AN ELEMENTARY INTRODUCTION TO DISTRIBUTIONS

TSOGTGEREL GANTUMUR

ABSTRACT. Textbooks on PDE usually introduce distributions as linear functionals satisfying certain properties, without saying much about where those conditions come from. The reason is that it would become a book by itself if one starts with the general setting of topological vector spaces. We take here an intermediate approach, that regards families of seminorms on vector spaces as the primary objects to generate topologies.

CONTENTS

1. INTRODUCTION

It is well known that differentiation of functions is not a well behaved operation. For instance, continuous, nowhere differentiable functions exist. The derivative of an integrable function maybe not locally integrable. A related difficulty is that if a sequence f_k converges to some function f pointwise or uniformly, then in general it is not true that f_k' converges to f' in the same sense. In order to use differentiation freely, one has to restrict to a class of functions that are many times differentiable, and in the extreme this process leads us to smooth and analytic classes. The latter classes alleviate the aforementioned difficulties somewhat, but they are too small and cumbersome for the purposes of studying PDEs. The idea behind *distributions* is that instead of restricting ourselves to a small subclass of functions, we should *expand* the class of functions to include hypothetical objects that are derivatives of ordinary functions. This will force us to extend the notion of functions, a process that is not dissimilar to extending the reals to complex numbers. The analogy can be pushed a bit further, in that by using distributions, we end up revealing deep and hidden truths even about ordinary functions that would otherwise be difficult to discover or could not be expressed naturally in the language of functions. A precise formulation of the theory of distributions was given by Laurent Schwartz during 1940's, with some crucial precursor ideas by Sergei Lvovich Sobolev.

To explain what distributions are, we start with a continuous function $u \in C(\mathbb{R})$ defined on the real line $\mathbb{R}$. Let $C_c^k(\mathbb{R})$ denote the space of k-times continuously differentiable functions with compact support, and define

$$T_u(\varphi) = \int u\varphi, \qquad \varphi \in C_c^k(\mathbb{R}). \tag{1}$$

We required φ to be compactly supported so that the above integral is finite for any continuous function u. It is clear that T_u is a linear functional acting on the space $C_c^k(\mathbb{R})$. Moreover,

Date: September 20, 2012.

this specifies u uniquely, meaning that if there is some $v \in C(\mathbb{R})$ such that $T_u(\varphi) = T_v(\varphi)$ for all $\varphi \in C_c^k(\mathbb{R})$, then $u = v$. If we replace the space $C(\mathbb{R})$ by the space $L^1_{\text{loc}}(\mathbb{R})$ of locally integrable functions, the conclusion would be that $u = v$ almost everywhere, which of course means that they are equal as the elements of $L^1_{\text{loc}}(\mathbb{R})$. So we can regard ordinary functions as linear functionals on $C_c^k(\mathbb{R})$. Then the point of departure now is to consider linear functionals that are not necessarily of the form (1) as *functions in a generalized sense*. For example, the *Dirac delta*, which is just the point evaluation

$$\delta(\varphi) = \varphi(0), \qquad \varphi \in C_c^k(\mathbb{R}), \tag{2}$$

is one such functional. In order to differentiate generalized functions, let us note that

$$T_{u'}(\varphi) = \int u'\varphi = -\int u\varphi' = -T_u(\varphi'), \qquad \varphi \in C_c^k(\mathbb{R}), \tag{3}$$

for any differentiable function u, and then make the observation that the right hand side actually makes sense even if u was just a continuous function. This motivates us to define the derivative of a generalized function T by

$$T'(\varphi) := -T(\varphi'), \qquad \varphi \in C_c^k(\mathbb{R}). \tag{4}$$

If we want to get more derivatives of T, we need k to be large, which leads us to consider the space $C_c^\infty(\mathbb{R})$ of compactly supported smooth functions as the space on which the functionals T act. This space is called the space of *test functions*. A *distribution* (on $\mathbb{R}$) is simply a continuous linear functional on $C_c^\infty(\mathbb{R})$, the latter equipped with a certain topology. In order to describe this topology, we need some preparation.

2. Locally convex spaces

In this section, we will discuss how to introduce a topology on a vector space by using a family of seminorms.

Definition 1. A function $p : X \to \mathbb{R}$ on a vector space X is called a *seminorm* if

i) $p(x + y) \leq p(x) + p(y)$ for $x, y \in X$, and
ii) $p(\lambda x) = |\lambda| p(x)$ for $\lambda \in \mathbb{R}$ and $x \in X$.

It is called a *norm* if in addition $p(x) = 0$ implies $x = 0$.

The property i) is *subadditivity* or the *triangle inequality*, and ii) is *positive homogeneity.*

Lemma 2. *Let p be a seminorm on a vector space X. Then we have*

a) $p(0) = 0$,
b) $p(x) \geq 0$,
c) $|p(x) - p(y)| \leq p(x - y)$, *and*
d) $\{x \in X : p(x) = 0\}$ *is a linear space.*

Proof. Part a) follows from positive homogeneity with $\lambda = 0$. Then we have

$$0 = p(0) = p(x - x) \leq p(x) + p(-x) = p(x) + p(x), \tag{5}$$

which gives b). While c) is obvious, d) is a consequence of

$$0 \leq p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y), \tag{6}$$

for $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. □

Let X be a vector space, and let $\mathscr{P}$ be a family of seminorms on X. Then given a finite collection $p_1, \ldots, p_k \in \mathscr{P}$ and given $\varepsilon > 0$, let us call the set

$$B_{y,\varepsilon}(p_1, \ldots, p_k) = \{x \in X : p_i(x - y) < \varepsilon,\ i = 1, \ldots, k\}, \tag{7}$$

the *semiball* of radius ε, centred at y, corresponding to the seminorms $p_1, \ldots, p_k$.

Definition 3. Let X be a vector space, and let $\mathscr{P}$ be a family of seminorms on X. Then we define a topology on X by calling $A \subset X$ *open* if for any $x \in A$, there exists a semiball $B_{x,\varepsilon}(p_1, \ldots, p_k) \subset A$ with $p_1, \ldots, p_k \in \mathscr{P}$ and $\varepsilon > 0$. We say that $(X, \mathscr{P})$ is a *locally convex space* (LCS).

The open sets in $(X, \mathscr{P})$ are precisely those which are the unions of semiballs. It is easy to verify that X itself is open, intersection of any two open sets is open, and that the union of any collection of open sets is open. The empty set is open, because any element of the empty set, of which there is none, satisfies any desired property. Therefore the preceding definition indeed defines a topology on X, making it a topological space. Note also that the topology on X does not change if we replace a seminorm $p \in \mathscr{P}$ by another seminorm p' satisfying $p(x) \leq cp'(x)$ for $x \in X$ and for some constant $c > 0$.

The reason we called X a locally convex space is that it agrees with the same notion from the theory of topological vector spaces. A topological vector space is a vector space which is also a topological space, with the property that the vector addition and scalar multiplication are continuous. Then a topological vector space X is called locally convex if $A \subset X$ is open and if $x \in A$ then there is a convex open set $C \subset A$ containing x. We choose not to go into details here, and use families of seminorms as primary objects to specify topological properties of X. This simplifies presentation and gives a quicker way to achieve our aim, and moreover does not lose generality, because of the (nontrivial) fact that any locally convex topological vector space has a family of seminorms that induces its topology.

