justification to the method of characteristics for first order equations. Then we briefly discuss weak solutions to conservation laws.

CONTENTS

1. VECTOR-VALUED CALCULUS

Let $I \subset \mathbb{R}$ be an interval, and let X be a Banach space. We want to develop a basic calculus for functions $f : I \to X$. First of all, let us define the *uniform norm*

$$\|f\|_\infty = \sup_{t\in I} \|f(t)\|, \tag{1}$$

where $\|\cdot\|$ denotes the norm in X, and the corresponding space of *bounded functions*

$$B(I,X) = \{f : I \to X : \|f\|_\infty < \infty\}. \tag{2}$$

We denote by $C(I,X)$ the space of *continuous functions* on I having values in X. Then the space of *bounded continuous functions* is defined by

$$C_b(I,X) \equiv BC(I,X) = B(I,X) \cap C(I,X). \tag{3}$$

For $f \in C(I,X)$, the norm $\|f(t)\|$ is a continuous function of $t \in I$. Therefore if I is a closed interval we have $C(I,X) = C_b(I,X)$.

Exercise 1. Show that $C_b(I,X)$ is a Banach space under the norm $\|\cdot\|_\infty$.

If for a function $f : I \to X$ and $a \in I$, there is some $\lambda \in X$ such that

$$f(t) - f(a) = \lambda(t-a) + o(|t-a|), \qquad \text{as} \quad t \to a, \tag{4}$$

then we say that f is *differentiable at* $a \in I$, and write

$$f'(a) = \dot f(a) = \frac{\mathrm{d}f}{\mathrm{d}t}(a) = \lambda, \tag{5}$$

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which is called the *derivative of f at a*. In case a is an endpoint of I, the limit in (4) should be understood as a one-sided limit. If f is differentiable everywhere on I, then f' is clearly a function on I having values in X. So the following definitions make sense

$$\begin{aligned} C^k(I,X) &= \{f \in C(I,X) : f' \in C^{k-1}(I,X)\}, \\ C^k_b(I,X) &= \{f \in C_b(I,X) : f' \in C^{k-1}_b(I,X)\}. \end{aligned} \tag{6}$$

Now we want to introduce the Riemann integral. A *partition* of an interval $[a,b]$ is a sequence $p = \{t_i\}_{i=0}^n$ satisfying

$$a = t_0 < t_1 < \ldots < t_n = b, \tag{7}$$

and we say that p is *tagged* by $\xi = \{\xi_i\}_{i=1}^n$ if $\xi_i \in [t_{i-1}, t_i]$ for $i = 1, \ldots n$. Given a tagged partition (p, ξ), the *Riemann sum* of $f \in B([a,b],X)$ with respect to (p,ξ) is defined by

$$S_{p,\xi}(f) = \sum_{i=1}^n (t_i - t_{i-1}) f(\xi_i). \tag{8}$$

Then we say that f is *Riemann integrable on* $[a,b]$ if $S_{p,\xi}(f)$ has a limit in X as $|p| \to 0$, where $|p|$ is the *width*

$$|p| = \max_i |t_i - t_{i-1}|. \tag{9}$$

If f is integrable, we take its *Riemann integral* over $[a,b]$ to be

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{|p|\to 0} S_{p,\xi}(f). \tag{10}$$

We also have the convention

$$\int_b^a f(t)\,\mathrm{d}t = -\int_a^b f(t)\,\mathrm{d}t. \tag{11}$$

Since there is no risk of confusion we simply drop the adjective "Riemann" from integrability and integral. We have a simple criterion on integrability, in terms of the quantity

$$\mathrm{osc}(f,p) = \sum_{i=1}^n (t_i - t_{i-1}) \mathrm{osc}(f, [t_{i-1}, t_i]), \tag{12}$$

which could be called the *oscillation of f* over the partition p, where

$$\mathrm{osc}(f, [t_{i-1}, t_i]) = \sup_{\xi,\eta \in [t_{i-1},t_i]} \|f(\xi) - f(\eta)\|, \tag{13}$$

is the oscillation of f over the interval $[t_{i-1}, t_i]$. Note that a function f is continuous at t if and only if $\mathrm{osc}(f,[r,s]) \to 0$ as $r \nearrow t$ and $s \searrow t$, and that if f is continuous, then the oscillation $\mathrm{osc}(f,[r,s])$ is a continuous function of r and s.

Lemma 1. *A function $f \in B([a,b],X)$ is integrable over $[a,b]$ if for any $\varepsilon > 0$ there is a partition p such that* $\mathrm{osc}(f,p) < \varepsilon$.

Proof. Note that if p' is a refinement of p, i.e., if $p \subset p'$, then $\mathrm{osc}(f,p') \le \mathrm{osc}(f,p)$ and

$$\|S_{p,\xi}(f) - S_{p',\xi'}(f)\| \le \sum_{i=1}^n (t_i - t_{i-1}) \mathrm{osc}(f, [t_{i-1}, t_i]) = \mathrm{osc}(f,p), \tag{14}$$

for any sets of tags ξ and ξ', respectively, of p and p'. Let $p_0, p_1, \ldots$ be a sequence of partitions with oscillations tending to 0. Replacing p_i by the common refinement $p_i \cup p_0$, we can assume that $p_i \subset p_{i+1}$ for all i. Then from (14) we see that the Riemann sums corresponding to the sequence $\{p_i\}$ converge to some element $x \in X$, independent of how the partitions are tagged. The convergence can be established first for some particular tagging of the sequence $\{p_i\}$, and then be extended to arbitrary tagging by (14). Now we need to show that as long as the width is small, any tagged partition gives rise to a Riemann sum that is close to x. Let (q, η) be a tagged partition with $|q|$ small. Let $q' = q \cup p_k$ with k large. We tag q' by η' such that η' coincides with η on the subintervals common to both q and q', meaning that

$$\|S_{q,\eta}(f) - S_{q',\eta'}(f)\| \leq \sum_I |q| \operatorname{osc}(f, I) \leq 2\#p_k|q|\|f\|_\infty, \tag{15}$$

where the sum is over all subintervals I of q satisfying $I \cap p_k \neq \varnothing$, and $\#p_k$ denotes the number of nodes in p_k. Given $\varepsilon > 0$, we choose k so large that $\|S_{p_k,\cdot}(f) - x\| < \varepsilon$. Then if $|q|$ is so small that the right hand side of (15) is less than ε, we get

$$\|S_{q,\eta}(f) - x\| \leq \|S_{q,\eta}(f) - S_{q',\eta'}(f)\| + \|S_{q',\eta'}(f) - x\| < 2\varepsilon, \tag{16}$$

establishing the claim. □

The following lemma produces a large class of integrable functions.

Corollary 2. *Functions in $C([a, b], X)$ are integrable.*

Proof. Since $C([a, b], X) \subset B([a, b], X)$, it suffices to produce a sequence of partitions with oscillations vanishing in the limit. The oscillation $\operatorname{osc}(f, [s, t])$ is a continuous function of (s, t) on the closed triangle $\{a \leq s \leq t \leq b\}$, and $\operatorname{osc}(f, [t, t]) = 0$. Hence the modulus of continuity $\omega(\delta) = \max_t \operatorname{osc}(f, [t, t + \delta])$ is a continuous function of δ with $\omega(0) = 0$. This means that for any given $\varepsilon > 0$, there exists a partition p of $[a, b]$ with $\operatorname{osc}(f, p) < \varepsilon$. □

Note that the proof shows that a continuous function on a closed interval is *uniformly continuous*, in the sense that $\omega(\delta) \to 0$ as $\delta \searrow 0$.

Obviously, integration is a linear operation. A few more simple properties are as follows.

Lemma 3. *If $f : [a, b] \to X$ is integrable, then we have the additivity*

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t, \qquad c \in [a, b], \tag{17}$$

and the bounds

$$\Big\|\int_a^b f(t)\,\mathrm{d}t\Big\| \leq (b - a)\|f\|_\infty, \qquad \text{and} \qquad \Big\|\int_a^b f(t)\,\mathrm{d}t\Big\| \leq \int_a^b \|f(t)\|\,\mathrm{d}t. \tag{18}$$

Moreover, if $\{f_n\}$ is a sequence of integrable functions on $[a, b]$ converging uniformly to f in $[a, b]$, then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t. \tag{19}$$

Proof. The additivity (17) and inequalities (18) are true because they are true when the integrals are replaced by the corresponding Riemann sums. For any partition p (with any tag), we have

$$\begin{aligned} \Big|\int_a^b f_n(t)\,\mathrm{d}t - S_p(f)\Big| &\leq \Big|\int_a^b f_n(t)\,\mathrm{d}t - S_p(f_n)\Big| + |S_p(f_n) - S_p(f)| \\ &\leq \Big|\int_a^b f_n(t)\,\mathrm{d}t - S_p(f_n)\Big| + (b - a)\|f_n - f\|_\infty, \end{aligned} \tag{20}$$

which implies that f is integrable and that the limit (19) holds. □

Theorem 4 (Fundamental theorem of calculus)**.** *a) If $u \in C^1([a, b], X)$ then*

$$u(b) - u(a) = \int_a^b u'(t)\,\mathrm{d}t. \tag{21}$$

b) If $f \in C([a,b],X)$ then the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \qquad x \in [a,b], \tag{22}$$

satisfies $F \in C^1([a,b],X)$ and $F' = f$ on $[a,b]$.

