Generalized solutions to Monge-Ampère Equations

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Contents

1 Introduction

In this report we will talk about generalized solutions to the Monge-Ampère Equation. We will follow closely [1], trying to provide more detail in the proofs where it felt needed.

2 The Normal Mapping

In this Section we introduce the concept of normal mapping of a function and study some of his properties. We will then use it in the next Section to introduce the notion of generalized solutions to the Monge-Ampère equation.

Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. Given $x_0 \in \Omega$, a supporting hyperplane to the function u at the point $(x_0, u(x_0))$ is an affine function $l(x) = u(x_0) + p \cdot (x - x_0)$ such that $u(x) \geq l(x)$ for all $x \in \Omega$ or $u(x) \leq l(x)$ for all $x \in \Omega$.

Definition 2.1. *The normal mapping of u, or subdifferential of u, is the set-valued function $\partial u : \Omega \to \mathcal{P}(\mathbb{R}^n)$ defined by*

$$\partial u(x_0) = \{p : u(x) \geq u(x_0) + p \cdot (x - x_0), \text{ for all } x \in \Omega\}$$

Given $E \subseteq \Omega$, we define $\partial u(E) = \cup_{x \in E} \partial u(x)$.

Remark 2.2. *In other words, $\partial u(x_0)$ is the set of supporting hyperplanes to u at the point $(x_0, u(x_0)$ from below.*

Remark 2.3. *Note that the set $\partial u(x_0)$ may be empty. Let $S = \{x \in \Omega : \partial u(x) \neq \emptyset\}$. If $u \in C^1(\Omega)$ and $x \in S$, then $\partial u(x) = Du(x)$ (we drop the brackets every time $\partial u(x)$ consists of a single point), where Du denotes the gradient of u at x. This means that when u is differentiable the normal mapping is basically the gradient. Moreover, if $u \in C^2(\Omega)$ and $x \in S$, then the Hessian of u is non-negative definite, that is $D^2u(x) \geq 0$. This means that when $u \in C^2$, S is the set where the graph of u is convex. Indeed, for $x \in S$ we have by Taylor's Theorem that*

$$u(x+h) = u(x) + Du(x) \cdot h + \frac{1}{2}\left\langle D^2u(\xi)h, h\right\rangle$$

where ξ lies on the segment between x and $x+h$. From the definition of the normal mapping of u we have also that

$$u(x+h) \geq u(x) + Du(x) \cdot h$$

for all h sufficiently small. Hence $\left\langle D^2u(\xi)h, h\right\rangle \geq 0$ for all h sufficiently small and the claim follows.

Remark 2.4. *Given $x_0 \in \Omega$, then $\partial u(x_0)$ is convex. However, if $K \subseteq \Omega$ is convex, then $\partial u(K)$ is not necessarily convex.*

Example 2.5. *Let's compute the normal mapping of a function whose graph is a cone in $\mathbb{R}^{n+1}$. For that effect, let $\Omega = B_R(x_0)$ in $\mathbb{R}^n$, $h > 0$ and $u(x) = h\frac{|x-x_0|}{R}$. The graph of u, for $x \in \Omega$, is a right circular cone in $\mathbb{R}^{n+1}$ with vertex at the point $(x_0, 0)$ and base on the hyperplane $x_{n+1} = h$. We will show that*

$$\partial u(x) = \begin{cases} \frac{h}{R}\frac{x-x_0}{|x-x_0|}, & \text{for } 0 < |x-x_0| < R, \\ \overline{B_{h/R}(0)}, & \text{for } x = x_0. \end{cases}$$

Let $x \in \Omega$. If $0 < |x - x_0| < R$, then the value of ∂u follows by computing the gradient of u at x. Otherwise, take $x = x_0$. By the definition of normal mapping, $p \in \partial u(x_0)$ if and only if

$$\frac{h}{R}|x - x_0| \geq p \cdot (x - x_0)$$

for all $x \in B_R(x_0)$. Clearly, if $p = 0$ then $p \in \partial u(x_0)$. If $p \neq 0$, taking $x = x_0 + R\frac{p}{|p|}$ in the above inequality leads to $|p| \leq \frac{h}{R}$. Using the Cauchy-Schwarz inequality, it is clear that $|p| \leq \frac{h}{R}$ implies that $p \in \partial u(x_0)$. Hence $\partial u(x_0) = \overline{B_{h/R}(0)}$.

Lemma 2.6. *If $\Omega \subseteq \mathbb{R}^n$ is open, $u \in C(\Omega)$ and $K \subseteq \Omega$ is compact, then $\partial u(K)$ is compact.*

Proof. We will prove that every sequence in $\partial u(K)$ has a convergent subsequence whose limit lies in $\partial u(K)$. Let $\{p_k\} \subseteq \partial u(K)$ be a sequence. The proof will go as follows: first we show that that the sequence $\{p_k\}$ is bounded; then by the Bolzano-Weierstrass theorem it has a convergence subsequence whose limit we prove to lie in $\partial u(K)$.

Step 1 : The sequence $\{p_k\}$ is bounded.

By the definition of normal mapping, for each k there exists $x_k \in K$ such that $p_k \in \partial u(x_k)$, i.e.,

$$u(x) \geq u(x_k) + p_k \cdot (x - x_k)$$

for all $x \in \Omega$. Since K is compact, we may assume, by passing if necessary to a subsequence, that $x_k \to x_0$ for some $x_0 \in K$. Also from the compactness of K, $K_\delta = \{x : \text{dist}(x, K) \leq \delta\}$ is compact and contained in Ω for all δ sufficiently small, For all $|w| = 1$ and for all k we have that $x_k + \delta w \in K_\delta$ and

$$u(x_k + \delta w) \geq u(x_k) + \delta p_k \cdot w.$$

If $p_k \neq 0$, taking $w = \frac{p_k}{|p_k|}$ in the above inequality leads to

$$u\left(x_k + \delta \frac{p_k}{|p_k|}\right) \geq u(x_k) + \delta |p_k|$$

and therefore

$$\max_{K_\delta} u(x) \geq \min_K u(x) + \delta |p_k|,$$

for all k. Since $u \in C(\Omega)$, u is bounded on any compact set contained in Ω and therefore the claim is proved.

Step 2 : There exists a convergent subsequence $\{p_{k_m}\}$ with limit $p_0 \in \mathbb{R}^n$.

This is a direct consequence of the Bolzano-Weierstrass since $\{p_k\}$ is bounded.

Step 3 : $p_0 \in \partial u(K)$.

We will prove that $p_0 \in \partial u(x_0)$. Since $p_{k_m} \in \partial u(x_{k_m})$, we have that

$$u(x) \geq u(x_{k_m}) + p_{k_m} \cdot (x - x_{k_m})$$

for all $x \in \Omega$. Since u is continuous, by letting $m \to \infty$, we obtain

$$u(x) \geq u(x_0) + p_0 \cdot (x - x_0)$$

for all $x \in \Omega$ and thus $p_0 \in \partial u(x_0)$, completing the proof of the Lemma. □

Lemma 2.7. *If u is a convex function in Ω and $K \subseteq \Omega$ is compact, then u is uniformly Lipschitz in K, i.e., there exists a constant $C = C(u, K)$ such that $|u(x) - u(y)| \leq C|x - y|$ for all $x, y \in K$.*

Proof. Since u is convex, u has a supporting hyperplane at any $x \in \Omega$. Let $C = \sup\{|p| : p \in \partial u(K)\}$. By Lemma 2.6, $C < \infty$. Let $x \in K$. Then

$$u(y) \geq u(x) + p \cdot (y - x)$$

for $p \in \partial u(x)$ and for all $y \in \Omega$. In particular, for $y \in K$, we have

$$u(y) - u(x) \geq -|p||y - x|.$$

By reversing the roles of x and y we prove the Lemma. □

Lemma 2.8. *If Ω is open and u is Lipschitz continuous in Ω, then u is differentiable a.e. in Ω.*