Recall that a sequence $\{x_k\} \subset X$ is said to converge to $x \in X$ if for any open set $\omega \subset X$ containing x, we have $x_k \in \omega$ for all large k. In terms of seminorms, this is equivalent to saying that $p(x_k - x) \to 0$ for any $p \in \mathscr{P}$.

Lemma 4. *a) Let Y be a normed space, and let X be as above. Then a function $f : X \to Y$ is continuous if and only if for any $x \in X$ and any $\varepsilon > 0$, there is a finite collection $p_1, \ldots, p_k \in \mathscr{P}$ and $\delta > 0$ such that*

$$z \in B_{x,\delta}(p_1, \ldots, p_k) \quad \Rightarrow \quad \|f(x) - f(z)\|_Y \leq \varepsilon. \tag{8}$$

b) In addition to what has been assumed, suppose that f is linear. Then f is continuous if and only if there is a finite collection $p_1, \ldots, p_k \in \mathscr{P}$ and a constant $C > 0$ such that

$$\|f(x)\|_Y \leq C \max_i p_i(x), \qquad x \in X. \tag{9}$$

Proof. Recall that a map is called continuous if the preimage of any open set is open. Suppose that f is continuous. Then for any $\varepsilon > 0$ and $y = f(x)$ with $x \in X$, the preimage of $B_{y,\varepsilon} \subset Y$ contains a semiball $B_{x,\delta}(p_1, \ldots, p_k)$ with $\delta = \delta(\varepsilon, x) > 0$. In the other direction, let $U \subset Y$ be open and let $x \in f^{-1}(U)$. Then with $y = f(x) \in U$, there exist a nonempty ball $B_{y,\varepsilon} \subset U$, and a nonempty semiball $B_{x,\delta}(p_1, \ldots, p_k)$ such that $f(B_{x,\delta}(p_1, \ldots, p_k)) \subset B_{y,\varepsilon}$. This means that $f^{-1}(U)$ is open.

For b), the condition associated to (9) immediately implies the condition associated to (8) by linearity. Now suppose that we have the condition associated to (8). Hence there is $\delta > 0$ and $p_1, \ldots, p_k \in \mathscr{P}$ such that

$$z \in B_{0,\delta}(p_1, \ldots, p_k) \quad \Rightarrow \quad \|f(z)\|_Y \leq 1. \tag{10}$$

Note that $z \in B_{0,\delta}(p_1, \ldots, p_k)$ is equivalent to $p(z) := \max_i p_i(z) < \delta$. Let $x \in X$, and define $z = \frac{\delta}{2p(x)} x$. Then we have $p(z) = \frac{\delta}{2} < \delta$, leading to

$$1 \geq \|f(z)\|_Y = \frac{\delta}{2p(x)} \|f(x)\|_Y, \tag{11}$$

which is (9) with $C = \frac{2}{\delta}$. □

Remark 5. The preceding lemma can easily be extended to the case where Y is a LCS endowed with a family $\mathscr{Q}$ of seminorms. For instance, part b) would read: f is continuous iff for any $q \in \mathscr{Q}$, there is a finite collection $p_1, \ldots, p_k \in \mathscr{P}$ and a constant $C > 0$ such that

$$q(f(x)) \leq C \max_i p_i(x), \qquad x \in X. \tag{12}$$

Notice how the quantifiers differ on the domain and the range of the function. If $X \subset Y$ as sets, by taking $f : X \to Y$ to be the inclusion map $f(x) = x$ we derive the following criterion: the embedding $X \subset Y$ is continuous iff for any $q \in \mathscr{Q}$, there is a finite collection $p_1, \ldots, p_k \in \mathscr{P}$ and a constant $C > 0$ such that

$$q(x) \leq C \max_i p_i(x), \qquad x \in X. \tag{13}$$

Remark 6. Part b) of Lemma 4 is valid for checking continuity of seminorms $q : X \to \mathbb{R}$, because of their positive homogeneity and the property in Lemma 2c). So a seminorm q on $(X, \mathscr{P})$ is continuous iff there is a finite collection $p_1, \ldots, p_k \in \mathscr{P}$ and a constant $C > 0$ such that

$$q(x) \leq C \max_i p_i(x), \qquad x \in X. \tag{14}$$

Comparing this with the previous remark, we conclude that the embedding $X \subset Y$ is continuous iff the restriction of every seminorm of $(Y, \mathscr{Q})$ to X is continuous on $(X, \mathscr{P})$.

Definition 7. Let $(X, \mathscr{P})$ be a locally convex space. We define the following notions.

- $\{x_k\}$ is *Cauchy* if for any $p \in \mathscr{P}$, $p(x_j - x_k) \to 0$ as $j, k \to \infty$.
- $A \subset X$ is *bounded* if for any $p \in \mathscr{P}$, $\sup_{x\in A} p(x) < \infty$.

A straightforward but useful observation is that every Cauchy sequence is bounded. Indeed, if $\{x_k\}$ is Cauchy then, with an arbitrary $p \in \mathscr{P}$, for a sufficiently large j we have $p(x_j - x_k) < 1$ hence $p(k) < p(j) + 1$ for all $k \geq j$.

Definition 8. The family $\mathscr{P}$ of seminorms on X is called *separating* if for any $x \in X \setminus \{0\}$, there exists $p \in \mathscr{P}$ such that $p(x) \neq 0$.

The significance of this is that if $(X, \mathscr{P})$ is a LCS with $\mathscr{P}$ separating, then the topology of X is *Hausdorff*, meaning that for any $x, y \in X$ distinct, there are open sets $A \subset X$ and $B \subset X$ with $x \in A$ and $y \in B$. Indeed, let $p \in \mathscr{P}$ be such that $\delta := p(x - y) > 0$. Then $A = \{z \in X : p(z - x) < \frac{\delta}{2}\}$ and $B = \{z \in X : p(z - y) < \frac{\delta}{2}\}$ satisfy the desired properties.

Lemma 9. *A locally convex space $(X, \mathscr{P})$ is metrizable if $\mathscr{P}$ is countable and separating.*

Proof. Let $\mathscr{P} = \{p_1, p_2, \ldots\}$, and let $\{\alpha_k\}$ be a sequence of positive numbers satisfying $\alpha_k \to 0$. Then we claim that

$$d(x, y) = \max_k \frac{\alpha_k p_k(x - y)}{1 + p_k(x - y)}, \tag{15}$$

defines a metric that induces the topology of X. First observe that the maximum is well-defined, since $p_k/(1 + p_k) < 1$ and $\alpha_k \to 0$. Also, because $\alpha_k > 0$ for all k, $d(x, y) = 0$ implies $p_k(x - y) = 0$ for all k, which then gives $x = y$ by the separating property. The triangle inequality for d follows from the elementary fact

$$a \leq b + c \qquad \Rightarrow \qquad \frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c} \qquad (a, b, c \geq 0), \tag{16}$$

which can easily be verified, e.g., by contradiction.