Proof. a) Consider the function

$$g(r,s) = u'(r) - \frac{u(s)-u(r)}{s-r}, \tag{23}$$

on the triangle $\{a \leq r < s \leq b\}$. This is continuous, and can be continuously extended to $\{r = s\}$ as $g(r,r) = 0$. Hence

$$g(h) = \max_r \|g(r, r+h)\|, \tag{24}$$

is continuous for $h \geq 0$ and $g(0) = 0$, which is to say that

$$\big\|u'(t) - \frac{u(t+h)-u(t)}{h}\big\| \leq g(h), \tag{25}$$

uniformly in t.

Let us take a partition p of evenly spaced points in $[a,b]$, with each subinterval tagged at its left endpoint. Let $h = |p|$ be the length of each subinterval. Then we have

$$\begin{aligned}\|S_p(u') - [u(b)-u(a)]\| &= \big\|\sum_{i=1}^n hu'(t_{i-1}) - \sum_{i=1}^n (u(t_i) - u(t_{i-1}))\big\| \\ &\leq \sum_{i=1}^n h\big\|u'(t_i) - \frac{u(t_i)-u(t_{i-1})}{h}\big\| \\ &\leq \sum_{i=1}^n hg(h) = g(h),\end{aligned} \tag{26}$$

which shows that $S_p(u') \to u(b) - u(a)$ as $h \to 0$. Since u' is integrable this limit must be equal to the integral.

b) By additivity, for any $a \leq x < y \leq b$ we have

$$F(y) - F(x) = \int_x^y f(t)\,\mathrm{d}t. \tag{27}$$

Let $\varepsilon > 0$. Then by uniform continuity, there is $\delta > 0$ such that $|s-t| < \delta$ implies $\|f(s) - f(t)\| < \varepsilon$. Let $a \leq x < y \leq b$ satisfy $y - x < \delta$, and take a partition p of n evenly spaced points in $[x,y]$, with each subinterval tagged at its left endpoint. We choose n so large that

$$\big\|F(y) - F(x) - S_p(f)\big\| = \big\|\int_x^y f(t)\,\mathrm{d}t - S_p(f)\big\| < \varepsilon|y-x|. \tag{28}$$

All this implies

$$\begin{aligned}\big\|\frac{F(y)-F(x)}{y-x} - f(x)\big\| &\leq \frac{\|F(y)-F(x)-S_p(f)\|}{y-x} + \big\|\frac{1}{y-x}\sum_{i=1}^n \frac{y-x}{n} f(t_{i-1}) - f(x)\big\| \\ &< \varepsilon + \frac{1}{n}\big\|\sum_{i=1}^n (f(t_{i-1}) - f(x))\big\| < 2\varepsilon,\end{aligned} \tag{29}$$

which establishes the claim. □

2. Banach's fixed point theorem

A *distance function*, or a *metric*, on a set M is a function $\rho : M\times M \to \mathbb{R}$ that is symmetric: $\rho(u, v) = \rho(v, u)$, nonnegative: $\rho(u, v) \geq 0$, nondegenerate: $\rho(u, v) = 0 \Leftrightarrow u = v$, and satisfies the triangle inequality: $\rho(u, v) \leq \rho(u, w)+\rho(w, v)$. Then a *metric space* is a set with a metric. If a sequence $\{u_n\}$ in M satisfies $\rho(u_n, u) \to 0$ as $n \to \infty$ for some $u \in M$, we say that the sequence *converges* to u, and write $u_n \to u$ in M. It is obvious that convergent sequences are *Cauchy*, meaning that $\rho(u_n, u_m) \to 0$ as $n, m \to \infty$. In general, however, Cauchy sequences do not have to converge in the space, as can be seen from, e.g, the example $M = \mathbb{Q}$ and $\rho(x, y) = |x - y|$. If the metric space M is such that every Cauchy sequence converges to an element of M, we call it a *complete metric space*. A mapping $\phi : M \to W$ between two metric spaces is called *continuous* if $u_n \to u$ in M implies $\phi(u_n) \to \phi(u)$ in W. With ϱ denoting the metric of W, if

$$\varrho(\phi(u), \phi(v)) \leq k\rho(u, v), \qquad u, v \in M, \tag{30}$$

with some constant $k \in \mathbb{R}$, then we say that ϕ is *Lipschitz continuous*. In this setting, ϕ is called a *nonexpansive* mapping if $k \leq 1$, and a *contraction* if $k < 1$.

Theorem 5. *Let M be a non-empty, complete metric space, and let $\phi : M \to M$ be a contraction. Then ϕ has a unique fixed point, i.e., there is a unique $u \in M$ such that $\phi(u) = u$.*

Proof. Uniqueness follows easily from nondegeneracy of the metric. For existence, starting with some $u_0 \in M$, define the sequence $\{u_n\}$ by $u_n = \phi(u_{n-1})$ for $n = 1, 2, \ldots$. Then this sequence is Cauchy, because

$$\rho(u_n, u_{n+1}) = \rho(\phi(u_{n-1}), \phi(u_n)) \leq k\rho(u_{n-1}, u_n) \leq \ldots \leq k^n\rho(u_0, u_1), \tag{31}$$

and so

$$\rho(u_n, u_m) \leq \rho(u_n, u_{n+1})+\ldots+\rho(u_{m-1}, u_m) \leq (k^n+\ldots+k^{m-1})\rho(u_0, u_1) \leq \frac{k^n\rho(u_0, u_1)}{1-k}, \tag{32}$$

for $n < m$. Since M is complete, there is $u \in M$ such that $u_n \to u$, which is a good candidate for the fixed point we are looking for. Indeed, we have

$$\rho(u, \phi(u)) \leq \rho(u, u_n) + \rho(\phi(u_{n-1}), \phi(u)) \leq \rho(u, u_n) + k\rho(u_{n-1}, u) \to 0, \tag{33}$$

as $n \to \infty$, showing that $u = \phi(u)$. □

Proof due to Dick Palais. The argument (32) can be replaced by the following more symmetric one. For any $u, v \in M$, we have

$$\rho(u, v) \leq \rho(u, \phi(u)) + \rho(\phi(u), \phi(v)) + \rho(\phi(v), v) \leq \rho(u, \phi(u)) + k\rho(u, v) + \rho(\phi(v), v), \tag{34}$$

implying that

$$(1 - k)\rho(u, v) \leq \rho(u, \phi(u)) + \rho(\phi(v), v). \tag{35}$$

Using this in combination with (31), we get

$$(1 - k)\rho(u_n, u_m) \leq \rho(u_n, u_{n+1}) + \rho(u_{m+1}, u_m) \leq (k^n + k^m)\rho(u_0, u_1), \tag{36}$$

hence $\{u_n\}$ is Cauchy. □

Remark 6. If the map ϕ depends on some parameter continuously, then the fixed point obtained by the preceding theorem also depends continuously on the parameter. This can be seen as follows. Let $u_s = \phi_s(u_s)$ and $u_t = \phi_t(u_t)$, with both ϕ_s and ϕ_t contractions. Then from the triangle inequality we have

$$\rho(u_s, u_t) \leq \rho(\phi_s(u_s), \phi_s(u_t)) + \rho(\phi_s(u_t), \phi_t(u_t)) \leq k\rho(u_s, u_t) + \rho(\phi_s(u_t), \phi_t(u_t)), \tag{37}$$

hence

$$(1 - k)\rho(u_s, u_t) \leq \rho(\phi_s(u_t), \phi_t(u_t)), \tag{38}$$

meaning that if $\phi_s(u_t)$ and $\phi_t(u_t)$ are close then so are u_s and u_t.