Proof. See [[2], p.81]. □

Lemma 2.9. *If u is convex or concave in Ω, then u is differentiable a.e. in Ω.*

Proof. Follows immediately from the previous two Lemmas. $\square$

Definition 2.10. *The Legendre transform of the function $u : \Omega \to \mathbb{R}$ is the function $u^* : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$u^*(p) = \sup_{x\in\Omega} \{x \cdot p - u(x)\} .$$

Remark 2.11. *If Ω is bounded and u is bounded in Ω, then u^* is finite. Also u^* is convex in $\mathbb{R}^n$. Indeed, let $p_1, p_2 \in \mathbb{R}^n$ and $0 \le t \le 1$. Then*

$$\begin{aligned} u^*(tp_1 + (1-t)p_2) &= \sup_{x\in\Omega} \{x \cdot (tp_1 + (1-t)p_2) - u(x)\} \\ &= \sup_{x\in\Omega} \{t(x \cdot p_1 - u(x)) + (1-t)(x \cdot p_2 - u(x))\} \\ &\le t \sup_{x\in\Omega} \{x \cdot p_1 - u(x)\} + (1-t) \sup_{x\in\Omega} \{x \cdot p_2 - u(x)\} \\ &= tu^*(p_1) + (1-t)u^*(p_2). \end{aligned}$$

Remark 2.12. *There is a close relation between the normal mapping of a function u and its Legendre transform: $p \in \partial u(x_0)$ if and only if $u^*(p) = p \cdot x_0 - u(x_0)$.*

Lemma 2.13. *If Ω is open and $u \in C(\Omega)$, then the set of points in $\mathbb{R}^n$ that belong to the image by the normal mapping of more than one point of Ω has Lebesgue measure zero. That is, the set*

$$S = \{p \in \mathbb{R}^n : \text{ there is } x_1, x_2 \in \Omega \text{ with } x_1 \neq x_2 \text{ and } p \in \partial u(x_1) \cap \partial u(x_2)\}$$

has measure zero. This also means that the set of supporting hyperplanes that touch the graph of u at more than one point has measure zero.

Proof. We start by proving that we can assume that Ω is bounded u is bounded in Ω. We write $\Omega = \cup_{k=1}^{\infty} \Omega_k$ where the $\Omega_k \subseteq \Omega_{k+1}$ are open and the $\overline{\Omega_k} \subseteq \Omega$ are compact. Let

$$S_m = \{p \in \mathbb{R}^n : \text{ there is } x_1, x_2 \in \Omega \text{ with } x_1 \neq x_2 \text{ and } p \in \partial u|_{\Omega_m}(x_1) \cap \partial u|_{\Omega_m}(x_2)\}$$

where $u|_{\Omega_m}$ is the restriction of u to Ω_m. Then $S \subseteq \cup_{m=1}^{\infty} S_m$. Indeed let $p \in S$. Then there exists $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$ and

$$u(z) \ge u(x_i) + p \cdot (z - x_i)$$

for all $z \in \Omega$ and $i = 1, 2$. Since the Ω_k increase with k, $x, y \in \Omega_m$ for some m and obviously the previous inequalities hold for all $z \in \Omega_m$. Hence $p \in S_m$.

Since $S \subseteq \cup_{m=1}^{\infty} S_m$ it is enough to prove that each S_m has measure zero. Hence we can in fact assume that Ω is bounded and u is bounded in Ω.

Let $E = \{p \in \mathbb{R}^n : u^* \text{ is not differentiable at } p\}$, where u^* is the Legendre transform of u. By Remark 2.11 and Lemma 2.9 u^* is finite and is differentiable a.e. and therefore E has Lebesgue measure zero. Thus, proving that $S \subseteq E$ completes the proof of the Lemma. Let $p \in S$. Then there exists $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$ and $p \in \partial u(x_1) \cap \partial u(x_2)$. Then by Remark 2.12, $u^*(p) = p \cdot x_i - u(x_i)$ for $i = 1, 2$. We have from the definition of u^* that

$$u^*(z) \geq x_i \cdot z - u(x_i)$$

for all $z \in \Omega$ and $i = 1, 2$, which we can rewrite as

$$u^*(z) \geq u^*(p) + x_i \cdot (z - p)$$

for all $z \in \Omega$ and $i = 1, 2$. Hence if u^* were differentiable at p we would have by Remark 2.11 $Du^*(p) = x_i$ for $i = 1, 2$, but since $x_1 \neq x_2$, u^* is not differentiable at p, i.e., $p \in E$. ☐

Theorem 2.14. *If Ω is open and $u \in C(\Omega)$, then the class*

$$\mathcal{S} = \{E \subseteq \Omega : \partial u(E) \text{ is Lebesgue measurable}\}$$

is a Borel σ-algebra. The set function $Mu : \mathcal{S} \to \overline{\mathbb{R}}$ defined by

$$Mu(E) = |\partial u(E)| \tag{1}$$

is a measure, finite on compacts, that is called the Monge-Ampère measure associated with the function u.

Proof. We need to show that $\mathcal{S}$ is closed under countable unions and complements. If $\{E_i\}_{i=1}^{\infty}$ is a sequence of subset of Ω then $\partial u(\cup_{i=1}^{\infty} E_i) = \cup_{i=1}^{\infty} \partial u(E_i)$ and so if $E_i \in \mathcal{S}$ for $i = 1, 2, \cdots$, then $\cup_{i=1}^{\infty} E_i \in \mathcal{S}$. By Lemma 2.6, the class $\mathcal{S}$ contains all compact subsets of Ω. Hence $\Omega \in \mathcal{S}$ since we can write $\Omega = \cup_{i=1}^{\infty} K_i$ with K_i compact.

It remains to show, in order to prove that $\mathcal{S}$ is a σ-algebra, that if $E \in \mathcal{S}$ then $\Omega \setminus E \in \mathcal{S}$. We first note that for any set $E \subseteq \Omega$

$$\partial u(\Omega \setminus E) = (\partial u(\Omega) \setminus \partial u(E)) \cup (\partial u(\Omega \setminus E) \cap \partial u(E)).$$

Let $E \in \mathcal{S}$. Clearly, $\partial u(\Omega) \setminus \partial u(E)$ is Lebesgue measurable. By Lemma 2.13, $|\partial u(\Omega \setminus E) \cap \partial u(E)| = 0$ and so it is also Lebesgue measurable. Hence by the formula above $\partial u(\Omega \setminus E)$ is Lebesgue measurable, i.e., $\Omega \setminus E \in \mathcal{S}$.

We now show that Mu is a measure. Clearly $Mu(\emptyset) = 0$ and so we only need to show that Mu is σ-additive. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of disjoint sets in $\mathcal{S}$ and set $H_i = \partial u(E_i)$. Since $\partial u(\cup_{i=1}^{\infty} E_i) = \cup_{i=1}^{\infty} H_i$ we need to show that

$$\left| \bigcup_{i=1}^{\infty} H_i \right| = \sum_{i=1}^{\infty} |H_i|.$$

Let us write

$$\bigcup_{i=1}^{\infty} H_i = \bigcup_{i=1}^{\infty} \tilde{H}_i$$

where $\tilde{H}_i = H_i \setminus \left(\cup_{j=1}^{i-1} H_j\right)$. Hence $\left\{\tilde{H}_i\right\}_{i=1}^{\infty}$ is a sequence of disjoint Lebesgue measurable sets and therefore

$$\left|\bigcup_{i=1}^{\infty} H_i\right| = \sum_{i=1}^{\infty} |\tilde{H}_i|.$$

We have $E_i \cap E_j = \emptyset$ for $i \neq j$. Then by Lemma 2.13$|H_i \cap H_j| = 0$ for $i \neq j$ and $|H_i \cap \left(\cup_{j=1}^{i-1} H_j\right)| = 0$. Hence $|H_i| = |\tilde{H}_i|$ and thus proving that Mu is σ-additive.

