MATH 580 ASSIGNMENT 4

DUE THURSDAY NOVEMBER 8

1. Here we will prove a version of the maximum principle for unbounded domains and for functions that are not necessarily continuous up to the boundary.
   (a) Exhibit a harmonic function violating the (weak) maximum principle on an unbounded domain.
   (b) Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $K_1 \subset K_2 \subset \ldots$ be a nested sequence of compact subsets of Ω, such that $\bigcup_j K_j = \Omega$. Suppose that $u \in C(\Omega)$ is a subharmonic function satisfying
   $$\limsup_{j\to\infty} \sup_{x\in\partial K_j} u(x) \leq 0.$$
   Show that $u \leq 0$ in Ω.
   (c) Let $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. We topologize it by identifying it with the n-sphere through a stereographical projection (which is required to be a homeomorphism by definition). In particular, we have $x_j \to \infty$ if $\{x_j\}$ eventually escapes any compact set of $\mathbb{R}^n$. For $\Omega \subset \mathbb{R}^n$ a domain, we denote its boundary in $\hat{\mathbb{R}}^n$ by $\hat{\partial}\Omega$. Show that $\hat{\partial}\Omega = \partial\Omega$ if Ω is bounded, and $\hat{\partial}\Omega = \partial\Omega \cup \{\infty\}$ if Ω is unbounded. Suppose that $u \in C(\Omega)$ is a subharmonic function satisfying
   $$\limsup_{\Omega\ni x\to z} u(x) \leq 0, \qquad \text{for each} \quad z \in \hat{\partial}\Omega.$$
   Show that $u \leq 0$ in Ω.
2. In this exercise, we will study a general version of Perron's method. Let $g : \hat{\mathbb{R}}^n \to \mathbb{R}$, and let
   $$S_g = \{v \in C(\Omega) : v \text{ subharmonic in } \Omega, \limsup_{\Omega\ni x\to z} u(x) \leq g(z) \text{ for each } z \in \hat{\partial}\Omega\}.$$
   Define the *Perron solution* $u : \Omega \to \mathbb{R}$ by
   $$u(x) = \sup_{v\in S_g} v(x), \qquad x \in \Omega.$$
   We call a subharmonic function $\varphi \in C(\Omega)$ a *barrier at* $z \in \hat{\partial}\Omega$ if $\varphi(x) \to 0$ as $\Omega \ni x \to z$, and $\sup_{\Omega\setminus B_\delta(z)} \varphi < 0$ for each $\delta > 0$.
   (a) Show that if g is bounded, then u is well-defined.
   (b) Prove that if u is well-defined, then $\Delta u = 0$ in Ω.
   (c) Supposing that u is well-defined, prove that if there is a barrier at $z \in \hat{\partial}\Omega$ and if g is continuous at z, then $u(x) \to g(z)$ as $\Omega \ni x \to z$.

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(d) Establish an analogue of the exterior sphere condition at ∞.