Remark 7. The continuous dependence on parameters can be updated to differentiability if there is enough structure. Let us assume that M is a closed subset of a Banach space X, and let $\{\phi_t\}$ be a one-parameter family of contractions on M, with the contraction factor k bounded away from 1 uniformly in t. Suppose that $u_t = \phi_t(u_t)$ are the fixed points and that u_0 is a point interior to M. We also assume that $\phi_t(u)$ is differentiable at $\{t = 0, u = u_0\}$ in the sense that there exist $\lambda \in X$ and a bounded linear operator $\Lambda : X \to X$ such that

$$\phi_t(u_0 + tv) = \phi_0(u_0) + \lambda t + t\Lambda v + o(|t|), \tag{39}$$

as $t \to 0$. For $t \neq 0$ small, the difference quotients Δ_t satisfy

$$\Delta_t := \frac{u_t - u_0}{t} = \frac{\phi_t(u_0 + t\Delta_t) - \phi_0(u_0)}{t} =: \Phi_t(\Delta_t). \tag{40}$$

The map Φ_t is a contraction on its domain because

$$|t| \cdot \|\Phi_t(x) - \Phi_t(y)\| \leq \|\phi_t(u_0 + xt) - \phi_t(u_0 + yt)\| \leq k|t| \cdot \|x - y\|. \tag{41}$$

Moreover, from the differentiability condition we have

$$\Phi_t(x) \to \lambda + \Lambda x, \qquad \text{as} \quad t \to 0, \tag{42}$$

for any $x \in X$, so $\Phi_0(x) = \lambda + \Lambda x$ is a contraction by continuity. Hence there exists a fixed point $\Delta_0 \in X$ of Φ_0 with $\Delta_t \to \Delta_0$ as $t \to 0$, and of course Δ_0 is the derivative of u_t with respect to t at $t = 0$. We can continue in this manner. For instance, the continuity of the derivative Δ_0 with respect to some parameter would follow if λ and Λ depend continuously on that parameter.

3. Inverse function theorem

Let X and Z be Banach spaces, and let $U \subset X$ be an open set. Then a mapping $f : U \to Z$ is called *Fréchet differentiable* at $x \in U$ if

$$f(x + h) = f(x) + \Lambda h + o(\|h\|), \qquad \text{as} \quad X \ni h \to 0, \tag{43}$$

for some bounded linear operator $\Lambda : X \to Z$. We call $Df(x) = \Lambda$ if it exists, the *Fréchet derivative* of f at x. The following results are fundamental.

Theorem 8 (Inverse function theorem). *Suppose that Df exists and is continuous in U, and that $Df(x)$ is invertible. Then there is an open neighbourhood of x on which f is invertible, and the inverse f^{-1} is C^1.*

The proof is left as an exercise. *Hint*: Given $z \in Z$ close to $f(x)$, consider the map $\phi : U \to X$ defined by $\phi(y) = y + [Df(x)]^{-1}(z - f(y))$.

The following corollary can be obtained by applying the inverse function theorem to the function $f(x, y) = (x, g(x, y))$.

Corollary 9 (Implicit function theorem). *Let X, Y and Z be Banach, and with $A \subset X \times Y$ an open set, let $g : A \to Z$ be a continuously differentiable mapping. Moreover, assume that the point $(a, b) \in A$ has the property that $g(a, b) = 0$ and that $D_y g(a, b)$ is invertible, where $D_y g(x, y)$ is the Fréchet derivative of $y \mapsto g(x, y)$, with fixed x. Then there is an open set $U \subset X$ and a function $h : U \to Y$ with $h(a) = b$, such that $g(x, h(x)) = 0$ for all $x \in U$.*

There is a weaker and more easily accessible derivative, called Gâteaux derivative, that plays an important role in calculus of variations. This is simply the Banach space version of the directional derivative. We define the *Gâteaux differential of $f : U \to Z$ at $x \in U$ in the direction $a \in X$*, to be $Df(z, a) = g'(0)$ if the latter exists, where

$$g(t) = f(x + at), \tag{44}$$

is a function of $t \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$. Assuming that $Df(x, a)$ exists for all $a \in X$ and for all $x \in U$, the totality of all Gâteaux differentials of f defines a function $(x, a) \mapsto Df(x, a)$ on $U \times X$. There is no obvious *a priori* structure on this function, except to say that $Df(x, a)$ is homogeneous in a, i.e., $Df(x, at) = tDf(x, a)$ for any $t \in \mathbb{R}$.

If f is Gâteaux differentiable at x and the map $a \mapsto Df(x, a) : X \to Z$ is a bounded linear map, we call it the *Gâteaux derivative* of f at x. Obviously, Fréchet differentiability implies the existence of the Gâteaux derivative. In the other direction, we have the following result, which gives a practical way to get a handle on Fréchet derivatives.

Lemma 10. *Suppose that the Gâteaux differential of f exists at each $x \in U$ as a bounded linear map $A(x) : X \to Z$, and that $A : U \to B(X, Z)$ is continuous, where $B(X, Z)$ is the set of bounded linear maps between X and Z, with its operator norm topology. Then f is Fréchet differentiable in U, with $Df = A$.*

Proof. We assume $0 \in U$ and will show that f is Fréchet differentiable at $0 \in U$. The continuity of $A : U \to B(X, Z)$ at 0 means that for any $\varepsilon > 0$, there is a small ball $B_\delta \subset U$ centred at 0, such that $x \in B_\delta$ implies

$$\|A(x)y - A(0)y\| \leq \varepsilon \|y\|, \tag{45}$$

for all $y \in X$. Pick $\varepsilon > 0$, let $x \in B_\delta$ with δ as above, and define $g(t) = f(xt)$. We compute

$$g'(t) = \lim_{\varepsilon \to 0} \frac{f(xt + x\varepsilon) - f(xt)}{\varepsilon} = Df(xt, x) = A(xt)x, \tag{46}$$

for $0 \leq t \leq 1$. We have

$$f(x) = g(1) = f(0) + \int_0^1 g'(t)\,\mathrm{d}t = f(0) + A(0)x + \int_0^1 [A(xt)x - A(0)x]\,\mathrm{d}t, \tag{47}$$

and hence

$$\|f(x) - f(0) - A(0)x\| \leq \sup_{1 \leq t \leq 1} \|A(xt)x - A(0)x\| \leq \varepsilon \|x\|, \tag{48}$$

showing that f is Fréchet differentiable at 0 with $Df(0) = A(0)$. Note that we could have set $f(0) = 0$ and $A(0) = 0$ in the beginning to simplify the formulas. □

Example 11. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We apply the preceding lemma to show that

$$f(u) = \int_\Omega |u|^p, \tag{49}$$

is Frećhet differentiable in the space $L^p(\Omega)$ with $2 < p < \infty$. First, we have to compute the Gâteaux derivative. Since $|u + t\psi|^p = |u|^p + tp|u|^{p-2}u\psi + o(t)$ pointwise, we expect the Gâteaux derivative to be

$$Df(u, \psi) = p \int_\Omega |u|^{p-2}u\psi. \tag{50}$$

We need to justify the limit when $\psi \in L^p(\Omega)$. From Taylor's theorem we have

$$|u + t\psi|^p = |u|^p + tp|u|^{p-2}u\psi + \tfrac{1}{2}t^2 p(p-1)|u + st\psi|^{p-2}\psi^2, \tag{51}$$

where $0 \leq s \leq 1$ is a function on Ω. We can use the inequality

$$(a + b)^\alpha \leq \max\{2^{1-\alpha}, 2^\alpha\}(a^\alpha + b^\alpha) \qquad \text{for} \qquad a, b \geq 0, \quad \alpha > 0, \tag{52}$$

to estimate

$$\tfrac{1}{2}t^2 p(p-1)|u + st\psi|^{p-2}\psi^2 \leq C(t^2|u|^{p-2}\psi^2 + |t|^p|\psi|^p), \tag{53}$$

with some constant $C > 0$. This implies that

$$\begin{aligned} \Big|\int_\Omega \big(\frac{|u+t\psi|^p - |u|^p}{t} - p|u|^{p-2}u\psi\big)\Big| &\leq C\int_\Omega (|t||u|^{p-2}\psi^2 + |t|^{p-1}|\psi|^p) \\ &\leq C|t|\|u\|_p^{p-2}\|\psi\|_p^2 + C|t|^{p-1}\|\psi\|_p^p, \end{aligned} \tag{54}$$

proving that (50) is indeed the Gâteaux derivative of f. Here $\|\cdot\|_q$ denotes the L^q-norm, and in the last line we have used the Hölder inequality with the exponents $\frac{p}{p-2}$ and $\frac{p}{2}$.