We only have left to prove that Mu is finite on compacts but that is just a simple consequence of Lemma 2.6. □

Example 2.15. *If $u \in C^2(\Omega)$ is a convex function, then the Monge-Ampère measure Mu associated with u satisfies*

$$Mu(E) = \int_E \det D^2 u(x) dx$$

for all Borel sets $E \subseteq \Omega$. To prove this we need to use the following result, which we won't prove.

Theorem 2.16. *(Sard's Theorem, see [4]) Let $\Omega \in \mathbb{R}^n$ be an open set and $g : \Omega \to \mathbb{R}^n$ a C^1 function in Ω. If $S_0 = \{x \in \Omega : \det g'(x) = 0\}$ then $|g(S_0)| = 0$, where $g'(x) = \left(\frac{\partial g_i(x)}{\partial x_j}\right)_{i,j=1}^{n}$.*

We first notice that since u is convex and $C^2(\Omega)$, then Du is one-to-one on the set

$$A = \left\{x \in \Omega : D^2 u(x) > 0\right\}.$$

Indeed let $x_1, x_2 \in A$ with $Du(x_1) = Du(x_2)$. We will show that $x_1 = x_2$. By convexity

$$u(z) \geq u(x_i) + Du(x_i) \cdot (z - x_i)$$

for all $z \in \Omega$ and $i = 1, 2$. Hence taking $z = x_2$ for $i = 1$ and $z = x_1$ for $i = 2$ we get that

$$u(x_1) - u(x_2) = Du(x_1) \cdot (x_1 - x_2) = Du(x_2) \cdot (x_1 - x_2)$$

By the Taylor's formula we can write

$$u(x_1) = u(x_2) + Du(x_2) \cdot (x_1 - x_2) + \int_0^1 t \left\langle D^2 u(x_2 + t(x_1 - x_2))(x_1 - x_2), x_1 - x_2 \right\rangle dt.$$

Therefore the integral is zero and the integrand must vanish for $0 \leq t \leq 1$. Since $x_2 \in A$ it follows that $x_2 + t(x_1 - x_2) \in A$ for t small. Therefore $x_1 = x_2$.

If $u \in C^2(\Omega)$ then $g = Du \in C^1(\Omega)$. Since $u \in C^2(\Omega)$ and u is convex, $\partial u(E) = Du(E)$ and so $Mu(E) = |Du(E)|$. Also

$$Du(E) = Du(E \cap S_0) \cup Du(E \setminus S_0).$$

Since $E \subseteq \mathbb{R}^m$ is a Borel set, $E \cap S_0$ and $E \setminus S_0$ are also Borel sets. Hence, by the formula of change of variables and Sard's Theorem, we get

$$Mu(E) = Mu(E \cap S_0) + Mu(E \setminus S_0) = \int_{E \setminus S_0} \det D^2 u(x) dx = \int_E \det D^2 u(x) dx.$$

Example 2.17. *If $u(x)$ is the cone of Example 2.5, then the Monge-Ampère measure associated with u is $Mu = |B_{h/R}|\delta_{x_0}$, where δ_{x_0} denotes the Dirac delta at x_0.*

3 Generalized Solutions

In this Section we introduce the notion of generalized solutions, using the Monge-Ampère measure defined in the previous Section, and study their stability .

Definition 3.1. *Let μ be a Borel measure defined in Ω an open and convex subset of $\mathbb{R}^n$. The convex function $u \in C(\Omega)$ is a generalized solution, or Aleksandrov solution, to the Monge-Ampère equation*

$$\det D^2 u = \mu \tag{2}$$

if the Monge-Ampère measure Mu associated with u is defined by (1) equals μ.

Remark 3.2. *Given $f \in C(\Omega)$ with $f \geq 0$, we will also say that the convex function $u \in C(\Omega)$ is a generalized solution to the Monge-Ampère equation*

$$\det D^2 u = f \tag{3}$$

if $Mu(E) = \int_E f(x)dx$ for all Borel subsets E of Ω.

The following Lemma addresses the stability of generalized solutions proving that this notion is closed under uniform limits. In other words, if u_j are generalized solutions to $\det D^2 u = \mu$ in Ω and $u_j \to u$ uniformly on compact subsets of Ω, then u is also a generalized solution to $\det D^2 u = \mu$ in Ω.

Lemma 3.3. *Let $u_j \in C(\Omega)$ be a convex functions such that $u_j \to u$ uniformly on compact subsets of Ω. Then*

i) if $K \subseteq \Omega$ is compact then

$$\limsup_{j\to\infty} \partial u_j(K) \subseteq \partial u(K)$$

and by Fatou's Lemma

$$\limsup_{j\to\infty} |\partial u_j(K)| \leq |\partial u(K)|$$

.

ii) if K is compact and U is open such that $K \subseteq U \subseteq \overline{U} \subseteq \Omega$ then

$$\partial u(K) \subseteq \liminf_{j\to\infty} \partial u_j(K)$$

where the inequality holds for almost every point on the set on the left-hand side and by Fatou's Lemma

$$|\partial u(K)| \leq \liminf_{j\to\infty} |\partial u_j(K)|.$$

iii) if u_j are generalized solutions to $\det D^2 u = \mu$ in Ω and $u_j \to u$ uniformly on compact subsets of Ω, then u is also a generalized solution to $\det D^2 u = \mu$ in Ω.

Proof.

i) Let $K \subseteq \Omega$ be a compact and $p \in \limsup_{j\to\infty} \partial u_j(K)$. By definition of lim sup for each n there exists j_n and $x_{j_n} \in K$ such that $p \in \partial u_{j_n}(x_{j_n})$. Passing to a subsequence x_i of x_{j_n} if necessary, we may assume that $x_i \to x_0$ for some $x_0 \in K$. Now, since $p \in \partial u(x_i)$ we have

$$u_i(x) \geq u_i(x_i) + p \cdot (x - x_i)$$

for all $x \in \Omega$. Letting $i \to \infty$, by the uniform convergence of u_i on compact subsets of Ω we obtain

$$u(x) \geq u(x_i) + p \cdot (x - x_0)$$

for all $x \in \Omega$, i.e., $p \in \partial u(x_0)$.

ii) Let

$$S = \{p \in \mathbb{R}^n : p \in \partial u(x_1) \cap \partial u(x_2) \text{ for some } x_1, x_2 \in \Omega \text{ with } x_1 \neq x_2\}$$

By Lemma 2.13 we know that $|S| = 0$. We will prove that

$$\partial u(K) \setminus S \subseteq \liminf_{j\to\infty} \partial u_j(U)$$

Let $p \in \partial u(K) \setminus S$. Then there exists a unique $x_0 \in K$ such that $p \in \partial u(x_0)$ and $p \notin \partial u(x_1)$ for all $x_1 \in \Omega$ with $x_1 \neq x_0$. Let U be an open set satisfying the assumptions and $x_1 \in \Omega$ with $x_1 \neq x_0$. Since $p \in \partial u(x_0)$, we have $u(x_1) \geq u(x_0) + p \cdot (x_1 - x_0)$. Moreover, we can say that the inequality is strict. Suppose $u(x_1) = u(x_0) + p \cdot (x_1 - x_0)$. Then using again the fact that $p \in \partial u(x_0)$, we have

$$\begin{aligned} u(x) &\geq u(x_0) + p \cdot (x - x_0) \\ &= u(x_1) - p \cdot (x_1 - x_0) + p \cdot (x - x_0) \\ &\geq u(x_1) + p \cdot (x - x_1) \end{aligned}$$

for all $x \in \Omega$ and therefore $p \in \partial u(x_1)$ which is a contradiction with the choice of p.

Now, let $l(x) = u(x_0) + p \cdot (x - x_0)$ and set $\delta = \min \{u(x) - l(x) : x \in \partial U\}$. Since $x_0 \notin \partial U$, $\delta > 0$ due to what we proved above. By the uniform convergence of u_j on compact subsets of Ω, there is j_0 such that $|u(x) - u_j(x)| < \delta/2$ for all $x \in \overline{U}$ and for all $j \geq j_0$. Let

$$\delta_j = \max_{x\in\overline{U}} \{l(x) - u_j(x) + \delta/2\}.$$

We have, for $j \geq j_0$, $\delta_j > 0$ and $u_j(x) - l(x) > \delta/2$ for $x \in \partial U$ due to the choice of j_0. Hence if $x_j \in \overline{U}$ is point the point where the maximum is attained, $x_j \notin \partial U$.