3. (a) Why is the existence of a barrier for any specific point on the boundary $\partial\Omega$ of a domain Ω a local property of that point? In other words, if $z \in \partial\Omega$ is regular for the Dirichlet problem on Ω, and if Ω' is a domain that coincides with Ω in a neighbourhood of z (hence in particular $z \in \partial\Omega'$), then can you conclude that z is also regular for the Dirichlet problem on Ω'?
   (b) Show that in the plane the Dirichlet problem is solvable for continuous boundary values if the domain can be touched at any of its boundary points by one end of a straight line segment, however short, having no other point in common with the domain.
   (c) Prove *Zaremba's criterion*: A boundary point z of a domain $\Omega \subset \mathbb{R}^n$ is regular if z is the vertex of a finite right circular cone, however small, which has no point in common with Ω. Take $n = 3$ if you prefer.
4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $\beta > 0$. For a given function f on Ω, define its *Riesz potential* by
$$(P_\beta f)(x) = \int_\Omega \frac{f(y)\,\mathrm{d}y}{|x-y|^{n-\beta}},$$
whenever it makes sense. Note that $\beta = 2$ corresponds to the Newtonian potential.
   (a) Prove that
$$\|P_\beta f\|_{L^q(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \qquad f \in L^p(\Omega),$$
for some constant $C > 0$, if $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ satisfy $n - \beta p < np/q$.
   (b) Using this result, investigate the question $P_2 f \in C^1(\Omega)$ for $f \in L^p(\Omega)$.
   (c) Prove that if f is compactly supported and Dini continuous in $\mathbb{R}^n$, then its *Riesz transform*
$$(R_i f)(x) = \lim_{\varepsilon \searrow 0} \int\limits_{|y-x|>\varepsilon} \frac{x_i - y_i}{|x-y|^{n+1}} f(y)\,\mathrm{d}y,$$
is continuous in $\mathbb{R}^n$.
   (d) Let K and K' be compact sets such that K is contained in the interior of K', and that $K' \subset \Omega$. Prove that
$$\|P_2 f\|_{C^{2,\alpha}(K)} \leq C\|f\|_{C^{0,\alpha}(K')}, \qquad f \in C^{0,\alpha}(K'),$$
where $C > 0$ is a constant and $0 < \alpha < 1$.
   (e) Let $\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$, with $f \in C^{0,\alpha}(\Omega)$. Show that $u \in C^{2,\alpha}(K)$ for any compact $K \subset \Omega$.
5. Let $\Omega \subset \mathbb{R}^n$ and let $\alpha > 0$. Recall that the *Hölder space* $C^{0,\alpha}(\Omega)$ is the space of functions $u \in C(\Omega)$ for which the *Hölder norm*
$$\|u\|_{C^{0,\alpha}(\Omega)} := \sup_{x\in\Omega} |u(x)| + \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$
is finite. We know that $C^{0,\alpha}(\Omega)$ is a Banach space.
   (a) Show that $C^{0,\alpha}(\Omega)$ consists of only constants if $\alpha > 1$.
   (b) Show that $C^1(\mathbb{R})$ is not dense in $C^{0,\alpha}(\mathbb{R})$ for any $0 < \alpha \leq 1$.

(c) Prove that for any $u \in C^{0,\alpha}(\mathbb{R}^n)$, there exists a sequence $\{u_j\} \subset C^\infty(\mathbb{R}^n)$ such that $u_j \to u$ uniformly and $\|u_j\|_{C^{0,\alpha}}$ uniformly bounded.

6. Let $\Omega \subset \mathbb{R}^n$ be an open set, let $k \geq 0$ be an integer, and let $1 \leq p \leq \infty$. Then the *Sobolev space* $W^{k,p}(\Omega)$ by definition consists of those $u \in \mathscr{D}'(\Omega)$ such that $\partial^\alpha u \in L^p(\Omega)$ for each α with $|\alpha| \leq k$. Equip it with the norm
$$\|u\|_{W^{k,p}(\Omega)} = N(\{\|\partial^\alpha u\|_{L^p(\Omega)} : |\alpha| \leq k\}),$$
where N is a norm on the finite dimensional space $\{\lambda_\alpha \in \mathbb{R} : |\alpha| \leq k\}$.
a) Show that the topology of $W^{k,p}(\Omega)$ does not depend on the choice of N.
b) Show that $W^{k,p}(\Omega)$ is a Banach space for any $k \geq 0$ and $1 \leq p \leq \infty$.
c) Prove that $\mathscr{D}(\mathbb{R}^n)$ is a dense subspace of $W^{k,p}(\mathbb{R}^n)$, for any $k \geq 0$ and $1 \leq p < \infty$.

7. Recall that the *Sobolev inequality*
$$\|u\|_{L^q} \leq C\|u\|_{W^{1,p}}, \qquad u \in \mathscr{D}(\mathbb{R}^n), \tag{1}$$
with some constant $C = C(p,q)$, is valid when $1 \leq p \leq q < \infty$, and $\frac{1}{p} \leq \frac{1}{q} + \frac{1}{n}$.
a) By way of a counterexample, show that the inequality (1) fails whenever $q < p$.
b) Show that (1) fails when $\frac{1}{p} > \frac{1}{q} + \frac{1}{n}$.
c) Show that (1) fails for $p = n$ and $q = \infty$ when $n \geq 2$.
d) Derive sufficient conditions on the exponents p, q, k, m under which the inequality
$$\|u\|_{W^{m,q}} \leq C\|u\|_{W^{k,p}}, \qquad u \in \mathscr{D}(\mathbb{R}^n),$$
is valid.