For fixed $u \in L^p(\Omega)$, the map $\psi \mapsto Df(u,\psi)$ is clearly linear, and bounded on L^p since

$$|Df(u,\psi)| \leq p\|u\|_p^{p-1}\|\psi\|_p, \tag{55}$$

by the Hölder inequality with the exponents $\frac{p}{p-1}$ and p.

Now we will show the continuity of $u \mapsto Df(u,\cdot) : L^p \to (L^p)'$, with the latter space taken with its norm topology. For any constant $a > 0$, the function $g(x) = |x|^a x$ is continuously differentiable with

$$g'(x) = (a+1)|x|^a, \tag{56}$$

implying that

$$\big||x|^a x - |y|^a y\big| \leq (a+1)\Big|\int_x^y |t|^a \mathrm{d}t\Big| \leq (a+1)\max\{|x|^a, |y|^a\}|x-y|. \tag{57}$$

Using this, we have

$$\begin{aligned} |f(u,\psi) - f(v,\psi)| &\leq p\int_\Omega \big||u|^{p-2}u - |v|^{p-2}v\big| \cdot |\psi| \\ &\leq p(p-1)\int_\Omega \left(|u|^{p-2} + |v|^{p-2}\right)|u-v|\cdot|\psi|. \end{aligned} \tag{58}$$

Finally, it follows from the Hölder inequality with the exponents $\frac{p}{p-2}$, p, and p that

$$|f(u,\psi) - f(v,\psi)| \leq p(p-1)\left(\|u\|_p^{p-2} + \|v\|_p^{p-2}\right)\|u-v\|_p \cdot \|\psi\|_p, \tag{59}$$

which establishes the claim.

4. Ordinary differential equations

Let X be a Banach space, and let $f : X \to X$ be a continuous map. We consider the *initial value problem*

$$\begin{cases} u'(t) = f(u(t)) & t \in I, \\ u(0) = x, \end{cases} \tag{60}$$

where $I \subset \mathbb{R}$ is an interval containing 0, and $x \in X$ is given, called the *initial datum*.

Definition 12. A *classical solution* of (60) is a function $u \in C^1(I,X)$ satisfying (60). On the other hand, a *strong solution* of (60) is a function $u \in C(I,X)$ satisfying

$$u(t) = x + \int_0^t f(u(s))\,\mathrm{d}s, \qquad t \in I. \tag{61}$$

Lemma 13. *A function $u \in C(I,X)$ is a strong solution of (60) if and only if it is a classical solution of (60).*

Proof. Suppose that $u \in C^1(I,X)$ is a classical solution. Then for any $0 \leq t \in I$, we have $u \in C^1([0,t],X)$, and hence by the fundamental theorem of calculus (Theorem 4a)

$$u(t) - x = u(t) - u(0) = \int_0^t u'(s)\,\mathrm{d}s = \int_0^t f(u(s))\,\mathrm{d}s. \tag{62}$$

Similarly, for any $0 \geq t \in I$, we have $u \in C^1([t, 0], X)$, and so

$$x - u(t) = u(0) - u(t) = \int_t^0 u'(s)\,\mathrm{d}s = -\int_0^t f(u(s))\,\mathrm{d}s. \tag{63}$$

Now suppose that $u \in C(I, X)$ is a strong solution. Then obviously $u(0) = x$, and since $f \circ u \in C(I, X)$, by Theorem 4b) we have $u \in C^1(I, X)$ and $u' = f \circ u$ on I. □

Now we want to establish local existence and uniqueness theorems for the initial value problem (60). To this end, we assume that f is *locally Lipschitz*, meaning that for any $R > 0$ there is $C_R > 0$ such that

$$\|f(x) - f(y)\| \leq C_R \|x - y\|, \qquad x, y \in B_R, \tag{64}$$

where $B_R = \{z \in X : \|x\| < R\}$.

Theorem 14 (Local existence, Picard-Lindelöf)**.** *Assume that f is locally Lipschitz. Then for any $r > 0$ there exists an open interval $I \ni 0$ such that as long as the initial datum satisfies $\|x\| \leq r$, the initial value problem (60) has a strong solution $u \in C(I, X)$.*

Proof. With $T > 0$, we consider the map $\phi : C([-T, T], X) \to C([-T, T], X)$, defined by

$$\phi(u)(t) = x + \int_0^t f(u(s))\,\mathrm{d}s, \qquad t \in [-T, T]. \tag{65}$$

Obviously, u is a strong solution on $[-T, T]$ iff $u = \phi(u)$. Our plan is to show that ϕ is a contraction on some closed subset of $C([-T, T], X)$ so that we can apply the Banach fixed point theorem. Suppose that $u, v \in U_R$, where $U_R = \{u \in C([-T, T], X) : \|u\|_\infty \leq R\}$ for some $R > 0$. Then we have

$$\|\phi(u) - \phi(v)\|_\infty \leq T\|f(u) - f(v)\|_\infty \leq TC_R\|u - v\|_\infty, \tag{66}$$

and similarly

$$\begin{aligned} \|\phi(u)\|_\infty &\leq \|x\| + T\|f(u)\|_\infty \leq r + T\|f(u) - f(0)\|_\infty + T\|f(0)\| \\ &\leq r + TC_R\|u\|_\infty + T\|f(0)\| \leq r + TC_R R + T\|f(0)\|. \end{aligned} \tag{67}$$

We see that for any $R > r$, by choosing $T > 0$ small enough we can ensure that $\phi(u) \in U_R$ for $u \in U_R$, and that ϕ is a contraction on U_R. Since U_R is a closed subset of a Banach space, it is a complete metric space (with the metric induced by the norm), and hence an application of the Banach fixed point theorem gives the existence of $u \in U_R$ satisfying $u = \phi(u)$. □

Remark 15. Since the fixed point map (65) depends on the initial datum x in a most straightforward manner, by Remark 6 the solution obtained in the Picard-Lindelöf theorem depends continuously on the initial datum.

Remark 16. If f is a C^1 function, for any fixed t, one can get continuous differentiability of $u(t)$ as a function of the initial datum x, by using an argument similar to Remark 7.

For the uniqueness result, we need the following important inequality.

Lemma 17 (Gronwall's inequality)**.** *Let y and b be continuous functions on $[0, T]$ with $b \geq 0$, and let A be a real constant. Assume*

$$y(t) \leq A + \int_0^t b(s)y(s)\,\mathrm{d}s, \qquad t \in [0, T]. \tag{68}$$

Then we have

$$y(t) \leq A \exp \int_0^t b(s)\,\mathrm{d}s, \qquad t \in [0, T]. \tag{69}$$

Proof. Let

$$g(t) = A + \int_0^t b(s)y(s)\,\mathrm{d}s, \qquad \text{and} \qquad z(t) = g(t)\exp\big(-\int_0^t b(s)\,\mathrm{d}s\big). \tag{70}$$

Then we have

$$g'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(A + \int_0^t b(s)y(s)\,\mathrm{d}s\big) = b(t)y(t) \leq b(t)\big(A + \int_0^t b(s)y(s)\,\mathrm{d}s\big) = b(t)g(t), \tag{71}$$

hence

$$z'(t) = g'(t)\exp\big(-\int_0^t b(s)\,\mathrm{d}s\big) - g(t)\exp\big(-\int_0^t b(s)\,\mathrm{d}s\big)b(t) \leq 0. \tag{72}$$

This implies $z(t) \leq A$ and

$$y(t) \leq g(t) \leq A\exp\int_0^t b(s)\,\mathrm{d}s, \tag{73}$$

which completes the proof. □

Theorem 18 (Uniqueness)**.** *If $I_1 \ni 0$ and $I_2 \ni 0$ are two intervals and $u_1 \in C(I_1, X)$ and $u_2 \in C(I_2, X)$ are two strong solutions of* (60)*, then $u_1 = u_2$ on $I_1 \cap I_2$.*

Proof. Let $T > 0$ be such that $T \in I_1 \cap I_2$. Then for $t \in [0, T]$ we have

$$\|u_1(t) - u_2(t)\| = \big\|\int_0^t (f(u_1(s)) - f(u_2(s)))\,\mathrm{d}s\big\| \leq C\int_0^t \|u_1(s) - u_2(s)\|\,\mathrm{d}s, \tag{74}$$

where C is the Lipschitz constant of f on B_R with $R = \sup_{0\leq t\leq T} \max\{\|u_1(t)\|, \|u_2(t)\|\}$. Now Gronwall's inequality applied to $y(t) = \|u_1(t) - u_2(t)\|$ gives $u_1 - u_2 = 0$ on $[0, T]$. The case $T < 0$ is the same because under the substitution $v(t) = u(-t)$, the problem $u' = f(u)$ becomes $v' = -f(v)$. □

Definition 19. Fix some initial datum $x \in X$, and let $\{I_\alpha\}$ be the set of all intervals such that there is a solution $u_\alpha \in C(I_\alpha, X)$ of the initial value problem (60), where α runs over some index set. Then the *maximal interval of existence* of (60) corresponding to the initial datum x, is defined by $I = I(x) = \bigcup_\alpha I_\alpha$, and the *maximal solution* $u \in C(I, X)$ is given by

$$u(t) = u_{\alpha_t}(t), \qquad t \in I, \tag{75}$$

where $\alpha_t \in \{\alpha : t \in I_\alpha\}$ for each $t \in I$.