For each k, we have

$$\frac{p_k(x - y)}{1 + p_k(x - y)} \leq d(x - y), \tag{17}$$

which tells us that any semiball contains a metric ball. To get the other direction, let $\varepsilon > 0$, and let n be an index such that $\alpha_k < \varepsilon$ for all $k > n$. Then we have

$$d(x-y) \leq \varepsilon + \max_{1\leq k\leq n} \frac{\alpha_k p_k(x-y)}{1+p_k(x-y)} \leq \varepsilon + \alpha \max_{1\leq k\leq n} p_k(x-y), \tag{18}$$

where $\alpha = \max \alpha_k$. This means that the semiball $B_{x,\varepsilon}(p_1,\ldots,p_n)$ is contained in the metric ball $B_{(1+\alpha)\varepsilon}(x) = \{y \in X : d(x-y) < (1+\alpha)\varepsilon\}$. □

3. Test functions

In this section, we will establish some basic properties of the so-called inductive limit topology on the space of test functions.

We start with introducing the multi-index notation, which is a convenient shorthand notation for partial derivatives and multivariate polynomials. A *multi-index* is a vector $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n$ whose components are nonnegative integers. Then we use

$$x^\alpha = x_1^{\alpha_1}\ldots x_n^{\alpha_n}, \qquad \text{and} \qquad \partial^\alpha = \partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}, \tag{19}$$

for multivariate monomials and partial derivatives. The *length* of a multi-index α is defined as $|\alpha| = \alpha_1 + \ldots + \alpha_n$, which corresponds to the total degree of a monomial or the order of a differential operator.

Given a set $A \subset \Omega$, and a function u on A, the *uniform norm* on A is

$$\|u\|_{C^0(A)} = \sup_{x\in A} |u(x)|, \tag{20}$$

and the C^k*-norm* on A is

$$\|u\|_{C^k(A)} = \max_{|\alpha|\leq k} \|\partial^\alpha u\|_{C^0(A)}, \tag{21}$$

whenever they make sense. If $\varphi : M \to \mathbb{R}$ is a continuous function, we define its *support* as

$$\operatorname{supp}\varphi = \overline{\{x \in M : \varphi(x) \neq 0\}}. \tag{22}$$

Furthermore, for $K \subset \mathbb{R}^n$ compact, we define the space

$$\mathscr{D}_K = \{\varphi \in C^\infty(\mathbb{R}^n) : \operatorname{supp}\varphi \subset K\}, \tag{23}$$

and endow it with the seminorms

$$p_m(\varphi) = \|\varphi\|_{C^m}, \qquad m = 0,1,\ldots. \tag{24}$$

The question arises if there exists any infinitely differentiable function with compact support. This is something we should check since a nonzero analytic function cannot have compact support, and being smooth is only slightly weaker than being analytic. We claim that the function φ on $\mathbb{R}^n$ defined by

$$\varphi(x) = \begin{cases} e^{-1/(1-|x|^2)} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases} \tag{25}$$

is in $C^\infty(\mathbb{R}^n)$. It is clear that $\varphi(x) \to 0$ as $|x| \nearrow 1$. As for the derivatives, we have

$$\partial^\alpha\varphi(x) = \frac{p(x)e^{-1/(1-|x|^2)}}{(1-|x|^2)^{|\alpha|}}, \qquad |x| < 1, \tag{26}$$

where p is some polynomial. From this it is also clear that $\partial^\alpha\varphi(x) \to 0$ as $|x| \nearrow 1$. So $\varphi \in C^\infty(\mathbb{R}^n)$. If K contains an open ball, we can fit infinitely many open balls inside K. Then scaling and translating φ, we can place them in K so that their supports are contained in K and do not intersect with each other. This implies that $\mathscr{D}_K$ is infinite dimensional.

Lemma 10. *Let $K \subset \mathbb{R}^n$ be a compact set. Then the space $\mathscr{D}_K$ is metrizable and complete, i.e., $\mathscr{D}_K$ is a Fréchet space.*

Proof. $\mathscr{D}_K$ is metrizable by Lemma 9, since $\{p_m\}$ is countable and separating.

Let $\{\varphi_k\}$ be a Cauchy sequence in $\mathscr{D}_K$. This means that each sequence $\{\partial^\alpha \varphi_k\}$ is Cauchy in the uniform norm, hence $\partial^\alpha \varphi_k \to \psi_\alpha \in C(\mathbb{R}^n)$ and $\operatorname{supp} \psi_\alpha \subset K$. Now the question is if $\partial^\alpha \psi_{(0,\ldots,0)} = \psi_\alpha$ holds, which then boils down to checking if $\partial_j \varphi = \psi_j$ holds given that $\varphi_k \to \varphi$ and $\partial_j \varphi_k \to \psi_j$, both uniformly. Let $x \in \mathbb{R}^n$ and let h be a (small) vector along the j-th coordinate axis. Then we have

$$\partial_j \varphi_k(\xi) = \frac{\varphi_k(x+h) - \varphi_k(x)}{h}, \tag{27}$$

for some ξ on the line segment $[x, x+h]$. Let $\varepsilon > 0$. By continuity, we have $|\psi_j(x) - \psi_j(\xi)| < \varepsilon$ if h is sufficiently small. Moreover, we have $|\psi_j(\xi) - \partial_j \varphi_k(\xi)| < \varepsilon$ and $\|\varphi_k - \varphi\|_{C^0} < \varepsilon |h|$ if k is sufficiently large. Combining all this we get

$$\big|\psi_j(x) - \frac{\varphi(x+h) - \varphi(x)}{h}\big| < 4\varepsilon, \tag{28}$$

for all sufficiently small h, which finishes the proof. □

A nonempty, open, and connected subset $\Omega \subset \mathbb{R}^n$ is called a *domain*.

Definition 11. Let $\Omega \subset \mathbb{R}^n$ be a domain. Then we define the *space of test functions* by

$$\mathscr{D}(\Omega) = \bigcup_{K \Subset \Omega} \mathscr{D}_K, \tag{29}$$

where we used the notation $K \Subset \Omega$ to mean that K is compact and is a subset of Ω.

Note that if $K_1 \subset K_2 \subset \ldots \subset \Omega$ are compact sets and $\bigcup_m K_m = \Omega$, then

$$\mathscr{D}(\Omega) = \bigcup_m \mathscr{D}_{K_m}. \tag{30}$$

Such a sequence $\{K_m\}$ can be constructed easily, for instance, by

$$K_m = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{m}\} \cap \overline{B}_m, \tag{31}$$

where

$$B_m = \{x \in \mathbb{R}^n : |x| < m\}, \tag{32}$$

is the open ball of radius m, centred at the origin.

Our next task is to introduce a topology on $\mathscr{D}(\Omega)$. In doing so, we want the inclusions $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ to be continuous. This means, by Remark 6 that for every seminorm p from $(\mathscr{D}(\Omega), \mathscr{P})$, where $\mathscr{P}$ is the hypothetical family inducing a topology on $\mathscr{D}(\Omega)$, the restriction $p|_{\mathscr{D}_K}$ must be continuous on $\mathscr{D}_K$. The family $\mathscr{P} = \{p_m\}$ has the desired property, but the following remark shows that it would not be a very convenient choice.