We now prove that $p \in \partial u_j(x_j)$ for all $j \geq j_0$ and therefore $p \in \liminf_{j\to\infty} \partial u_j(U)$. Let $j \geq j_0$. We have that

$$\delta_j = l(x_j) - u_j(x_j) + \delta/2 = u(x_0) + p \cdot (x_j - x_0) - u_j(x_j) + \delta/2$$

and therefore, by the definition of δ_j,

$$u(x_j) + p \cdot (x_j - x_0) - u_j(x_j) + \delta/2 \geq u(x_0) + p \cdot (x - x_0) - u_j(x) + \delta/2$$

for all $x \in \overline{U}$, which we can rewrite as

$$u_j(x) \geq u_j(x_j) + p \cdot (x - x_j)$$

for all $x \in \overline{U}$. Since u_j is convex in Ω and U is open the inequality above is true for all $x \in \Omega$ and therefore $p \in \partial u_j(x_j)$.

iii) It follows from i) and ii). $\square$

Lemma 3.4. *If u_k are convex functions in Ω such that $u_k \to u$ uniformly on compact subsets of Ω then the associated Monge-Ampère measures Mu_k tend to Mu weakly, that is*

$$\int_\Omega f(x) dMu_k(x) \to \int_\Omega f(x) dMu(x)$$

for every f continuous with compact support in Ω.

Proof. It is a consequence of i) and ii) of the previous Lemma. $\square$

4 Viscosity Solutions

Definition 4.1. *Let $u \in C(\Omega)$ be a convex function and $f \in C(\Omega)$ with $f \geq 0$.*

i) u is a viscosity subsolution of the equation $\det D^2 u = f$ if for all convex functions $\phi \in C^2(\Omega)$ if $x_0 \in \Omega$ is a local maximum point of $u - \phi$ then

$$\det D^2\phi \geq f(x_0).$$

ii) u is a viscosity supersolution of the equation $\det D^2 u = f$ if for all convex functions $\phi \in C^2(\Omega)$ if $x_0 \in \Omega$ is a local minimum point of $u - \phi$ then

$$\det D^2\phi \leq f(x_0).$$

We say that u is a viscosity solution of the equation $\det D^2 u = f$ if it is both a viscosity sub and supersolution of $\det D^2 u = f$.

One of our goals is to compare the notions of viscosity solutions and generalized solutions. In this Section we will see if u is a generalized solution then u is a viscosity solution. The converse result will be proved later.

Before we prove the result we make some remarks about the class of test functions used in the definition of viscosity solutions.

Remark 4.2. *If $u \in C(\Omega)$ is convex, $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum at $x_0 \in \Omega$, then $D^2\phi(x_0) \geq 0$. Indeed, since $\phi \in C^2(\Omega)$, we have*

$$\phi(x) = \phi(x_0) + D\phi(x_0) \cdot (x - x_0) + \frac{1}{2}\left\langle D^2\phi(x - x_0), x - x_0 \right\rangle + o(|x - x_0|^2)$$

Let N be a neighbourhood of x_0 where x_0 is a maximum point of $u - \phi$. Then for $x \in N$,

$$\begin{aligned} u(x) &\leq u(x_0) - \phi(x_0) + \phi(x) \\ &= u(x_0) + D\phi(x_0) \cdot (x - x_0) + \frac{1}{2} \left\langle D^2\phi(x - x_0), x - x_0 \right\rangle + o(|x - x_0|^2) \end{aligned}$$

Since u is convex, there exists $p \in \mathbb{R}^n$ such that

$$u(x) \geq u(x_0) + p \cdot (x - x_0)$$

for all $x \in \Omega$. Hence we get

$$p \cdot (x - x_0) \leq D\phi(x_0) \cdot (x - x_0) + \frac{1}{2} \left\langle D^2\phi(x - x_0), x - x_0 \right\rangle + o(|x - x_0|^2)$$

for all $x \in N$. Now, fix $\omega \in \partial B_1(0)$ Choose $x \in N$ such that $x - x_0 = \rho\omega$ for $\rho > 0$ sufficiently small. Then the above inequality becomes

$$\rho p \cdot w \leq \rho D\phi(x_0)\omega + \frac{1}{2}\rho^2 \left\langle D^2\phi(x_0)\omega, \omega \right\rangle + o(\rho^2).$$

Diving by ρ and letting $\rho \to 0$ leads to

$$(D\phi(x_0) - p) \cdot \omega \geq 0$$

Since ω was chosen arbitrarily in $\partial B_1(0)$ we conclude that $p = D\phi(x_0)$. Then we have

$$o(\rho^2) + \frac{1}{2}\rho^2 \left\langle D^2\phi(x_0)\omega, \omega \right\rangle \geq 0$$

Dividing by ρ^2 and letting $\rho \to 0$ shows that $\left\langle D^2\phi(x_0)\omega, \omega \right\rangle \geq 0$ and so we are done.

Lemma 4.3. *We can restrict the class of test functions used in the definition of viscosity solutions to the class of strictly convex quadratic polynomials.*

Proof. We consider first the subsolution case. It is enough to prove that if $\det D^2P(x_0) \geq f(x_0)$ holds for all strictly convex quadratic polynomials P and $x_o \in \Omega$ local maximum of $u - P$ implies that then u is a viscosity subsolution of the equation $\det D^2u = f$ in Ω. Let then $\phi \in C^2(\Omega)$ be convex such that $u - \phi$ has a local maximum at $x_0 \in \Omega$. We want to show that $\det D^2\phi(x_0) \geq f(x_0)$. Let P be the quadratic polynomial given by

$$P(x) = \phi(x_0) + D\phi(x_0) \cdot (x - x_0) + \frac{1}{2} \left\langle D^2\phi(x - x_0), x - x_0 \right\rangle.$$

Then since $\phi \in C^2(\Omega)$, $\phi(x) = P(x) + o(|x - x_0|^2)$. Let $\varepsilon > 0$ and consider the quadratic polynomial $P_\varepsilon(x) = P(x) + \varepsilon|x - x_0|^2$. We have

$$D^2P_\varepsilon(x_0) = D^2P(x_0) + 2\varepsilon Id$$

and so P_ε is strictly convex. We have

$$\phi(x) - P_\varepsilon(x) = o(|x - x_0|^2) - \varepsilon|x - x_0|^2$$

and therefore for x sufficiently close to x_0, $\phi(x) - P_\varepsilon(x) \leq 0 = \phi(x_0) - P_\varepsilon(x_0)$ and so x_0 is a local maximum of $\phi - P_\varepsilon$. Then by assumption

$$\det D^2 P_\varepsilon(x) = \det\big(D^2\phi(x_0) + 2\varepsilon Id\big) \geq f(x_0).$$

By letting $\varepsilon \to 0$, we obtain the desired inequality.

As for the supersolution case, let $\phi \in C^2(\Omega)$ be convex such that $u - \phi$ has a local minimum at x_0. If $D^2\phi(x_0)$ has some zero eigenvalue, then $\det D^2\phi(x_0) = 0 \leq f(x_0)$ since $f \geq 0$ in Ω. If all eigenvalues are positive, consider $P(x)$ as in the subsolution case. Then $P_\varepsilon(x) = P(x) - \varepsilon|x - x_0|^2$ is strictly convex for all $\varepsilon > 0$ sufficiently small. Hence, proceeding as in the case of subsolutions, we can show that $u - P_\varepsilon$ has a local minimum at x_0 and consequently $\det D^2\phi(x_0) \leq f(x_0)$ by letting $\varepsilon \to 0$ as before □

Proposition 4.4. *Let $f \in C(\Omega)$ with $f \geq 0$ in Ω. If u is a generalized solution to* $\det D^2 u = f$*, then u is a viscosity solution* $\det D^2 u = f$.