Note that for each $t \in I$, the set $\{\alpha : t \in I_\alpha\}$ is nonempty, hence the existence of α_t satisfying $\alpha_t \in \{\alpha : t \in I_\alpha\}$ is guaranteed by the axiom of choice. Moreover, u is well-defined thanks to Theorem 18, and u solves (60) since it agrees with solutions everywhere.

Theorem 20 (Blow-up criterion)**.** *The maximal interval of existence is necessarily open, i.e., it has the form $I = (a, b)$ for some $-\infty \leq a < 0 < b \leq \infty$. If $b < \infty$, then $\|u(t)\| \to \infty$ as $t \nearrow b$, and similarly, if $a > -\infty$, then $\|u(t)\| \to \infty$ as $t \searrow a$.*

Proof. Suppose that $I \cap [0, \infty) = [0, b]$ for some $b > 0$. Then $\|u(b)\| < \infty$ by continuity. Hence the initial value problem

$$v' = f(v), \qquad v(0) = u(b), \tag{76}$$

has a solution on some $[-\varepsilon, \varepsilon]$ with $\varepsilon > 0$. The function $t \mapsto u(b + t)$ satisfies the same equation for $t \leq 0$ with the same initial datum, so $u(b + t) = v(t)$ for $-\varepsilon \leq t \leq 0$. This means that if we define

$$w(t) = \begin{cases} u(t) & t \leq b, \\ v(t - b) & b < t \leq b + \varepsilon, \end{cases} \tag{77}$$

it solves (60) on $[0, b + \varepsilon]$, which contradicts the maximality of b.

The second part can be proven by essentially the same argument. Suppose that $b < \infty$, and let $\{t_k\}$ be a sequence satisfying $t_k \nearrow b$. Assume $\|u(t_k)\| \leq r$ for some constant $r > 0$. Then by the local existence theorem, u can be continued up to time $t_k + T$ for each k, with $T > 0$ independent of k. Finally, choosing k large enough that $t_k + T > b$, we contradict the maximality of b. □

Example 21. Let $K \in L^1(\mathbb{R})$, and let $A : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ be defined by

$$(Au)(x) = \int_{\mathbb{R}} K(y)u(x - y)\,\mathrm{d}y. \tag{78}$$

The map A is Lipschitz, since

$$\|Au - Av\|_\infty \leq \|K\|_{L^1(\mathbb{R})}\|u - v\|_\infty, \qquad u, v \in C_b(\mathbb{R}). \tag{79}$$

Therefore, given any $u_0 \in C_b(\mathbb{R})$, there is a unique maximal solution $u \in C^1(I, C_b(\mathbb{R}))$ of the initial value problem

$$u' = Au, \qquad u(0) = u_0, \tag{80}$$

with $I = (a, b)$. We want to prove that $I = \mathbb{R}$, i.e., the global solvability of the problem, by using the blow-up criterion in Theorem 20. Suppose that (80) has a solution on $[0, b)$. Then for $t \in [0, b)$, we have

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + \int_0^t \|Au(s)\|_\infty\,\mathrm{d}s \leq \|u_0\|_\infty + \|K\|_{L^1}\int_0^t \|u(s)\|_\infty\,\mathrm{d}s, \tag{81}$$

which in combination with Gronwall's inequality implies that

$$\|u(t)\|_\infty \leq \|u_0\|_\infty \exp(t\|K\|_{L^1}) \leq \|u_0\|_\infty \exp(b\|K\|_{L^1}). \tag{82}$$

The right hand side is clearly finite for any $b < \infty$, so it cannot happen that $\|u(t)\|_\infty \to \infty$ as $t \nearrow b$. By Theorem 20 this means that $b = \infty$.

Given initial datum $x \in X$, let $u_x \in C(I_x, X)$ be the maximal solution of the initial value problem (60), where I_x is the maximal interval of existence corresponding to x. Then we define the *flow map* to be $\Phi_t(x) = u_x(t)$ for $t \in I_x$.

Theorem 22 (Group property of flow maps)**.** *We have*

$$\Phi_0 = \mathrm{id}, \qquad \textit{and} \qquad \Phi_{t+s} = \Phi_t \circ \Phi_s, \tag{83}$$

whenever they make sense.

Proof. We will prove $\Phi_{t+s} = \Phi_t \circ \Phi_s$. Let $x \in X$ and let $t, s, t+s \in I_x$. Denote $u(t) = \Phi_{t+s}(x)$ and $v(t) = \Phi_t(\Phi_s(x))$. Then we have

$$u'(t) = \frac{\mathrm{d}\Phi_{t+s}(x)}{\mathrm{d}t} = \frac{\mathrm{d}\Phi_{t+s}(x)}{\mathrm{d}(t+s)} = f(\Phi_{t+s}(x)) = f(u(t)), \tag{84}$$

and $u(0) = \Phi_s(x)$. On the other hand, we have

$$v'(t) = \frac{\mathrm{d}\Phi_t(\Phi_s(x))}{\mathrm{d}t} = f(\Phi_t(\Phi_s(x))) = f(v(t)), \tag{85}$$

and $v(0) = \Phi_s(x)$. We see that u and v satisfy the same equations, with the same initial data. Hence by uniqueness, $u(t) = v(t)$. □

5. Semilinear equations

Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\alpha : \Omega \to \mathbb{R}^n$ be a vector field in Ω. We associate to α a differential operator $A = \sum_i \alpha_i \partial_i$. Then given a function $\beta : \Omega : \mathbb{R} \to \mathbb{R}$, we consider the first order equation

$$Au(x) \equiv \sum_i \alpha_i(x)\partial_i u(x) = \beta(x, u(x)), \qquad x \in \Omega. \tag{86}$$

We will solve it by the *method of characteristics*.

Definition 23. A *parametric characteristic curve* of A is a map $\gamma \in C^1(I_\gamma, \Omega)$ with some open interval $I_\gamma \subset \mathbb{R}$, satisfying

$$\gamma'(t) = \alpha(\gamma(t)), \qquad t \in I_\gamma. \tag{87}$$

We call the image $[\gamma] = \gamma(I_\gamma) \subset \Omega$ a *characteristic curve* of A.

It is immediate from the definition that if γ is a parametric characteristic curve of A and if u is differentiable at the points of $[\gamma]$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) = \sum_i \partial_i u \cdot \gamma_i'(t) = \sum_i \partial_i u \cdot \alpha_i(\gamma(t)) = Au(\gamma(t)), \tag{88}$$

for all $t \in I_\gamma$. This transforms the equation (86) into an ODE problem.

Lemma 24. *Let $\alpha \in C^1(\Omega, \mathbb{R}^n)$ and let u be differentiable in Ω. Then (86) holds if and only if*

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) = \beta(\gamma(t), u(\gamma(t))), \qquad t \in I_\gamma, \tag{89}$$

for every parametric characteristic curve γ of A.

Proof. If (86) holds and if γ is a parametric characteristic curve, then

$$\frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t)) = Au(\gamma(t)) = \beta(\gamma(t), u(\gamma(t))), \qquad t \in I_\gamma. \tag{90}$$

On the other hand, let $x \in \Omega$. Then by ODE theory there exists a parametric characteristic curve γ of A with $\gamma(0) = x$. Now if (89) holds then we have

$$Au(x) = Au(\gamma(t))|_{t=0} = \frac{\mathrm{d}u(\gamma(t))}{\mathrm{d}t}\Big|_{t=0} = \beta(\gamma(t), u(\gamma(t)))|_{t=0} = \beta(x, u(x)), \tag{91}$$

which completes the proof. □

The equation (89) means that once the value of u at some point of γ is fixed, then the value of u along γ is completely determined. Moreover, as long as u is differentiable, it can behave in an arbitrary fashion in directions transversal to the characteristic curves, since the lemma says that all (86) requires is (89). So we expect that the space of all solutions of (86) could be parameterized by a space of functions on a surface transversal to the characteristic curves. This leads us to the *Cauchy problem*: We are given a differentiable surface $\Gamma \subset \Omega$ and a function $g : \Gamma \to \mathbb{R}$, and consider the problem

$$\begin{cases} Au(x) = \beta(x, u(x)) & x \in \Omega, \\ u(\xi) = g(\xi) & \xi \in \Gamma. \end{cases} \tag{92}$$

We assume that Γ is *noncharacteristic*, meaning that $\alpha(\xi)$ is not tangent to Γ at any $\xi \in \Gamma$. The surface Γ is called the *Cauchy surface*, and g the *Cauchy data*.