Remark 12. $\mathscr{D}(\Omega)$ is *not* complete with respect to the topology induced by $\{p_m\}$. We illustrate it in the case $\Omega = \mathbb{R}$. Take a nonzero function $\varphi \in \mathscr{D}(\mathbb{R})$ whose support is small and concentrated near 0, and consider the sequence

$$\varphi_k(x) = \varphi(x) + 2^{-1}\varphi(x-1) + \ldots + 2^{-k}\varphi(x-k), \qquad k = 1, 2, \ldots. \tag{33}$$

Obviously, this sequence is Cauchy with respect to the family $\{p_m\}$, but the support of the limit function is not compact.

This failure indicates that the family $\{p_m\}$ has not enough seminorms to prevent Cauchy sequences from "leaking" towards the boundary of Ω. So we can add more seminorms to the family, and hope that things get better. Having a large family of seminorms will have the added benefit that it becomes easier for a function $f : \mathscr{D}(\Omega) \to Y$ to be continuous, meaning that we will have a large supply of continuous functions on $\mathscr{D}(\Omega)$. Of course there is a limit in expanding the family $\mathscr{P}$ because of the aforementioned requirement that $p|_{\mathscr{D}_K}$ be continuous. These two competing requirements give rise to a unique family $\mathscr{P}$ as follows.

Definition 13. We define the collection $\mathscr{P}$ of seminorms on $\mathscr{D}(\Omega)$ by the condition that a seminorm p on $\mathscr{D}(\Omega)$ is in $\mathscr{P}$ iff $p|_{\mathscr{D}_K}$ is continuous for each compact $K \subset \Omega$.

The topology generated by $\mathscr{P}$ on $\mathscr{D}(\Omega)$ is called the *inductive limit topology*. Looking back, this topology is completely natural, given that $\mathscr{D}(\Omega)$ is the union of $\{\mathscr{D}_K : K \Subset \Omega\}$, and that each $\mathscr{D}_K$ has its own topology.

Lemma 14. *The topology of $\mathscr{D}_K$ is exactly the one induced by the embedding $\mathscr{D}_K \subset \mathscr{D}(\Omega)$.*

Proof. Let $A \subset \mathscr{D}(\Omega)$ be open and let $K \subset \Omega$ be compact. We will show that $A \cap \mathscr{D}_K$ is open in $\mathscr{D}_K$. Let $\psi \in A \cap \mathscr{D}_K$. Let us denote the semiballs in $\mathscr{D}_K$ by $B_{\psi,\varepsilon}(p_m; \mathscr{D}_K)$ etc., and the semiballs in $\mathscr{D}(\Omega)$ by $B_{\psi,\varepsilon}(p)$ etc. Then there exists $p \in \mathscr{P}$ such that $B_{\psi,\varepsilon}(p) \subset A$ with $\varepsilon > 0$. By construction, there exists p_m such that $p \leq cp_m$ on $\mathscr{D}_K$, with some constant $c > 0$. Hence $B_{\psi,\varepsilon/c}(p_m, \mathscr{D}_K) \subset B_{\psi,\varepsilon}(p) \cap \mathscr{D}_K \subset A \cap \mathscr{D}_K$, showing that $A \cap \mathscr{D}_K$ is open in $\mathscr{D}_K$.

On the other hand, since $\{p_m\} \subset \mathscr{P}$, any semiball $B_{\psi,\varepsilon}(p_m; \mathscr{D}_K)$ in $\mathscr{D}_K$ is equal to the intersection of the semiball $B_{\psi,\varepsilon}(p_m)$ in $\mathscr{D}(\Omega)$ with $\mathscr{D}_K$, i.e.,

$$B_{\psi,\varepsilon}(p_m; \mathscr{D}_K) = B_{\psi,\varepsilon}(p_m) \cap \mathscr{D}_K. \tag{34}$$

This immediately implies that any open set in $\mathscr{D}_K$ can be written as the intersection of an open set of $\mathscr{D}(\Omega)$ with $\mathscr{D}_K$. □

Let us ask the question: Does $\mathscr{P}$ have any seminorm that is not one of $\{p_m\}$? An example of such a seminorm is given by

$$p(\varphi) = \sup_j c_j |\varphi(x_j)|, \qquad \varphi \in \mathscr{D}(\Omega), \tag{35}$$

where $\{x_j\} \subset \Omega$ is a sequence having no accumulation points in Ω, and $\{c_j\}$ is a sequence of positive numbers. We can easily check that p is a seminorm, and that $p|_{\mathscr{D}_K}$ is continuous on $\mathscr{D}_K$ for any compact $K \subset \Omega$, so that $p \in \mathscr{P}$. Seminorms such as this give a very strong control near the boundary of Ω, because $\{x_j\}$ concentrate towards the boundary and c_j can grow arbitrarily fast. The following result illustrates this phenomenon.

Theorem 15. *The set $A \subset \mathscr{D}(\Omega)$ is bounded if and only if there is a compact set $K \subset \Omega$ such that $A \subset \mathscr{D}_K$ and that A is bounded in $\mathscr{D}_K$. Recall that the latter means that each p_m is bounded on A.*

Proof. Suppose that A is bounded in $\mathscr{D}_K$ for some compact set $K \subset \Omega$. We claim that continuity of the embedding $\mathscr{D}_K \subset \mathscr{D}(\Omega)$ implies that A is also bounded in $\mathscr{D}(\Omega)$. To prove it, let $p \in \mathscr{P}$. Then there is p_m such that

$$p(\varphi) \leq Cp_m(\varphi), \qquad \varphi \in \mathscr{D}_K. \tag{36}$$

By assumption, $p_m(\varphi) \leq M$ for $\varphi \in A$ and for some constant M, which implies that p is bounded on A.

To prove the other direction, suppose that $A \not\subset \mathscr{D}_K$ for any compact $K \subset \Omega$. Then there exist sequences $\{\varphi_m\} \subset A$ and $\{x_m\} \subset \Omega$ such that $\varphi(x_m) \neq 0$, and that $\{x_m\}$ has no accumulation points in Ω. Let

$$p(\varphi) = \sup_m \frac{m|\varphi(x_m)|}{|\varphi_m(x_m)|}, \qquad \varphi \in \mathscr{D}(\Omega). \tag{37}$$

Obviously it is a seminorm, and $p \in \mathscr{P}$ because for any compact $K' \subset \Omega$ there is a constant C such that

$$p(\varphi) \leq C\|\varphi\|_{C^0}, \qquad \varphi \in \mathscr{D}_{K'}. \tag{38}$$

However, we have $p(\varphi_m) \geq m$, so p is not bounded on A, leading to a contradiction. □

Corollary 16. *a) The sequence $\{\varphi_j\}$ is Cauchy in $\mathscr{D}(\Omega)$ iff $\{\varphi_j\} \subset \mathscr{D}_K$ for some compact $K \subset \Omega$, and $\|\varphi_j - \varphi_k\|_{C^m} \to 0$ as $j, k \to \infty$, for each m.*

b) If $\varphi_j \to 0$ in $\mathscr{D}(\Omega)$ iff $\{\varphi_j\} \subset \mathscr{D}_K$ for some compact $K \subset \Omega$, and $\|\varphi_j\|_{C^m} \to 0$ as $j \to \infty$, for each m.

c) $\mathscr{D}(\Omega)$ is sequentially complete.