Proof. Let $\phi \in C^2(\Omega)$ be a strictly convex function such that $u - \phi$ has a local maximum at x_0. By the previous Lemma, to show that u is a viscosity subsolution we need to prove that $\det D^2\phi(x_0) \geq f(x_0)$. Without loss of generality we can assume that $u(x_0) = \phi(x_0)$ and that $u(x) < \phi(x)$ for all $0 < |x - x_0| \leq \delta$ for $\delta > 0$ sufficiently small. The first assumption can be made by considering if necessary $\tilde{\phi}(x) = \phi(x) + u(x_0) - \phi(x_0)$ and observing that $\tilde{\phi}$ is still a $C^2(\Omega)$ convex function such that $u - \tilde{\phi}$ has a local maximum at x_0 and $\det D^2\tilde{\phi} = \det D^2\phi$. As for the second assumption, we can consider instead $\phi_r(x) = \phi(x) + r|x - x_0|^2$, prove that $\det D^2\phi_r \geq f$ and then let $r \to 0$.

Assume then that x_0 is a strict local maximum with $u(x_0) = \phi(x_0)$. Then there is $\delta > 0$ such that $u(x) < \phi(x)$ for all $0 < |x - x_0| \leq \delta$. Let

$$m = \min_{\frac{\delta}{2} \leq |x - x_0| \leq \delta} \phi(x) - u(x)$$

We have $m > 0$. Let $0 < \varepsilon < m$ and $S_\varepsilon = \{x \in B_\delta(x_0) : u(x) + \varepsilon > \phi(x)\}$. If $\frac{\delta}{2} \leq |x - x_0| \leq \delta$, then $\phi(x) - u(x) \geq m$ by definition of m and so $x \notin S_\varepsilon$. Hence $S_\varepsilon \subseteq B_{\frac{\delta}{2}}(x_0)$. Now, let $z \in \partial S_\varepsilon$. Then there exists sequences $\{x_n\} \subseteq S_\varepsilon$ and $\{\overline{x_n}\} \subseteq B_\delta(x_0) \setminus S_\varepsilon$ such that $x_n \to z$ and $\overline{x_n} \to z$. We then have $u(x_n) + \varepsilon > \phi(x_n)$ and $u(\overline{x_n}) + \varepsilon \leq \phi(\overline{x_n})$ and therefore taking the limit as $n \to +\infty$, leads to $u(z) + \varepsilon \geq \phi(z)$ and $u(z) + \varepsilon \leq \phi(z)$. Hence $u + \varepsilon = \phi$ on ∂S_ε. Since $u + \varepsilon$ and ϕ are convex, by Lemma 5.1 (which will be proved and presented in the next Section), we have that $\partial(u + \varepsilon)(S_\varepsilon) \subseteq \partial\phi(S_\varepsilon)$. Note that $\partial(u + \varepsilon)(S_\varepsilon) = \partial u(S_\varepsilon)$. Hence, since u is a generalized solution, we have

$$\int_{S_\varepsilon} f(x)dx = Mu(S_\varepsilon) = |\partial u(S_\varepsilon)| = |\partial(u + \varepsilon)(S_\varepsilon) \leq |\partial\phi(S_\varepsilon)| = \int_{S_\varepsilon} \det D^2\phi(x)dx$$

Then by continuity of f, we obtain that $D^2\phi(x_0) \geq f(x_0)$.

A similar argument shows u is a viscosity supersolution. □

5 Maximum Principles

Lemma 5.1. *Let $\Omega \subseteq \mathbb{R}$ be a bounded open set and $u, v \in C(\overline{\Omega})$. If $u = v$ on $\partial\Omega$ and $v \geq u$ in Ω, then*

$$\partial v(\Omega) \subseteq \partial u(\Omega)$$

Proof. Let $p \in \partial v(\Omega)$. Then there exists $x_0 \in \Omega$ such that

$$v(x) \geq v(x_0) + p \cdot (x - x_x 0)$$

for all $x \in \Omega$. Let

$$a = \sup_{x \in \Omega} \{v(x_0) + p \cdot (x - x_0) - u(x)\}.$$

Since Ω is bounded and u and v are continuous in $C(\overline{\Omega})$, a is well-defined and there is $x_1 \in \overline{\Omega}$ such that

$$a = v(x_0) + p \cdot (x_1 - x_0) - u(x_1).$$

Also, since $v(x_0) \geq u(x_0)$, we have $a \geq 0$ and by definition of a

$$u(x) \geq v(x_0) + p \cdot (x - x_0) - a$$

for all $x \in \Omega$. We now consider two cases: $a = 0$ and $a > 0$. If $a = 0$ then the above inequality becomes

$$u(x) \geq v(x_0) + p \cdot (x - x_0)$$

for all $x \in \Omega$ and so $p \in \partial u(x_0) \subseteq \partial u(\Omega)$. If $a > 0$, then we can rewrite the above inequality as

$$u(x) \geq u(x_1) + p \cdot (x - x_1)$$

for all $x \in \Omega$. Thus proving that $x_1 \in \Omega$ ends the proof since in that case $p \in \partial u(x_1) \subseteq \nu(\Omega)$. Indeed, since $p \in \partial v(x_0)$ and $v \in C(\overline{\Omega})$, we have

$$v(x_1) \geq v(x_0) + p \cdot (x_1 - x_0)$$

which due to the choice of x_1 we can rewrite as $v(x_1) \geq u(x_1) + a$. Since $u = v$ on $\partial\Omega$, we have in fact $x_1 \in \Omega$. □

Theorem 5.2 (Aleksandrov's maximum principle)**.** *If $\Omega \subseteq \mathbb{R}^n$ is a bounded open and convex set with diameter Δ, and $u \in C(\overline{\Omega})$ is convex with $u = 0$ on $\partial\Omega$, then*

$$|u(x_0)|^n \leq C_n \Delta^{n-1} \operatorname{dist}(x_0, \partial\Omega) |\partial u(\Omega)|$$

for all $x_0 \in \Omega$, where C_n is a constant depending only on the dimension n.

Proof. Fix $x_0 \in \Omega$ and let v be the convex function whose graph is the upside-down cone with vertex $(x_0, u(x_0))$ and base Ω, with $v = 0$ on $\partial\Omega$. Since u is convex, $v \geq u$ in Ω. Then by Lemma 5.1

$$\partial v(\Omega) \subseteq \partial u(\Omega).$$

The idea is then to estimate the measure of $\partial v(\Omega)$ from below. We first notice that $\partial v(\Omega) = \partial v(x_0)$ and therefore $\partial v(\Omega)$ is convex by Remark 2.4. Indeed, let $p \in \partial v(\Omega)$. Then there exists $x_1 \in \Omega$ such that

$$v(x) \geq v(x_1) + p \cdot (x - x_1), \quad \forall x \in \Omega.$$

If $x_1 = x_0$ we are done. Otherwise since the graph of v is a cone with vertex $(x_0, u(x_0))$ we have $v(x_0) + p \cdot (x_1 - x_0) = v(x_1)$ and then we can write the above inequality as

$$v(x) \geq v(x_0) + p \cdot (x - x_0), \quad \forall x \in \Omega.$$

Thus $p \in \partial v(x_0)$. Geometrically we proved that any supporting hyperplane of v at $(x_1, v(x_1))$ is also a supporting hyperplane of v at $(x_0, v(x_0))$.

We now notice that there exists $p_0 \in \partial v(\Omega)$ such that $|p_0| = \frac{-u(x_0)}{\text{dist}(x_0, \partial\Omega)}$. This follows because Ω is convex. Indeed, we take $x_1 \in \partial\Omega$ such that $|x_1 - x_0| = \text{dist}(x_0, \partial\Omega)$ and H is supporting hyperplane to the set Ω at x_1. The hyperplane in $\mathbb{R}^{n+1}$ generated by H and the point $(x_0, u(x_0))$ is a supporting hyperplane to v that has the desired slope.