For $\xi \in \Gamma$, let γ_ξ be the parametric characteristic curve with $\gamma_\xi(0) = \xi$, and let I_ξ be its maximal interval of existence (in Ω). Then we solve

$$\begin{cases} v_\xi'(t) = \beta(\gamma_\xi(t), v_\xi(t)), \\ v_\xi(0) = g(\xi). \end{cases} \tag{93}$$

It is possible that the maximal interval of existence of v_ξ is smaller than I_ξ. Let us call it $J_\xi \subset I_\xi$. We expect that the solution u should be given by the prescription

$$u(\gamma_\xi(t)) = v_\xi(t), \tag{94}$$

as ξ varies over Γ and t runs over J_ξ. However, there are some obstacles to our program:

- It may happen that one characteristic curve crosses the Cauchy surface Γ multiple times. In this case, there will be $x \in \Omega$ such that $x = \gamma_\xi(t) = \gamma_{\xi'}(t')$ with $\xi \neq \xi'$, and hence unless $g(\xi)$ is compatible with $g(\xi')$, there can exist no solutions.
- It may happen that some of the characteristic curves do not cross Γ. In this case there will be $x \in \Omega$ that is not reachable by any characteristic curve starting at Γ, and hence one cannot control the value of u at x. This leads to nonuniqueness.

In terms of the mapping $\varphi(\xi, t) = \gamma_\xi(t)$ that maps some region of the (ξ, t)-space into Ω, the first obstacle can be described as noninjectivity, while the second one is about nonsurjectivity. The following lemma says that in a certain sense those are the only obstacles.

Lemma 25. *Let Γ be a C^1 surface, and let g and α be C^1 functions. Let $\Sigma \subset \Gamma \times \mathbb{R}$ be an open set, such that $(\xi, t) \in \Sigma$ implies $t \in J_\xi$. Suppose that the mapping $\varphi : \Sigma \to \Omega$, defined by $\varphi(\xi, t) = \gamma_\xi(t)$, is invertible and that the inverse $(\xi, t) := \varphi^{-1} : \Omega \to \Sigma$ is C^1. Then the function $u(x) = v_{\xi(x)}(t(x))$ for $x \in \Omega$ solves the Cauchy problem (92).*

Proof. If $\eta \in \Gamma$ then $\xi(\eta) = \eta$ and $t(\eta) = 0$, so $u(\eta) = v_\eta(0) = g(\eta)$. If $x \in \Omega$, then by assumption there is a unique pair (ξ, t) with $\xi \in \Gamma$ and $t \in J_\xi$. Moreover, by ODE uniqueness theory any parametric characteristic curve going through x agrees with γ_ξ, up to a time translation. Hence Lemma 24 in combination with (93) guarantees that u solves the Cauchy problem, provided that u is differentiable in Ω. But u is C^1, as follows from ODE theory, since $\xi(x)$ and $t(x)$ are both C^1 functions by assumption. □

If we have local invertibility, then global injectivity would guarantee global invertibility. While in general it is hard to say anything about global injectivity, local invertibility can be approached through the inverse function theorem.

Theorem 26. *Let Γ be a noncharacteristic C^1 surface, and let g and α be C^1 functions.*

a) Let $\xi \in \Gamma$. Then there exists a neighbourhood $U \subset \Omega$ of ξ, such that the Cauchy problem (92) with Ω replaced by U and Γ replaced by $\Gamma \cap U$, has a unique solution.

b) Let $\Sigma \subset \Gamma \times \mathbb{R}$ be an open set, such that $(\xi, t) \in \Sigma$ implies $t \in J_\xi$. Suppose that the mapping $\varphi : \Sigma \to \Omega$, defined by $\varphi(\xi, t) = \gamma_\xi(t)$, is injective, and let $(\xi, t) := \varphi^{-1} : U \to \Sigma$ be its inverse on $U = \varphi(\Sigma)$. Then the function $u(x) = v_{\xi(x)}(t(x))$ for $x \in U$ solves the Cauchy problem (92), and it is unique in U.

Proof. First of all, from ODE existence theory it is clear that there exists an open neighbourhood of $\Gamma \times \{0\}$ in $\Gamma \times \mathbb{R}$, on which φ is well defined. Let (ξ, t) be a point in that neighbourhood, and let us compute the derivative of φ at (ξ, t). Note that $\varphi(\xi, t) = \Phi_t(\xi)$ for $\xi \in \Gamma$, where Φ_t is the flow map of the vector field α. We have $\Phi_{t+s}(\xi) = \Phi_t(\Phi_s(\xi))$ for all small s. Differentiating this with respect to s gives

$$\frac{\partial}{\partial t}\Phi_{t+s}(\xi) = \frac{\partial}{\partial s}\Phi_{t+s}(\xi) = D\Phi_t(\Phi_s(\xi))\frac{\partial}{\partial s}\Phi_s(\xi), \tag{95}$$

and putting $s = 0$, we get

$$\frac{\partial}{\partial t}\Phi_t(\xi) = D\Phi_t(\xi)\alpha(\xi). \tag{96}$$

It follows that

$$\Phi_{t+s}(\xi+\eta) = D\Phi_t(\xi)\eta + D\Phi_t(\xi)\alpha(\xi)s + o(|\eta| + |s|), \tag{97}$$

for small $\eta \in \mathbb{R}^n$ and $s \in \mathbb{R}$, in particular showing that $\varphi \in C^1$. Moreover, by restricting $\xi \in \Gamma$ and $\eta \in T_\xi\Gamma$, the latter the tangent space of Γ at ξ, it tells us that

$$D\varphi(\xi,t)(\eta,s) = D\Phi_t(\xi)(\eta + \alpha(\xi)s), \tag{98}$$

with which we mean the derivative $D\varphi$ at (ξ,t) applied to the vector (η,s). Since $D\Phi_t(\xi)$ is invertible, the invertibility of $D\varphi(\xi,t)$ boils down to checking if one can recover the pair (η,s) uniquely from $\eta+\alpha(\xi)s$. But this is guaranteed by the noncharacteristic condition $\alpha(\xi) \notin T_\xi\Gamma$.

Part a) is established, since we can conclude that there is a neighbourhood of $(\xi,0)$ in $\Gamma\times\mathbb{R}$ such that the mapping φ on that neighbourhood is continuously differentiable with $D\varphi(\xi,0)$ invertible. Then the inverse function theorem would finish the job.

Part b) is also done, because we have local invertibility of φ by the inverse function theorem, which is updated to global invertibility by the injectivity assumption. □

Remark 27. When can we take $\Sigma = \bigcup_{\xi\in\Gamma}\{\xi\}\times J_\xi$, i.e., is it possible to include in U all points in Ω that are reachable by characteristics starting at Γ? This would be possible if characteristics starting at Γ do not return to Γ again. Indeed, if $\varphi(\xi,t) = \varphi(\eta,s)$, from ODE uniqueness theory we would have $\eta = \varphi(\xi,t-s)$. One case where characteristic are guaranteed not to return to the Cauchy surface is the case of *evolution equations*, which read

$$\partial_n u(x) + \sum_{i=1}^{n-1}\alpha_i(x)\partial_i u(x) = \beta(x,u(x)), \qquad x\in\Omega, \tag{99}$$

with the Cauchy surface $\Gamma = \Omega\cap\{x_n = 0\}$. The coordinate x_n is interpreted as time, and since $\alpha_n \equiv 1$, we see that for any parametric characteristic curve, we have $\gamma_n(t) = t$. Hence $\gamma(t)\in\Gamma$ necessarily implies $t = 0$.