Proof. a) If $\{\varphi_j\} \subset \mathscr{D}_K$ is Cauchy in $\mathscr{D}_K$ for some compact $K \subset \Omega$, then it is Cauchy in $\mathscr{D}(\Omega)$ by continuity of the embedding $\mathscr{D}_K \subset \mathscr{D}(\Omega)$. Now let $\{\varphi_j\} \subset \mathscr{D}(\Omega)$ be Cauchy in $\mathscr{D}(\Omega)$. Since Cauchy sequences are bounded, by the preceding theorem we have $\{\varphi_j\} \subset \mathscr{D}_K$ for some compact $K \subset \Omega$. But then $\{p_m\} \subset \mathscr{P}$, which means that $p_m(\varphi_j - \varphi_k) \to 0$ as $j, k \to \infty$, for each p_m.

b) Left as an exercise.

c) Let $\{\varphi_j\} \subset \mathscr{D}(\Omega)$ be Cauchy in $\mathscr{D}(\Omega)$. Then by a) it is Cauchy in some $\mathscr{D}_K$. But $\mathscr{D}_K$ is Fréchet, so the limit exists in $\mathscr{D}_K$. This limit is valid also in $\mathscr{D}(\Omega)$, since a convergent sequence in $\mathscr{D}_K$ is convergent in $\mathscr{D}(\Omega)$. □

Theorem 17. *Let $(Y, \mathscr{Q})$ be a locally convex space, and let $f : \mathscr{D}(\Omega) \to Y$ be a linear map. Then the followings are equivalent.*

a) *f is continuous.*
b) *$\varphi_j \to 0$ in $\mathscr{D}(\Omega)$ implies $f(\varphi_j) \to 0$ in Y.*
c) *For any compact $K \subset \Omega$, $f : \mathscr{D}_K \to Y$ is continuous.*

Proof. a) ⇒ b). The continuity of f means that for any $q \in \mathscr{Q}$, there is $p \in \mathscr{P}$ such that

$$q(f(\varphi)) \leq p(\varphi), \qquad \varphi \in \mathscr{D}(\Omega). \tag{39}$$

Since $\varphi_j \to 0$ in $\mathscr{D}(\Omega)$, we have $p(\varphi_j) \to 0$, hence $q(f(\varphi_j)) \to 0$. As $q \in \mathscr{Q}$ is arbitrary, we conclude that $f(\varphi_j) \to 0$ in Y.

b) ⇒ c). Let $K \subset \Omega$ be compact. If b) holds then for any sequence $\varphi_j \to 0$ in $\mathscr{D}_K$ we have $f(\varphi_j) \to 0$ in Y. Then continuity of $f : \mathscr{D}_K \to Y$ follows from the general fact that for a metric space X and a topological space Y, a map $f : X \to Y$ is continuous if whenever $x_j \to x$ in X we have $f(x_j) \to f(x)$ in Y. To prove this fact, supposing that f is *not* continuous at $x \in X$, we want to show that there is a sequence $x_n \to x$ with $f(x_n) \not\to f(x)$. Let $U \subset Y$ be an open set such that $f(x) \in U$ and that $f^{-1}(U)$ is not open. Hence $f^{-1}(U)$ does not contain any metric ball $B_\varepsilon(x) = \{z \in X : d(z, x) < \varepsilon\}$ with $\varepsilon > 0$, where d is the metric of X. This means that for any $\varepsilon > 0$, there is $z \in B_\varepsilon(x)$ with $f(z) \notin U$, i.e., there exists a sequence $x_n \to x$ with $f(x_n) \notin U$ for all n.

c) ⇒ a). We want to show that for any $q \in \mathscr{Q}$, there is $p \in \mathscr{P}$ such that (39) holds. Given q, let us define the function

$$p(\varphi) = q(f(\varphi)), \qquad \varphi \in \mathscr{D}(\Omega). \tag{40}$$

It is a seminorm on $\mathscr{D}(\Omega)$, and moreover for each compact $K \subset \Omega$, the restriction $p|_{\mathscr{D}_K}$ is continuous since

$$q(f(\varphi)) \leq C p_m(\varphi), \qquad \varphi \in \mathscr{D}_K, \tag{41}$$

for some C and m possibly depending on K. Therefore $p \in \mathscr{P}$, which clearly implies (39). □

Example 18. The partial differentiation operator $\partial_j : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ is continuous, since for any compact $K \subset \Omega$ and any m, we have

$$p_m(\partial_j \varphi) \leq p_{m+1}(\varphi), \qquad \varphi \in \mathscr{D}_K. \tag{42}$$

Remark 19. $\mathscr{D}(\Omega)$ is *not* metrizable. We illustrate this in the case $\Omega = \mathbb{R}$. Pick a function $\varphi \in \mathscr{D}(\mathbb{R})$ with $\operatorname{supp} \varphi = [-1, 1]$, and define the double-indexed sequence

$$\varphi_{km}(x) = \frac{1}{m} \varphi\left(\frac{x}{k}\right), \qquad k, m = 1, 2, \ldots. \tag{43}$$

It is clear that for each fixed k, the sequence $\varphi_{k,1}, \varphi_{k,2}, \ldots$ converges to 0 in $\mathscr{D}(\mathbb{R})$. Then if $\mathscr{D}(\mathbb{R})$ was metrizable, say with metric d, we can extract a sequence $m_1, m_2, \ldots$, such that $\varphi_{k,m_k} \to 0$ in $\mathscr{D}(\mathbb{R})$. This can be done, for instance, by choosing m_k sufficiently large so that $d(\varphi_{k,m_k}, 0) < \frac{1}{k}$, for each k. But it is not possible for such a sequence to converge in $\mathscr{D}(\mathbb{R})$, because the support of φ_{k,m_k} is $[-k, k]$, which eventually becomes larger than any compact set in $\mathbb{R}$.

4. Basic operations on distributions

From now on the space $\mathscr{D}(\Omega)$ is equipped with its inductive limit topology.

Definition 20. A *distribution* on Ω is a continuous linear functional on $\mathscr{D}(\Omega)$. The space of all distributions on Ω is denoted by $\mathscr{D}'(\Omega)$.

We denote the action $u(\varphi)$ of $u \in \mathscr{D}'(\Omega)$ also by $\langle u, \varphi \rangle$. Theorem 17 tailored to distributions is the following.