Now notice that the ball B with center at the origin and radius $\frac{-u(x_0)}{\Delta}$ is contained in $\partial v(\Omega)$, and $|p_0| \geq \frac{-u(x_0)}{\Delta}$. Hence the convex hull of B and p_0 is contained in $\partial v(\Omega)$ and it has measure

$$C_n \left(\frac{-u(x_0)}{\Delta} \right)^{n-1} |p_0|,$$

which proves the Theorem. □

6 Aleksandrov-Bakelman-Pucci's maximum principle

Consider $u \in C(\Omega)$ with Ω convex and the class of functions

$$\mathcal{F}(u) = \{v : v \leq u \text{ in } \Omega \text{ and } v \text{ convex in } \Omega\}$$

$$\mathcal{G}(u) = \{w : w \geq u \text{ in } \Omega \text{ and } w \text{ concave in } \Omega\}.$$

Definition 6.1. *Let*

$$u_*(x) = sup_{v \in \mathcal{F}(u)} v(x), \quad u^*(x) = inf_{w \in \mathcal{G}(w)} w(x)$$

We call these functions the convex and concave envelopes of u in Ω, respectively.

It is easy to see that u_* and u^* are, respectively, convex and concave in Ω (using a similar reasoning to what we did in Remark 2.11 and that the inequalities

$$u_*(x) \leq u(x) \leq u^*(x)$$

hold for any $x \in \Omega$. Also $\mathcal{F}(-u) = -\mathcal{G}(u)$ and therefore

$$-(u^*)(x) = (-u)_*(x)$$

for all $x \in \Omega$.

Definition 6.2. *We call the sets*

$$\mathcal{C}_*(u) = \{x \in \Omega : u_*(x) = u(x)\}, \quad \mathcal{C}^*(u) = \{x \in \Omega : u^*(x) = u(x)\}.$$

the sets of contact points.

Since $-(u^*)(x) = (-u)_*(x)$ we have that

$$\mathcal{C}_*(u) = \mathcal{C}^*(-u).$$

Since u_* is convex,u_* has a supporting hyperplane at $x_0 \in C_*(u)$. Additionally, $u_*(x_0) = u(x_0)$ and, by the definition of the convex envelope, $u_*(x) \leq u(x)$ for all $x \in \Omega$ and so this hyperplane is also a supporting hyperplane to u at the same point. We have then just proved that

$$\partial(u_*)(x_0) \subseteq \partial u(x_0)$$

for all $x_0 \in \mathcal{C}_*(u)$ and so

$$\partial(u_*)(\mathcal{C}_*(u)) \subseteq \partial u(\mathcal{C}_*(u)).$$

If $x_0 \notin \mathcal{C}_*(u)$, then $\partial u(x_0) = \emptyset$. We argue by contradiction. Suppose there exists $p \in \partial u(x_0)$ such that

$$u(x) \geq l(x)$$

for all $x \in \Omega$ where $l(x) = u(x_0) + p \cdot (x - x_0)$. Then $l \in \mathcal{F}(u)$ and therefore $u_*(x_0) \leq l(x_0) = u(x_0)$. But since $x_0 \in \mathcal{C}_*(u)$ and $u_* \leq u$ in Ω we have $u_*(x_0) < u(x_0)$ and so we get a contradiction. Hence $\partial u(\Omega \setminus C_*(u)) = \emptyset$ and therefore

$$\begin{aligned}\partial u(\Omega) &= \partial u(\mathcal{C}_*(u) \cup (\Omega \setminus \mathcal{C}_*(u))) \\ &= \partial u(\mathcal{C}_*(u)) \cup \partial u(\Omega \setminus \mathcal{C}_*(u)) \\ &= \partial u(\mathcal{C}_*(u))\end{aligned}$$

since given any sets A,B, we always have $\partial u(A \cup B) = \partial u(A) \cup \partial u(B)$.

From the definition of u_* and $C_*(u)$ it is easy see that

$$\partial u(\mathcal{C}_*(u) \subseteq \partial(u_*)(\mathcal{C}_*(u)).$$

Thus

$$\partial u(\Omega) = \partial u(\mathcal{C}_*(u)) = \partial(u_*)(\mathcal{C}_*(u)).$$

Now let

$$\Phi_u(x_0) = \{p : u(x) \leq u(x_0) + p \cdot (x - x_0), \quad \forall x \in \Omega\}.$$

Notice that $\Phi_{-u}(x_0) = -\partial u(x_0)$.

Lemma 6.3. *Let $u \in C^(\overline{\Omega})$ such that $u(x) \leq 0$ on $\partial\Omega$, and $x_0 \in \Omega$ with $u(x_0) > 0$. Then*

$$\Omega(x, u(x_0)) \subseteq \Phi_{u^*}(\mathcal{C}^*(u)),$$

where $\Omega(x,t) = \left\{y : y \cdot (\xi - x) + t > 0, \forall \xi \overline{\Omega}\right\}$ and

$$\frac{\omega_n u(x_0)^n}{(\mathrm{diam}(\Omega))^n} \leq |\partial((-u)_*)(\mathcal{C}_*(-u))|.$$

Proof. Let $y \in \Omega(x_0, u(x_0))$. Then

$$y \cdot (\xi - x_0) + u(x_0) > 0 \tag{4}$$

for all $\xi \in \overline{\Omega}$. Let

$$\lambda_0 = \inf \left\{ \lambda : \lambda + y \cdot (\xi - x_0) \geq u(\xi), \ \forall \xi \in \overline{\Omega} \right\}.$$

Clearly λ_0 is finite and by continuity we have

$$\lambda_0 + y \cdot (\xi - x_0) \geq u(\xi) \tag{5}$$

for all $\xi \in \overline{\Omega}$. Consider now the minimum

$$\min_{\xi \in \overline{\Omega}} \left\{ \lambda_0 + y \cdot (\xi - x_0) - u(\xi) \right\}.$$

Since $\overline{\Omega}$ is compact and $u \in C(\overline{\Omega})$, this minimum is attained at some point $\overline{\xi} \in \overline{\Omega}$. We claim that

$$\lambda_0 + y \cdot (\overline{\xi} - x_0) - u(\overline{\xi}) = 0. \tag{6}$$

We argue by contradiction: if there existed $\varepsilon > 0$ such that

$$\lambda_0 + y \cdot (\xi - x_0) - u(\xi) \geq \varepsilon$$

for all $\xi \in \overline{\Omega}$, then λ_0 would not be the infimum.

We want to show now that $y \in \Phi_{u^*}(\overline{\xi})$ with $\xi \in \mathcal{C}^*(u)$ and if so we proved the first part of the Lemma. We first prove that $\xi \in \Omega$. Since $u \leq 0$ on $\partial\Omega$ it is enough to show that $u(\overline{\xi}) > 0$. Taking $\xi = x_0$ in (5) we get $\lambda_0 \geq u(x_0)$ and consequently from (4) we get

$$y \cdot (\xi - x_0) + \lambda_0 > 0$$

for all $\xi \in \overline{\Omega}$. In particular for $\xi = \overline{\xi}$ we get $u(\overline{\xi}) = y \cdot (\overline{\xi} - x_0) + \lambda_0 > 0$.

So far we have proved that if $y \in \Omega(x_0, u(x_0))$ then there exists $\overline{\xi} \in \Omega$ such that

$$u(\overline{\xi}) = y \cdot (\overline{\xi} - x_0) + \lambda_0$$

and

$$u(\xi) \leq y \cdot (\xi - x_0) - \lambda_0$$

for all $\xi \in \overline{\Omega}$. By the definition of u^*, we have

$$u(\xi) \leq u^*(\xi) \leq y \cdot (\xi - x_0) + \lambda_0$$

for all $\xi \in \overline{\Omega}$. From (6), we can rewrite the inequality above as

$$u(\xi) \leq u^*(\xi) \leq u(\overline{\xi}) + y \cdot (\xi - \overline{\xi})$$

for all $\xi \in \overline{\Omega}$. In particular for $\xi = \overline{\xi}$ we get $u(\overline{\xi}) = u^*(\overline{\xi})$ and therefore $y \in \Phi_{u^*}(\overline{\xi})$ and $\overline{\xi} \in \mathcal{C}^*(u)$.

