MATH 580 ASSIGNMENT 3

DUE THURSDAY OCTOBER 25

1. Derive fundamental solutions for the following operators.
   a) Δ^m in $\mathbb{R}^n$, where m is a positive integer.
   b) $-\Delta + c$ in $\mathbb{R}^3$, where $c > 0$ is a real constant.
2. Let Ω be an open subset of $\mathbb{R}^n$. Show that if $u \in C^2(\Omega)$ is harmonic in Ω then
$$\int_{\partial B} \partial_\nu u = 0,$$
   for any ball B whose closure is contained in Ω. Here $\partial_\nu u$ is the normal derivative of u. Conversely, prove that if $u \in C^1(\Omega)$ satisfies the above property for any ball B whose closure is contained in Ω, then u is harmonic in Ω.
3. Let u be an entire harmonic function in $\mathbb{R}^n$. Prove the followings.
   (a) If $u \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$ then $u \equiv 0$.
   (b) If u satisfies $u(x) \geq -C(1 + |x|)^m$ for some constants C and $m \in \mathbb{N}$, then u is a polynomial of degree less or equal to m.
   (c) Any tangent hyperplane to the graph of u intersects the graph more than once.
4. We say $u \in C(\Omega)$ is *subharmonic* in Ω if for each $y \in \Omega$ there exists $r^* > 0$ such that
$$u(y) \leq \frac{1}{|B_r|} \int_{B_r(y)} u, \qquad \forall r \in (0, r^*).$$
   Prove the following statements.
   (a) A function $u \in C^2(\Omega)$ is subharmonic in Ω iff $\Delta u \geq 0$ in Ω.
   (b) If u is harmonic in Ω, then $|\nabla u|^2$ is subharmonic in Ω.
   (c) A function subharmonic in $\mathbb{R}^2$ and bounded from above must be constant. Is this statement true in $\mathbb{R}^n$ for $n \geq 3$?
5. Let Ω be a domain, and let $\Sigma = \partial\Omega \cap B$ be a smooth and nonempty portion of the boundary, where B is an open ball. Let $u \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma)$ satisfy $\Delta u = 0$ in Ω and $u = \partial_\nu u = 0$ on Σ. Show that u is identically zero in Ω.
6. Let u be a harmonic function, and define
$$q(r) = \int_{\partial B_r} u^2, \qquad \text{for} \quad r > 0.$$
   Prove that
   a) q is monotone and convex.
   b) q is log-convex, i.e., $\log q(r)$ is a convex function of $\log r$.

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7. Prove the *Hopf lemma*: Let $u \in C^2(B_r) \cap C(\overline{B}_r)$ be a function harmonic in B_r, which attains its maximum at $z \in \partial B_r$. Show that unless u is constant, there exists $c > 0$ such that $u(z) - u(zt) \geq (1-t)c$ for all $0 < t < 1$.
8. Consider the equation
$$(\Delta + \lambda)u = 0,$$
in a bounded domain $\Omega \subset \mathbb{R}^n$, where λ is a real parameter. We say that the maximum principle holds for the particular value λ if $(\Delta + \lambda)u = 0$ in Ω implies
$$u(x) \leq \sup_{\partial\Omega} u, \qquad \text{for all} \quad x \in \Omega.$$
Try to identify the set of $\lambda \in \mathbb{R}$ for which the maximum principle holds by proving the maximum principle for some values of λ and exhibiting counterexamples for other values.