Lemma 21. *A linear functional $u : \mathscr{D}(\Omega) \to \mathbb{R}$ is in $\mathscr{D}'(\Omega)$ iff any of the followings holds.*

a) *$\varphi_j \to 0$ in $\mathscr{D}(\Omega)$ implies $u(\varphi_j) \to 0$.*
b) *For any compact $K \subset \Omega$, there exist m and C such that*

$$|u(\varphi)| \leq C\|\varphi\|_{C^m} \qquad \text{for} \quad \varphi \in \mathscr{D}_K. \tag{44}$$

Definition 22. Let $u \in \mathscr{D}'(\Omega)$. If we have

$$|u(\varphi)| \leq C\|\varphi\|_{C^m} \qquad \text{for} \quad \varphi \in \mathscr{D}_K, \tag{45}$$

with the same m for all compact $K \subset \Omega$, with C possibly depending on K, then u is said to be a *distribution of order* $\leq m$. The smallest such m is called the *order of* u.

Example 23. For $u \in C(\Omega)$, the functional $T_u : \varphi \mapsto \int u\varphi$ is a distribution of order 0 since

$$|T_u(\varphi)| \leq \operatorname{vol}(K)\|u\|_{C^0(K)}\|\varphi\|_{C^0}, \qquad \text{for} \quad \varphi \in \mathscr{D}_K. \tag{46}$$

Similarly, δ is a distribution of order 0, and the derivative evaluation $\varphi \mapsto \varphi'(0)$ is a distribution of order 1.

Definition 24. The *weak topology* on $\mathscr{D}'(\Omega)$ is the one induced by the family of seminorms $\mathscr{P}' = \{p_\varphi : \varphi \in \mathscr{D}(\Omega)\}$, where $p_\varphi(u) = |u(\varphi)|$.

Thus $u_j \to 0$ in the weak topology of $\mathscr{D}'(\Omega)$ iff

$$\langle u_j, \varphi \rangle \to 0 \qquad \text{for each} \quad \varphi \in \mathscr{D}(\Omega). \tag{47}$$

We see that this is simply the pointwise convergence. The family $\mathscr{P}'$ is separating, since if $u \in \mathscr{D}'(\Omega)$ is nonzero, there is $\varphi \in \mathscr{D}(\Omega)$ such that $u(\varphi) \neq 0$. Hence the weak topology is Hausdorff.

For $u \in L^1_{\text{loc}}(\Omega)$, the functional $T_u : \varphi \mapsto \int u\varphi$ is a distribution of order 0 since

$$|T_u(\varphi)| \leq \|u\|_{L^1(K)}\|\varphi\|_{C^0}. \qquad \text{for} \quad \varphi \in \mathscr{D}_K, \tag{48}$$

We have seen that the map $u \mapsto T_u : L^1_{\text{loc}}(\Omega) \to \mathscr{D}'(\Omega)$ is an injection, so that $L^1_{\text{loc}}(\Omega)$ can be regarded as a subspace of $\mathscr{D}'(\Omega)$. Thus we will identify T_u with u. Then with the (Fréchet) topology on $L^1_{\text{loc}}(\Omega)$ defined by the seminorms $\{\|\cdot\|_{L^1(K)} : K \Subset \Omega\}$, from the above inequality we infer that $u_j \to 0$ in $L^1_{\text{loc}}(\Omega)$ implies $\langle u_j, \varphi \rangle \to 0$ for any fixed $\varphi \in \mathscr{D}(\Omega)$. Hence the embedding $L^1_{\text{loc}}(\Omega) \subset \mathscr{D}'(\Omega)$ is continuous. We can also infer the continuity of the embedding $C(\Omega) \subset \mathscr{D}'(\Omega)$ either directly or through the continuous embedding $C(\Omega) \subset L^1_{\text{loc}}(\Omega)$.

Example 25. Consider $u_j(x) = \sin(jx)$. Then $u_j \to 0$ in $\mathscr{D}'(\mathbb{R})$, since

$$\int \sin(jx)\varphi(x)\mathrm{d}x \to 0 \qquad \text{as} \quad j \to \infty, \tag{49}$$

for any $\varphi \in \mathscr{D}(\mathbb{R})$, by the Riemann-Lebesgue lemma.

Now we want to extend some basic operations on functions to distributions. This is usually achieved by means of a simple duality device that can be described as follows. Suppose that $T, T' : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ are continuous linear maps, satisfying

$$\int (T\psi)\varphi = \int \psi(T'\varphi), \qquad \psi, \varphi \in \mathscr{D}(\Omega). \tag{50}$$

Then we define $\tilde{T} : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$, which is the intended extension of T, by

$$\langle \tilde{T}u, \varphi \rangle = \langle u, T'\varphi \rangle, \qquad u \in \mathscr{D}'(\Omega),\, \varphi \in \mathscr{D}(\Omega). \tag{51}$$

It is easily checked that $\tilde{T}u \in \mathscr{D}'(\Omega)$, since by continuity of T', $\varphi_j \to 0$ in $\mathscr{D}(\Omega)$ implies $T'\varphi_j \to 0$ in $\mathscr{D}(\Omega)$, which then implies that $\langle u, T'\varphi_j \rangle \to 0$. Moreover, $\tilde{T} : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is continuous, because

$$p_\varphi(\tilde{T}u) = |\langle \tilde{T}u, \varphi \rangle| = |\langle u, T'\varphi \rangle| = p_\psi(u), \tag{52}$$

where $\psi = T'\varphi \in \mathscr{D}(\Omega)$. If $u \in \mathscr{D}(\Omega)$, then

$$\langle \tilde{T}u, \varphi \rangle = \langle u, T'\varphi \rangle = \int u(T'\varphi) = \int (Tu)\varphi = \langle Tu, \varphi \rangle, \tag{53}$$

hence $\tilde{T}$ is indeed an extension of T. In fact, $\tilde{T}$ is the *unique* continuous extension of T. To see this, we will use the (nontrivial) fact that $\mathscr{D}(\Omega)$ is sequentially dense in $\mathscr{D}'(\Omega)$, i.e., for any $u \in \mathscr{D}'(\Omega)$, there exists a sequence $\{u_j\} \subset \mathscr{D}(\Omega)$ such that $u_j \to u$ in $\mathscr{D}'(\Omega)$. Let T_1 and T_2 be two continuous extensions of T. Then with u and $\{u_j\}$ as above, since $T_1 u_j = T_2 u_j$, we have

$$T_1 u - T_2 u = T_1(u - u_j) + T_2(u_j - u), \tag{54}$$

which implies for any $\varphi \in \mathscr{D}(\Omega)$ that

$$\begin{aligned} |\langle T_1 u - T_2 u, \varphi \rangle| &\leq p_\varphi(T_1(u - u_j)) + p_\varphi(T_2(u - u_j)) \\ &\leq C_1 p_{\psi_1}(u - u_j) + C_2 p_{\psi_2}(u - u_j), \end{aligned} \tag{55}$$

with some $\psi_1, \psi_2 \in \mathscr{D}(\Omega)$, and some constants $C_1, C_2 > 0$. Now sending $j \to \infty$ we get $\langle T_1 u - T_2 u, \varphi \rangle = 0$ for any $\varphi \in \mathscr{D}(\Omega)$, hence $T_1 u = T_2 u$.

Let us consider now some applications of this device.