As for the second part we first observe that

$$\Phi_{u^*}(\mathcal{C}^*(u)) = -\partial(-(u^*))(\mathcal{C}^*(u)) = -\partial((-u)_*)(\mathcal{C}_*(-u))$$

and so we just have to prove that

$$|\Omega(x_0, t)| \geq \frac{\omega_n t^n}{(\operatorname{diam}(\Omega))^n}.$$

Since

$$y \cdot (\xi - x_0) + t = t\left(\frac{y}{t} \cdot (\xi - x_0) + 1\right)$$

we have

$$\Omega(x_0, t) = t\Omega(x_0, 1).$$

Thus if we prove that

$$B_{1/\operatorname{diam}(\Omega)}(0) \subseteq \Omega(x_0, 1)$$

we are done. Let $\xi \in \overline{\Omega}$ and $y \in B_{1/\operatorname{diam}(\Omega)}(0)$. Then

$$\begin{aligned} y \cdot (\xi - x_0) + 1 &= |y||\xi - x_0| \cos(\phi) + 1 \\ &= |y| \operatorname{diam}(\Omega) \frac{|\xi - x_0|}{\operatorname{diam}(\Omega)} \cos(\phi) + 1 \\ &\geq -|y| \operatorname{diam}(\Omega) \frac{|\xi - x_0|}{\operatorname{diam}(\Omega)} + 1 \\ &> 0 \end{aligned}$$

and therefore $y \in \Omega(x_0, 1)$. □

We can prove the following maximum principle:

Theorem 6.4 (Aleksandrov-Bakelman-Pucci's maximum principle)**.** *If $u \in C(\overline{\Omega})$ and $u \leq 0$ on $\partial\Omega$, then*

$$\max_{\Omega} u(x) \leq \omega_n^{-1/n} \operatorname{diam}(\Omega) |\partial((-u)_*)(\mathcal{C}_*(-u))|^{1/n}.$$

If in addition $u \in C^2(\Omega)$ (without any assumptions on the sign of u on $\partial\Omega$), then

$$\max_{\Omega} u(x) \leq \max_{\partial\Omega} u(x) + \omega_n^{-1/n} \operatorname{diam}(\Omega) \left(\int_{\mathcal{C}_*(-u)} |\det D^2 u(x)| dx \right)^{1/n}.$$

Proof. Using Lemma 6.3, we only have to prove that the second inequality. Let $u \in C^2(\Omega)$. Subtracting from u the maximum on the boundary, we may assume that $u \leq 0$ on $\partial\Omega$. We now notice that

$$\partial((-u)_*)(\mathcal{C}_*(-u)) = \partial(-u)(\mathcal{C}_*(-u))$$

and, since $u \in C^2(\Omega)$, $D^2(-u)(z) \geq 0$ if $z \in \mathcal{C}_*(-u)$. Thus by the formula for change of variables we obtain

$$|\partial(-u)(\mathcal{C}_*(-u))| \leq \int_{\mathcal{C}_*(-u)} |\det D^2 u(x)| dx$$

and so we are done. □

7 Comparison Principle

In this Section we discuss a comparison principle for generalized solutions, from which we can deduce their uniqueness.

Theorem 7.1. *Let $u, v \in C(\overline{\Omega})$ be a convex functions such that*

$$|\partial u(E)| \leq |\partial v(E)|, \qquad \textit{for every Borel set } E \subseteq \Omega.$$

Then

$$\min_{x \in \overline{\Omega}} \{u(x) - v(x)\} = \min_{x \in \partial\Omega} \{u(x) - v(x)\}.$$

Proof. We argue by contradiction. Let

$$a = \min_{x \in \overline{\Omega}} u(x) - v(x)$$

and

$$b = \min_{x \in \partial\Omega} u(x) - v(x).$$

Suppose $a < b$. Since $u, v \in C(\overline{\Omega})$, there exists $x_0 \in \Omega$ such that

$$a = u(x_0) - v(x_0).$$

Pick $\delta > 0$ sufficiently small such that

$$\delta(\mathrm{diam}(\Omega)^2 < \frac{b-a}{2}$$

and let

$$w(x) = v(x) + \delta|x - x_0|^2 + \frac{b+a}{2}.$$

Consider the set $G = \{x \in \overline{\Omega} : u(x) < w(x)\}$. We have

$$\begin{aligned} u(x_0) &= v(x_0) + a \\ &< v(x_0) + \frac{a+b}{2} \\ &= u(x_0) \end{aligned}$$

since $a < b$ and so $x_0 \in G$. Also $G \cap \partial\Omega = \emptyset$. In fact if $x \in G \cap \partial\Omega$, then $u(x) - v(x) \geq b$ and so

$$\begin{aligned} w(x) &\leq u(x) + \delta|x - x_0|^2 - \frac{b-a}{2} \\ &\leq u(x) + \delta(\mathrm{diam}(\Omega))^2 - \frac{b-a}{2} \\ &< u(x). \end{aligned}$$

Hence $x \notin G$ and therefore we have a contradiction. We then have that $\partial G = \{x \in \Omega : u(x) = w(x)\}$. By Lemma 5.1 we obtain $\partial w(G) \subseteq \partial u(G)$. If A and B are symmetric and non-negative definite matrices, then $\det(A + B) \geq \det(A) + \det(B)$. So since $\partial w = \partial(v + \delta|x - x_0|^2)$, if $v \in C^2(\Omega)$ we have the inequality

$$\partial(v + \delta|x - x_0|^2)(G)| \geq |\partial v(G)| + |\partial(\delta|x - x_0|^2)(G)|.$$

If v is not smooth, we can approximate v be a sequence $v_k \in C^2(\Omega)$ of convex functions converging uniformly on compact subsets of Ω. This can be achieved by taking a smooth function $\phi \geq 0$ with support in $B_1(0)$ and $\int \phi = 1$ and then letting $v_\varepsilon = v * \phi_\varepsilon$, where $\phi_\varepsilon(x) = \phi(x/\varepsilon)$. Hence the inequality above now follows from Lemma 3.3. Therefore

$$\partial u(G)| \geq |\partial v(G)| + |\partial(\delta|x - x_0|^2)(G)| = |\partial v(G)| + (2\delta)^n |G|,$$

which contradicts the assumption of the theorem. □

Corollary 7.2. *If $u, v \in C(\overline{\Omega})$ are convex functions such that $|\partial u(E)| = \partial v(E)$ for every Borel set $E \subseteq \Omega$ and $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

8 The Dirichlet problem

In this Section we will prove the existence of a solution to the (homogeneous) Dirichlet problem of the Monge-Ampère equation.

Definition 8.1. *The open set $\Omega \subseteq \mathbb{R}^n$ is strictly convex if for all $x, y \in \overline{\Omega}$ the open segment joining x and y lies in Ω.*

Theorem 8.2. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded and strictly convex, and $g \in C(\partial\Omega)$. Then there exists a unique convex function $u \in C(\overline{\Omega})$ generalized solution of the problem*

$$\begin{cases} \det D^2 u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{7}$$

Proof. Let $\mathcal{F} = \{a(x) : a \text{ is an affine function and } a \leq g \text{ on } \partial\Omega\}$. Since g is continuous, $\mathcal{F} \neq \emptyset$. Define

$$u(x) = \sup \{a(x) : a \in \mathcal{F}\}.$$

We will prove that u is a solution to (7) and then that is in fact the unique solution.

Step 1 : u is convex and $u = g$ on $\partial\Omega$.