6. Quasilinear and fully nonlinear equations

By allowing the vector field α to depend on the function value u as well, we arrive at the following *quasilinear equation*

$$\sum_i \alpha_i(x,u(x))\partial_i u(x) = \alpha_{n+1}(x,u(x)), \qquad x\in\Omega, \tag{100}$$

where $\alpha = (\alpha_1,\ldots,\alpha_{n+1})$ is considered as a vector field defined on some domain $\tilde\Omega\subset\mathbb{R}^{n+1}$. Now it is no longer possible to solve for the characteristics and the solution value separately, as in (87) and (93). Instead, we need to solve them simultaneously, resulting in curves that live in the graph of the solution. A *graph characteristic* of α is a map $\gamma\in C^1(I_\gamma,\tilde\Omega)$ with some open interval $I_\gamma\subset\mathbb{R}$, satisfying

$$\gamma'(t) = \alpha(\gamma(t)), \qquad t\in I_\gamma. \tag{101}$$

Exercise 2. Let $\alpha\in C^1(\tilde\Omega,\mathbb{R}^{n+1})$, and let u be differentiable in Ω. Suppose that the graph of u is a subset of $\tilde\Omega$. Then u is a solution of (100) if and only if every graph characteristic that starts at a point on the graph of u stays in the graph of u for at least a short time.

Given a Cauchy surface $\Gamma\subset\Omega$ and initial datum $g:\Gamma\to\mathbb{R}$, the *Cauchy problem* takes the form

$$\begin{cases}\sum_i \alpha_i(x,u(x))\partial_i u(x) = \beta(x,u(x)) & x\in\Omega,\\ u(\xi) = g(\xi) & \xi\in\Gamma,\end{cases} \tag{102}$$

and *noncharacteristicity* of Γ reads as the vector $(\alpha_1(\xi, g(\xi)), \ldots, \alpha_n(\xi, g(\xi))) \in \mathbb{R}^n$ being not tangent to Γ at any $\xi \in \Gamma$.

In addition to the difficulties we had for semilinear equations with regard to global solvability, we encounter here a new obstacle that is caused by potential multi-valuedness of the function defined by the graph characteristics. Let us illustrate this phenomenon by *Burgers' equation*

$$\partial_t u + u\partial_x u = 0. \tag{103}$$

Assume that the initial datum $g \in C^1(\mathbb{R})$ is given. We can take the t coordinate as the parameter in the characteristic equations, since the equation for t would be $t' = 1$. Hence the characteristic equations are

$$\begin{cases} x'(t) = z(t), \\ z'(t) = 0, \end{cases} \tag{104}$$

where we decomposed the graph characteristics γ as $\gamma = (x, z)$. The initial conditions for the graph characteristic starting at $\xi \in \mathbb{R}$ are $x(0) = \xi$ and $z(0) = g(\xi)$. This is easily solvable and the solution is $x_\xi(t) = \xi + g(\xi)t$ and $z_\xi(t) = g(\xi)$. So the graph characteristics are straight lines orthogonal to the u-axis, with slopes in the xt-plane equal to their u-coordinate, and the solution can be written implicitly as $u(\xi + g(\xi)t) = g(\xi)$ wherever it is well-defined. Let $U(t) = \{(x, u(x,t)) : x \in \mathbb{R}\}$ be the graph of u at the time moment t. Then from the implicit formula we see that $U(t) = A(t)U(0)$, where $A(t)$ is the linear transformation of the (x, u) plane given by

$$A(t)(x, u) = (x + ut, u). \tag{105}$$

It shows that the points with higher u-coordinates move to the right with faster rate than the points with lower u-coordinates. Thus if g has a region where it decreases, as t grows the graph of g would become increasingly steep in that region, and eventually the upper part would go past the lower part of the graph, making itself a non-graph. This phenomenon is called *wave breaking* or *shock formation*. We will revisit it in the next section.

Exercise 3. Let Γ be a C^1 surface, and let g and α be C^1 functions. Assume that Γ is noncharacteristic, and assume that the graph of g is in $\tilde{\Omega}$. Then there exists an open neighbourhood of Γ on which there exists a unique C^1 solution of the Cauchy problem (102).

The most general form of a first order equation is

$$F(x, u(x), \nabla u(x)) = 0, \qquad x \in \Omega, \tag{106}$$

where F is a C^1 function defined on a domain $\tilde{\Omega}$ of $\mathbb{R}^{2n+1}$. Interestingly, this can also be solved by a method of characteristics, where the characteristics are now curves in $\mathbb{R}^{2n+1}$. Let us label the arguments of F by $x \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $p \in \mathbb{R}^n$, so that, for example, $\partial_z F$ means the derivative of $F(x, z, p)$ with respect to z, and $\nabla_p F$ is the vector consisting of the derivatives of $F(x, z, p)$ with respect to $p_1, \ldots, p_n$. Suppose that u solves (106), and that $x = x(t)$ is a curve in Ω. Let $z(t) = u(x(t))$ and $p(t) = \nabla u(x(t))$. We want to derive equations for the curve $(x(t), z(t), p(t))$. First, note that

$$z'(t) = \nabla u(x(t)) \cdot x'(t) = p(t) \cdot x'(t). \tag{107}$$

Next, differentiating (106) with respect to x_i gives

$$0 = \frac{\partial F}{\partial x_i} = \partial_i F + \partial_z F \cdot \partial_i u + \nabla_p F \cdot \partial_i \nabla u, \tag{108}$$

and we have

$$p_i'(t) = \partial_i \nabla u(x(t)) \cdot x'(t) = \partial_i \nabla u \cdot \nabla_p F = -\partial_i F - \partial_z F \cdot \partial_i u = -\partial_i F - \partial_z F \cdot p_i, \tag{109}$$

where we have made the choice $x'(t) = \nabla_p F$. Collecting the relevant equations, we conclude

$$\begin{cases} x' = \nabla_p F \\ z' = p \cdot \nabla_p F \\ p' = -\nabla_x F - p\,\partial_z F. \end{cases} \tag{110}$$

Now supposing that a Cauchy surface $\Gamma \subset \Omega$ and initial datum $g \in C^1(\Gamma)$ are given, we attempt to solve the Cauchy problem of (106) with the initial condition $u|_\Gamma = g$ in the following manner. For each $\xi \in \Gamma$, we solve the characteristic equations (110) with the initial data $x(0) = \xi$, $z(0) = g(\xi)$, and $p(0) = h(\xi)$, the latter given implicitly by

$$\tau \cdot \nabla g = h(\xi) \cdot \tau \quad \text{for all} \quad \tau \in T_\xi \Gamma, \qquad \text{and} \qquad F(\xi, g(\xi), h(\xi)) = 0. \tag{111}$$

The first condition fixes the component of $h(\xi)$ that is tangential to Γ, while the second condition is supposed to give the normal component. At this level of generality, however, the second equation may not be solvable for $h(\xi)$, or even if it is solvable, the solution may not be unique. So we simply assume that it is solvable, and for each ξ, one of the solutions is chosen, which we denote by $h(\xi)$. Given such data, *noncharacteristicity* of Γ is expressed as

$$\nabla_p F(\xi, g(\xi), h(\xi)) \notin T_\xi \Gamma. \tag{112}$$

Under the noncharacteristic condition, one can prove a local existence result for the Cauchy problem of (106) similar to the semilinear and quasilinear cases. We refer to Evans §3.2.4 for a detailed proof, and end this section with some examples.

Example 28. Consider the *eikonal equation*

$$|\nabla u|^2 = 1. \tag{113}$$

We take $F(x, z, p) = \frac{1}{2}(|p|^2 - 1)$, and so $\nabla_p F = p$, $\nabla_x F = 0$, and $\partial_z F = 0$. The characteristic equations are

$$x' = p, \qquad z' = 1, \qquad p' = 0. \tag{114}$$

Given an initial datum g on a Cauchy surface Γ, the initial data for the characteristic curve starting at $\xi \in \Gamma$ are $x(0) = \xi$, $z(0) = g(\xi)$, and $p(0) = \nabla_\Gamma g(\xi) + \nu$ with $|p(0)| = 1$, where $\nu \perp T_\xi \Gamma$. We see that if $|\nabla_\Gamma g(\xi)| \geq 1$ then there is no solution, and if $|\nabla_\Gamma g(\xi)| < 1$ there are two possible choices for $p(0)$ depending on the direction of ν.