Differentiation: $T = \partial_j$. As we have already discussed, the operator $\partial_j : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ is continuous, and integration by parts gives

$$\int \varphi \partial_j \psi = -\int \psi \partial_j \varphi, \qquad \psi, \varphi \in \mathscr{D}(\Omega). \tag{56}$$

Hence $T' = -\partial_j$, and the derivative of $u \in \mathscr{D}'(\Omega)$ is given by

$$\langle \partial_j u, \varphi \rangle = -\langle u, \partial_j \varphi \rangle, \qquad \varphi \in \mathscr{D}(\Omega). \tag{57}$$

For any multi-index α, this generalizes to

$$\langle \partial^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle, \qquad \varphi \in \mathscr{D}(\Omega). \tag{58}$$

Multiplication by a smooth function: $T\psi = a\psi$, where $a \in C^\infty(\Omega)$. One can easily see that $T' = T$, so up to showing continuity of T on $\mathscr{D}(\Omega)$, we infer

$$\langle au, \varphi \rangle = \langle u, a\varphi \rangle, \qquad u \in \mathscr{D}'(\Omega),\, \varphi \in \mathscr{D}(\Omega). \tag{59}$$

The continuity of $T : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ is left as an exercise.

Translation: $(T\psi)(x) = \psi(x+a)$, where $a \in \mathbb{R}^n$. We take $\Omega = \mathbb{R}^n$. By change of variables, we have

$$\int \psi(x+a)\varphi(x)\mathrm{d}x = \int \psi(x)\varphi(x-a)\mathrm{d}x, \qquad \psi,\varphi \in \mathscr{D}(\mathbb{R}^n), \tag{60}$$

so with $(\tau_a\psi)(x) = \psi(x+a)$, we infer

$$\langle \tau_a u, \varphi\rangle = \langle u, \tau_{-a}\varphi\rangle, \qquad u \in \mathscr{D}'(\mathbb{R}^n),\, \varphi \in \mathscr{D}(\mathbb{R}^n). \tag{61}$$

The continuity of τ_a on test functions is left as an exercise.

Convolution with a test function: $T\psi = a * \psi$, where $a \in \mathscr{D}(\mathbb{R}^n)$. We have

$$\begin{aligned} \int (a*\psi)\varphi &= \int\int a(x-z)\psi(z)\varphi(x)\mathrm{d}z\mathrm{d}x \\ &= \int \psi(\tilde{a}*\varphi), \qquad \psi,\varphi \in \mathscr{D}(\mathbb{R}^n), \end{aligned} \tag{62}$$

where $\tilde{a}(x) = a(-x)$ denotes the reflection through the origin. Again leaving the continuity question as an exercise, we get

$$\langle a*u, \varphi\rangle = \langle u, \tilde{a}*\varphi\rangle, \qquad u \in \mathscr{D}'(\mathbb{R}^n),\, \varphi \in \mathscr{D}(\mathbb{R}^n). \tag{63}$$

Example 26. Let $\theta \in L^1_{\mathrm{loc}}(\mathbb{R})$ be the Heaviside step function, defined by $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. Then its distributional derivative is given by

$$\langle \theta', \varphi\rangle = -\langle \theta, \varphi'\rangle = -\int_0^\infty \varphi'(x)\mathrm{d}x = \varphi(0), \tag{64}$$

for any $\varphi \in \mathscr{D}(\mathbb{R})$. Hence $\theta' = \delta$.

5. Local structure of distributions

Definition 27. Let $u \in \mathscr{D}'(\Omega)$ and let $\omega \subset \Omega$ be open. The *restriction* $u|_\omega \in \mathscr{D}'(\omega)$ of u to ω is defined by

$$\langle u|_\omega, \varphi\rangle = \langle u, \varphi\rangle, \qquad \varphi \in \mathscr{D}(\omega). \tag{65}$$

We say that $u = 0$ on ω if $u|_\omega = 0$.

This gives us a possibility to talk about distributions locally, meaning that we can focus on small open sets, one at a time. In order for this to be meaningful, we expect some natural properties to be satisfied by the restriction process. First, let us check if the above definition indeed makes sense, i.e., if $u|_\omega \in \mathscr{D}'(\omega)$. So let $\varphi_j \to 0$ in $\mathscr{D}(\omega)$. Then $\varphi_j \to 0$ in $\mathscr{D}(\Omega)$, because there is a compact $K \subset \omega$ such that $\varphi_j \to 0$ in $\mathscr{D}_K$. Since $u \in \mathscr{D}'(\Omega)$, we have $\langle u|_\omega, \varphi_j\rangle = \langle u, \varphi_j\rangle \to 0$, showing that $u|_\omega \in \mathscr{D}'(\omega)$. Note that the same argument also demonstrates that the embedding $\mathscr{D}(\omega) \subset \mathscr{D}(\Omega)$ is continuous. However, unless $\omega = \Omega$, the topology of $\mathscr{D}(\omega)$ is *strictly finer* than that induced by the embedding $\mathscr{D}(\omega) \subset \mathscr{D}(\Omega)$, i.e., there are more open sets in $\mathscr{D}(\omega)$ than those inherited from $\mathscr{D}(\Omega)$. The reason is that for instance, the seminorm $p(\varphi) = \sup j|\varphi(x_j)|$ with $\{x_j\}$ having no accumulation points in ω, is compatible with the topology of $\mathscr{D}(\omega)$, while it is in general not with the topology of $\mathscr{D}(\Omega)$. This results in the fact that not every distribution in $\mathscr{D}'(\omega)$ is the restriction of some distribution in $\mathscr{D}'(\Omega)$.

The following theorem shows that as far as restrictions are concerned, we can work with distributions as if they were functions. The properties a)-d) in the theorem are called the *sheaf properties*.

Theorem 28. *Let* $u \in \mathscr{D}'(\Omega)$.

a) $u|_\Omega = u$.
b) $(u|_\omega)|_\sigma = u|_\sigma$ *for open sets* $\sigma \subset \omega \subset \Omega$.

c) *If* $\{\omega_\alpha\}$ *is an open cover of* Ω*, then*

$$\forall\alpha,\, u|_{\omega_\alpha} = 0 \qquad \Rightarrow \qquad u = 0. \tag{66}$$

d) *With* $\{\omega_\alpha\}$ *as in c), let* $u_\alpha \in \mathscr{D}'(\omega_\alpha)$ *is given for each* α*, satisfying*

$$u_\alpha|_{\omega_\alpha\cap\omega_\beta} = u_\beta|_{\omega_\alpha\cap\omega_\beta} \qquad \forall\alpha,\beta. \tag{67}$$

Then there exists a unique $u \in \mathscr{D}'(\Omega)$ *such that* $u|_\alpha = u_\alpha$ *for each* α*.*

Proof. a) and b) are trivial.

For c), let $\varphi \in \mathscr{D}(\Omega)$, and let $K = \operatorname{supp}\varphi$. Let $\{\chi_\alpha\}$ be a $\mathscr{D}(\Omega)$-partition of unity over K subordinate to $\{\omega_\alpha\}$. This means that

- $\chi_\alpha \in \mathscr{D}(\Omega)$ is nonnegative for each α,
- χ_α is nonzero for only finitely many α,
- there is an open set $V \supset K$ such that $\sum_\alpha \chi_\alpha = 1$ on V, and
- $\operatorname{supp}\chi_\alpha \subset \omega_\alpha$ for each α.

