MATH 580 ASSIGNMENT 2

DUE THURSDAY OCTOBER 11

1. Let X and Z be Banach spaces, and let $U \subset X$ be an open set. Then a mapping $f : U \to Z$ is called *Fréchet differentiable* at $x \in U$ if

$$f(x+h) = f(x) + \Lambda h + o(\|h\|), \qquad \text{as} \quad X \ni h \to 0,$$

for some bounded linear operator $\Lambda : X \to Z$. We call $Df(x) = \Lambda$ if it exists, the *Fréchet derivative* of f at x. Prove the followings.

   a) The *inverse function theorem*: Suppose that Df exists and is continuous in U, and that $Df(x)$ is invertible. Then there is an open neighbourhood of x on which f is invertible. *Hint*: Given $z \in Z$ close to $f(x)$, consider the map $\phi : U \to X$ defined by $\phi(y) = y + [Df(x)]^{-1}(z - f(y))$.
   b) The *implicit function theorem*: Let X, Y and Z be Banach, and with $A \subset X \times Y$ an open set, let $g : A \to Z$ be a continuously differentiable mapping. Moreover, assume that the point $(a, b) \in A$ has the property that $g(a, b) = 0$ and that $D_y g(a, b)$ is invertible, where $D_y g(x, y)$ is the Fréchet derivative of $y \mapsto g(x, y)$, with fixed x. Then there is an open set $U \subset X$ and a function $h : U \to Y$ with $h(a) = b$, such that $g(x, h(x)) = 0$ for all $x \in U$. *Hint*: Consider the function $f(x, y) = (x, g(x, y))$.
   c) In a), prove that the inverse f^{-1} is C^1. You can take $X = Y = \mathbb{R}^n$ if you prefer.

2. Let X be a Banach space, and with $U \subset X$ open, let $f : U \to X$ be a locally Lipschitz map, which means that f is Lipschitz on any closed bounded subset of U. Consider the initial value problem

$$u'(t) = f(u(t)), \qquad u(0) = x \in U. \tag{1}$$

   a) Prove that for any $x \in U$ there exists a unique maximal solution $u : (a, b) \to U$ to (1), and if $b < \infty$ then $u(t)$ leaves any closed bounded subset of U as $t \nearrow b$.
   b) Extend the above result to *nonautonomous* problems, where $U \subset X \times \mathbb{R}$ is open, $f : U \to X$ is locally Lipschitz, and the problem is replaced by

$$u'(t) = f(u(t), t), \qquad u(t_0) = x, \quad \text{with} \quad (x, t_0) \in U.$$

   c) Prove a theorem that makes sense of the following: If f depends on a parameter continuously, then the solution to (1) depends on that parameter continuously.
   d) In a), prove that u is C^2 if f is C^1. You can take $U = X = \mathbb{R}^n$ if you prefer.

Date: Fall 2012.

3. Prove that the *Lorenz system*

$$\begin{cases} x' = \sigma(y - x) \\ y' = x(\rho - z) - y \\ z' = xy - \beta z \end{cases}$$

where σ, ρ, and β are positive parameters, is globally well-posed, meaning that for any given initial data, a unique solution exists for all time $t \in \mathbb{R}$, which depends continuously on the initial data.

4. Consider the equation

$$xyu_x + (2y^2 - x^6)u_y = 0, \qquad x > 0,\ y > 0.$$

Determine and sketch the characteristics. For $n = 1, 2, 3$, and $\alpha > 0$, consider the initial condition

$$u(x, \alpha x^n) = x^2.$$

For which $\alpha > 0$ does the problem have a solution? Give an explicit expression for the solution. For which $\alpha > 0$ is the solution uniquely determined?

5. Consider the quasilinear first-order evolution equation

$$\partial_t u(t,x) + \sum_{i=1}^{n} \alpha_i(t, x, u(x,t))\partial_i u(t,x) = \alpha_{n+1}(t, x, u(t,x)), \tag{2}$$

where α_i, $i = 1, \dots, n+1$, are C^1 functions of the $n+2$ variables (t, x, u). We impose the initial condition

$$u(0, x) = g(x),$$

with a C^1 function g.

a) Suppose that $\gamma(t, \xi)$ is a function with values in $\mathbb{R}^{n+1}$, satisfying

$$\partial_t \gamma(t,\xi) = \alpha(t, \gamma(t,\xi)), \qquad \gamma(0,\xi) = (\xi, g(\xi)), \qquad \xi \in \mathbb{R}^n.$$

Show that if u solves (2) with the initial condition g, then

$$u(t, \hat{\gamma}(t,\xi)) = \gamma_{n+1}(t,\xi),$$

where $\hat{\gamma} = (\gamma_1, \dots, \gamma_n)$.

b) Prove a local existence theorem for the initial value problem for (2).

6. Consider the initial value problem

$$u_t + uu_x = u^2, \qquad u(x, 0) = g(x).$$

Prove that a solution u satisfies

$$u(x,t) = \frac{g(\xi)}{1 - tg(\xi)}, \qquad \text{with} \qquad x = \xi - \log(1 - tg(\xi)).$$

Prove that if $g \in C^1(\mathbb{R})$ and $\|g\|_{\infty,\mathbb{R}} + \|g'\|_{\infty,\mathbb{R}} < \infty$, then there exists $T > 0$ such that the initial value problem has a unique C^1 solution defined on $\mathbb{R} \times (-T, T)$. Show that if g is given by

$$g(x) = \begin{cases} 1, & x \leq 0, \\ 1-x, & 0 \leq x \leq 1, \\ 0, & x \geq 1, \end{cases}$$

then not only we have the issues caused by the multi-valuedness, but also that $u(x,t) \to \infty$ for $x < 0$ as $t \to 1$.

7. Consider the problem

$$\begin{aligned} u_t + h(u)u_x &= 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) &= g(x), \qquad c \in \mathbb{R}, \end{aligned} \tag{3}$$

where $h \in C^\infty(\mathbb{R})$ and $g \in C^1(\mathbb{R})$.

a) Determine a condition on g and h so that (3) has a unique C^1 solution u on $\mathbb{R} \times [0,T]$ for some $T > 0$ small enough.
b) What is the least upper bound T_c on T?
c) In the case we cannot expect that all solutions remain C^1 we need to introduce the concept of weak solutions. Give a definition of weak solutions to (3).
d) Derive the equivalent of the Rankine-Hugoniot jump conditions for (3).