Example 29. An important class of first order equations is *Hamilton-Jacobi equations*, that are of the form

$$\partial_t u + H(x, t, \nabla u) = 0, \tag{115}$$

where H is a function of $2n + 1$ variables, and t is singled out as a time coordinate. To compare it with the general form (106), we introduce $\tilde{x} = (x, t)$ and $\tilde{p} = (p, \tau)$. With these variables, we can write $F(\tilde{x}, \tilde{p}) = H(x, t, p) + \tau$, and we have

$$\nabla_{\tilde{p}} F = (\nabla_p H, 1), \qquad \nabla_{\tilde{x}} F = (\nabla_x H, \partial_t H), \qquad \text{and} \qquad \partial_z F = 0. \tag{116}$$

We can use t as the parameter in the characteristic curves, since its equation would be $t' = 1$. Moreover, we can eliminate τ from the characteristic equations because $\tau = \partial_t u = -H$ by the equation. The end result is

$$\begin{cases} x' = \nabla_p H \\ p' = -\nabla_x H \\ z' = p \cdot \nabla_p H - H. \end{cases} \tag{117}$$

Given an initial datum g on $\{t = 0\}$, the initial data for the characteristic curve starting at $\xi \in \mathbb{R}^n \times \{0\}$ are $x(0) = \xi$, $p(0) = \nabla_x g(\xi)$, and $z(0) = g(\xi)$.

7. Conservation laws

Let $F_\alpha : \mathbb{R}^m \to \mathbb{R}^n$ ($\alpha = 1, \ldots, m$) be differentiable functions, and let $u \in C^1(\Omega, \mathbb{R}^m)$ with some domain $\Omega \subset \mathbb{R}^n$. Then an *integral conservation law* is

$$\int_{\partial\omega} F_\alpha(u) = 0, \qquad \alpha = 1, \ldots, m, \tag{118}$$

that is supposed to hold for all bounded smooth domains ω such that $\overline{\omega} \subset \Omega$. By the divergence theorem, we can derive from it the differential *conservation law*

$$\nabla \cdot F_\alpha(u) = 0, \qquad \alpha = 1, \ldots, m. \tag{119}$$

As the name suggests, conservation laws express conservation of physical (or other) quantities, such as mass, charge, momentum, and energy. The vector fields F_α are then would correspond to the *fluxes* of those quantities.

Example 30 (Euler equations)**.** An example is given by the Euler equations in fluid dynamics, which read

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho u) &= 0, \\ \partial_t(\rho u_k) + \partial_k p + \nabla \cdot (\rho u_k u) &= 0, \qquad (k = 1, \ldots, n) \\ \partial_t E + \nabla \cdot ((E + p)u) &= 0, \end{aligned} \tag{120}$$

where ρ is the fluid density, u is the fluid velocity, p is the pressure, and E is the total energy density.

The simplest conservation laws are *scalar conservation laws in 1 dimension*

$$\partial_t u + \partial_x f(u) = \partial_t u + f'(u)\partial_x u = 0. \tag{121}$$

Burgers' equation (103) is an example with $f(u) = u^2/2$. As we have seen, they exhibit wave breaking. If we follow the graph characteristics beyond wave breaking, the solution would become multi-valued. From the viewpoint of conservation laws, this is not acceptable, as for instance, charge density cannot have many values at the same time. This discredits the use of graph characteristics beyond the time of wave breaking. On the other hand, as the shock develops (i.e., the graph becomes vertical at some point), the equation (121) no longer makes sense because of loss of differentiability. So we have to allow for a wider class of functions as solutions, and interpret the differentiation in (121) in a generalized sense.

Definition 31. A *weak solution* of (119) in Ω is a locally integrable function $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that $F(u)$ is also locally integrable in Ω, and (119) is satisfied in the sense of distributions.

Another justification of the preceding definition is the fact that $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is a weak solution of (119) if an only if it is a solution of the integral conservation law (118), which should be regarded as more fundamental than the differential law (119).

For piecewise smooth functions, one can derive an explicit criterion for determining if they are divergence free. This is important, since piecewise smooth functions are the first nondifferentiable functions encountered when one starts with a nice initial datum.

Lemma 32 (Rankine-Hugoniot condition)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $\Gamma \subset \Omega$ be an orientable C^1 hypersurface such that it cuts Ω into two disjoint domains Ω_1 and Ω_2. Suppose that $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is a vector field such that $f|_{\Omega_1} = f_1 \in C^1(\Omega_1) \cap C(\overline{\Omega}_1)$ and $f|_{\Omega_2} = f_2 \in C^1(\Omega_2) \cap C(\overline{\Omega}_2)$. Then $\nabla \cdot f = 0$ in the sense of distributions if and only if $\nabla \cdot f_1 = 0$ in Ω_1 and $\nabla \cdot f_2 = 0$ in Ω_2, and*

$$f_1 \cdot \nu = f_2 \cdot \nu \quad on \quad \Gamma, \tag{122}$$

where ν is a unit vector field normal to Γ.

Proof. First, note that if $\nabla\cdot f = 0$ as a distribution, then f_1 and f_2 are divergence free in Ω_1 and Ω_2, respectively. Hence for $\varphi \in \mathscr{D}(\Omega,\mathbb{R}^n)$ we have

$$\begin{aligned}\langle \nabla\cdot f,\varphi\rangle &= \int f\cdot\nabla\varphi = \int_{\Omega_1} f_1\cdot\nabla\varphi + \int_{\Omega_1} f_2\cdot\nabla\varphi\\ &= \int_\Gamma \varphi f_1\cdot\nu - \int_{\Omega_1}\varphi\nabla\cdot f_1 + \int_\Gamma \varphi f_2\cdot(-\nu) - \int_{\Omega_1}\varphi\nabla\cdot f_2\\ &= \int_\Gamma \varphi(f_1\cdot\nu - f_2\cdot\nu),\end{aligned} \tag{123}$$

where we chose ν to be the unit normal to Γ pointing outward from Ω_1. Since φ is arbitrary and $f_1 - f_2$ is continuous it follows that $f_1\cdot\nu = f_2\cdot\nu$ on Γ. The other direction of the equivalence can be proven by a similar reasoning. □

We shall apply this criterion to scalar conservation laws. Let $\gamma(t)$ be a C^1 curve at which a function $u(x,t)$ has discontinuity, i.e., $u \in L^1_{\text{loc}}(\Omega)$ with $u|_{\Omega^\pm} = u^\pm \in C^1(\Omega^\pm)\cap C(\overline{\Omega^\pm})$, where $\Omega\subset\mathbb{R}^2$ is a domain, and $\Omega^\pm = \{(x,t)\in\Omega : x \gtrless \gamma(t)\}$. In other words, the curve $\gamma(t)$ describes the evolution of a shock. The equation (121) says that the space-time divergence of the vector field $(f(u),u)$ must vanish. A space-time normal to the curve γ is given by $\nu = (1,-\gamma')$. Hence we need

$$f(u^-) - \gamma' u^- = f(u^+) - \gamma' u^+, \tag{124}$$

at the shock curve γ, which leads to the Rankine-Hugoniot condition

$$\gamma' = \frac{f(u^+) - f(u^-)}{u^+ - u^-}, \tag{125}$$

on the shock speed. This condition at the shocks, and the differential equation (121) everywhere else, completely characterize the weak solutions of (121), provided the solutions are piecewise smooth.

Remark 33 (Entropy conditions)*.* It turns out that in general weak solutions are not unique. So if we want uniqueness, we need to impose additional conditions that select one of the weak solutions as the admissible one. Note that such admissibility conditions may depend on the particular physics or other considerations behind the differential equation. In the case of conservation laws, though, it is generally accepted that the correct way to approach this problem is to treat (121) as the limit of a higher order equation, such as

$$\partial_t u + \partial_x f(u) = \varepsilon\partial_x^2 u, \qquad \text{as} \quad \varepsilon\searrow 0. \tag{126}$$

If the solution of the preceding equation has a limit as $\varepsilon\searrow 0$, we say it is the *viscosity solution* of (121). It is motivated by physics, where one considers (126) as more fundamental, and regard the conservation law (121) as a mathematical idealization of (126) when ε is small. Now, solving (126) for every $\varepsilon > 0$ and sending $\varepsilon\to 0$ is not always easy, so one wants to have practical criteria analogous to the Rankine-Hugoniot conditions that determine whether or not a given weak solution is a viscosity solution. This has led to many criteria, usually called *entropy conditions* or *admissibility conditions*, that separate admissible shocks from nonadmissible ones. We mention here a couple of those.

- The Lax condition: Characteristics run into the shock, not out from it.
- Stability: Shocks should not disintegrate under small perturbations.

Finally, it needs to be mentioned that conservation laws are an active field of research, and most of the rigorous results are established only for model cases. One dimensional scalar conservation laws are one such case where essentially everything is known. For more on conservation laws I recommend Joel Smoller's book, and Alberto Bressan's homepage.