Note that we use the same index set for $\{\chi_\alpha\}$ as that of $\{\omega_\alpha\}$ at the expense of keeping some unnecessary zero functions in $\{\chi_\alpha\}$. We employ the existence of such a partition of unity without proof. We compute

$$\langle u,\varphi\rangle = \langle u, \sum_\alpha \chi_\alpha\varphi\rangle = \sum_\alpha\langle u,\chi_\alpha\varphi\rangle = \sum_\alpha\langle u|_{\omega_\alpha},\chi_\alpha\varphi\rangle = 0, \tag{68}$$

showing that $u = 0$, since $\varphi \in \mathscr{D}(\Omega)$ was arbitrary.

The uniqueness part of d) follows immediately from c). For existence, let $\varphi \in \mathscr{D}(\Omega)$, and keep the setting of the previous paragraph. We define

$$u(\varphi) := \sum_\alpha \langle u_\alpha, \chi_\alpha\varphi\rangle. \tag{69}$$

Before anything, we need to show that this definition does not depend on the partition of unity $\{\chi_\alpha\}$. Let $\{\xi_\alpha\}$ be another such partition of unity. Then we have

$$\sum_\alpha\langle u_\alpha,\chi_\alpha\varphi\rangle = \sum_{\alpha,\beta}\langle u_\alpha,\xi_\beta\chi_\alpha\varphi\rangle = \sum_{\alpha,\beta}\langle u_\beta,\xi_\beta\chi_\alpha\varphi\rangle = \sum_\beta\langle u_\beta,\xi_\beta\varphi\rangle, \tag{70}$$

where in the second step we used the property (67). Linearity can be verified for $\varphi_1,\varphi_2 \in \mathscr{D}(\Omega)$ by taking a partition of unity on $\operatorname{supp}\varphi_1 \cup \operatorname{supp}\varphi_2$. For continuity, let $K \subset \Omega$ be a compact set, and let $\varphi \in \mathscr{D}_K$. Then by using the fact that $u_\alpha \in \mathscr{D}'(\omega_\alpha)$ and $\chi_\alpha\varphi \in \mathscr{D}(\omega_\alpha)$, we have

$$|u(\varphi)| \leq \sum_\alpha |\langle u_\alpha,\chi_\alpha\varphi\rangle| \leq \sum_\alpha C_\alpha\|\chi_\alpha\varphi\|_{C^{m_\alpha}} \leq C\|\varphi\|_{C^m}, \tag{71}$$

showing that $u \in \mathscr{D}'(\Omega)$. □

Definition 29. The *support* of $u \in \mathscr{D}'(\Omega)$ is given by

$$\operatorname{supp} u = \Omega \setminus \bigcup\{\omega \subset \Omega \text{ open} : u|_\omega = 0\}. \tag{72}$$

Lemma 30. *It is easy to check that the following properties hold.*

i) $\operatorname{supp} u$ *is relatively closed in* Ω*.*
ii) $x \in \operatorname{supp} u$ *iff* $x \in \Omega$ *and* x *does not have any open neighbourhood on which* u *vanishes.*
iii) $\operatorname{supp} u$ *agrees with the usual notion when* u *is a continuous function.*
iv) $u|_{\Omega\setminus\operatorname{supp} u} = 0$.
v) $\operatorname{supp} u = \varnothing$ *implies* $u = 0$*.*
vi) $\operatorname{supp}(u+v) \subset \operatorname{supp} u + \operatorname{supp} v$.
vii) $\operatorname{supp}\partial^\alpha u \subset \operatorname{supp} u$.
viii) $\operatorname{supp}(au) \subset \operatorname{supp} a \cap \operatorname{supp} u$.

Example 31. $\operatorname{supp}\delta = \{0\}$.

We end these notes by proving a theorem that says that locally every distribution is a (possibly high order) derivative of a function. This means that distributions are not much more than derivatives of functions.

Theorem 32. *Let $u \in \mathscr{D}'(\Omega)$ and let ω be a bounded open set with $\overline{\omega} \subset \Omega$. Then there exist a function $f \in L^\infty(\omega)$ and a multi-index α such that $u = \partial^\alpha f$ on ω.*

Proof. Since $\overline{\omega}$ is compact, there exist m and C such that

$$|u(\varphi)| \leq C\|\varphi\|_{C^m} = C \max_{|\alpha|\leq m} \|\partial^\alpha \varphi\|_{C^0}, \qquad \varphi \in \mathscr{D}(\omega). \tag{73}$$

For any $\psi \in \mathscr{D}(\omega)$, we have

$$\|\psi\|_{C^0} \leq C\|\partial_j \psi\|_{C^0}, \tag{74}$$

with some constant $C > 0$, because ω is bounded. So we can replace the derivatives in the right hand side of (73) by higher order derivatives so as to have only one term in the maximum. This term would of course be the norm of $\partial^\beta\varphi$ with $\beta = (m, m, \dots, m)$, i.e.,

$$|u(\varphi)| \leq C\|\partial^\beta \varphi\|_{C^0}, \qquad \varphi \in \mathscr{D}(\omega). \tag{75}$$

We want to replace the uniform norm in the right hand side by the L^1-norm of a derivative of φ. For any $\psi \in \mathscr{D}(\omega)$ and for $x \in \mathbb{R}^n$, we have

$$\psi(x) = \int_{y<x} \partial_1 \dots \partial_n \psi(y)\, \mathrm{d}y, \tag{76}$$

where $y < x$ should be read componentwise. Using this, we finally get

$$|u(\varphi)| \leq C \int |\partial^\beta \varphi|, \qquad \varphi \in \mathscr{D}(\omega), \tag{77}$$

now with $\beta = (m+1, m+1, \dots, m+1)$. This inequality in particular implies that the distribution u cannot distinguish two functions $\varphi, \psi \in \mathscr{D}(\omega)$ if they satisfy $\partial^\beta\varphi = \partial^\beta\psi$. Therefore the map

$$T(\partial^\beta\varphi) := u(\varphi), \tag{78}$$

as a linear functional on the space $X = \{\partial^\beta\psi : \psi \in \mathscr{D}(\omega)\}$, is well-defined. Then the estimate (77) simply says that

$$|T(\xi)| \leq C\|\xi\|_{L^1(\omega)}, \qquad \xi \in X, \tag{79}$$

and so we can employ the Hahn-Banach theorem to extend T as a bounded linear functional on all of $L^1(\omega)$. Then by the duality between L^1 and L^∞, there is $g \in L^\infty(\omega)$ such that

$$T(\xi) = \int g\xi, \qquad \xi \in L^1(\omega). \tag{80}$$

Finally, putting $\xi = \partial^\beta\varphi$ with $\varphi \in \mathscr{D}(\omega)$ and unraveling the definitions, we get

$$u(\varphi) = T(\partial^\beta\varphi) = \int g\partial^\beta\varphi = (-1)^{|\beta|}\langle \partial^\beta g, \varphi\rangle, \tag{81}$$

which means that $u = (-1)^{|\beta|}\partial^\beta g$ on ω. □