Since u is the supremum of convex functions $\leq g$ on $\partial\Omega$, u is convex and $u \leq g$ on $\partial\Omega$. Now let $\xi \in \partial\Omega$. We only need to prove that $u(\xi) \geq g(\xi)$. Let $\varepsilon > 0$. Since $g \in C(\partial\Omega)$, there exists $\delta > 0$ such that $|g(x) - g(\xi)| < \varepsilon$ for $x \in \partial\Omega \cap B_\delta(\xi)$. Let $P(x) = 0$ be the equation of the supporting hyperplane to Ω at the point ξ. Such P exists since Ω is convex and without loss of generality we assume that $\Omega \subseteq \{x : P(x) \geq 0\}$. Since Ω is strictly convex, there exists $\eta > 0$ such that $S = \left\{x \in \overline{\Omega} : P(x) \leq \eta\right\} \subseteq B_\delta(\xi)$. Let

$$M = \min \{g(x) : x \in \partial\Omega, P(x) \geq \eta\}$$

and consider

$$a(x) = g(\xi) - \varepsilon - AP(x),$$

where A is a constant satisfying

$$A \geq \max \left\{ \frac{g(\xi) - \varepsilon - M}{\eta}, 0 \right\}.$$

a is an affine function and so if $a \leq g$ on $\partial\Omega$ we have $a \in \mathcal{F}$. Let $x \in \partial\Omega \cap S$. Then

$$g(\xi) - \varepsilon \leq g(x) \leq g(\xi) + \varepsilon$$

by the choice of δ and since $S \subseteq B_\delta(\xi)$. Hence

$$\begin{aligned} g(x) &\geq g(\xi) - \varepsilon - AP(x) + AP(x) \\ &\geq g(\xi) - \varepsilon - AP(x) \\ &= a(x) \end{aligned}$$

since $\overline{\Omega} \subseteq \{x : P(x) \geq 0\}$ and $A \geq 0$. If $x \in \partial\Omega \cap S^c$ then $P(x) > \eta$ and by definition of M and the choice of A we have

$$\begin{aligned} g(x) &\geq M \\ &= a(x) + M - g(\xi) - \varepsilon + AP(x) \\ &\geq a(x) + M - g(\xi) + \varepsilon + A\eta \\ &\geq a(x). \end{aligned}$$

Therefore $a \in \mathcal{F}$, and in particular $u(\xi) \geq a(\xi) = g(\xi) - \varepsilon$ for every $\varepsilon > 0$ and therefore $u(\xi) \geq g(\xi)$ as desired.

Step 2 : $u \in C(\overline{\Omega})$.

Since u is convex in Ω, u is continuous in Ω. To prove the continuity on $\partial\Omega$, let $\xi \in \partial\Omega$, $\{x_n\} \subseteq \overline{\Omega}$ with $x_n \to \xi$. We will show that $u(x_n) \to g(\xi)$. Let a be the affine function constructed in Step 1. Then $u \geq a$ in $\overline{\Omega}$ and in particular $u(x_n) \geq a(x_n)$. Hence

$$\begin{aligned} \liminf u(x_n) &\geq \liminf a(x_n) \\ &= \liminf (g(\xi) - \varepsilon - AP(x_n) \\ &= g(\xi) - \varepsilon \end{aligned}$$

for all $\varepsilon > 0$. Hence $\liminf u(x_n) \geq g(\xi)$. We now prove that $\limsup u(x_n) \leq g(\xi)$. Since Ω is convex, there exists h harmonic in Ω such that $h \in C(\overline{\Omega})$ and $h = g$ on $\partial\Omega$. If $a \in \mathcal{F}$, then a is harmonic and by the maximum principle $a \leq h$ in Ω. Taking the supremum over a we obtain $u(x) \leq h(x)$ for $x \in \Omega$. In particular, $u(x_n) \leq h(x_n)$ and therefore $\limsup u(x_n) \leq \limsup h(x_n) = g(\xi)$ and we are done.

Step 3 : $\partial u(\Omega) \subseteq \{p \in \mathbb{R}^n :$ there is $x, y \in \Omega$ with $x \neq y$ and $p \in \partial u(x) \cap \partial u(y)\}$

If $p \in \partial u(\Omega)$, then there exists $x_0 \in \Omega$ such that

$$u(x) \geq u(x_0) + p \cdot (x - x_0) = a(x)$$

for all $x \in \Omega$. Since $u = g$ on $\partial\Omega$, we have $g(x) \geq a(x)$ for all $x \in \partial\Omega$. There exists $\xi\partial\Omega$ such that $g(\xi) = a(\xi)$. Otherwise, there exists some $\varepsilon > 0$ such that $g(x) \geq a(x) + \varepsilon$ for all $x \in \partial\Omega$ and then $u(x) \geq a(x) + \varepsilon$ for all $x \in \Omega$ and in particular $u(x_0) \geq a(x_0) + \varepsilon = u(x_0) + \varepsilon$, a contradiction. Since Ω is convex, the open segment I joining x_0 and ξ is contained in Ω. Now $u(x_0) = a(x_0)$ and $u(\xi) = a(\xi)$. If $z \in I$, then $z = tx_0 + (1-t)\xi$ and by convexity $u(z) \leq tu(x_0) + (1-t)u(\xi) = ta(x_0) + (1-t)a(\xi) = a(z)$.

But $u(x) \geq a(x)$ for all $x \in \Omega$ and so a is a supporting hyperplane to u at any point on the segment I, therefore $p \in \partial u(z)$ for all $z \in I$ and (1.5.2) is then proved.

Step 4 : u is a solution of (7).

From Step 3, $|\partial u(\Omega)| = 0$ by Lemma 2.13.

Step 4 : the solution of (7) is unique.

It follows from Corollary 7.2. □

9 The non-homogeneous Dirichlet problem

In this Section we solve the non-homogeneous Dirichlet problem for the Monge-Ampère operator using the Perron method and Theorem 1.5.2. Let Ω be an open bounded convex set, μ a Borel measure in Ω, and $g \in C(\partial\Omega)$. Set

$$\mathcal{F}(\mu, g) = \left\{ v \in C(\overline{\Omega}) : v \text{ convex }, Mv \geq \mu \text{ in } \Omega, v = g \text{ on } \partial\Omega \right\}.$$

Suppose that $\mathcal{F}(\mu, g) \neq \emptyset$ and let $v \in \mathcal{F}(\mu, g)$. Assume that Ω is strictly convex. By Theorem 8.2, let $W \in C(\overline{\Omega})$ be the unique convex solution to $MV = 0$ in Ω and $W = g$ on $\partial\Omega$. We have $0 = MW \leq \mu \leq Mv$ in Ω and therefore we have that $v \leq W$ in Ω by Theorem 7.1. Hence all functions in $\mathcal{F}(\mu, g)$ are uniformly bounded above and we can define

$$U(x) = \sup \{ v(x) : v \ \in \mathcal{F}(\mu, g) \}. \tag{8}$$

The idea to solve the non-homogeneous Dirichlet problem is firs to construct U when the measure is a combination of delta masses and then to approximate a general measure μ by a sequence of measures of this form, and in this way construct the desired solution. With this in mind we start begin with two Lemmas.

Lemma 9.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex open set and u a convex function in Ω such that $u \leq 0$ in $\partial\Omega$. If $x_0 \in \Omega$ and $p \in \partial u(x_0)$ then*

$$|p| \leq \frac{-u(x_0)}{\text{dist}(x_0, \Omega)}.$$

Proof. Let $p \in \partial u(x)$ and assume that $p \neq 0$. We have

$$u(x) \geq u(x_0) + p \cdot (x - x_0)$$

for all $x \in \Omega$. Let $r > 0$ be such that $r < \text{dist}(x_0, \partial\Omega)$. Then $x = x_0 + r\frac{p}{|p|} \in \Omega$ and therefore

$$0 \geq u(x) \geq u(x) + r|p|$$

from where the Lemma follows. □

Lemma 9.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open strictly convex domain, μ_j, μ be Borel measures in Ω, $u_j \in C(\overline{\Omega})$, and $g \in C(\partial\Omega)$ such that*

1. *$u_j = g$ on $\partial\Omega$,*

2. $Mu_j = \mu_j$ *in* Ω*,*

3. $\mu_j \to \mu$ *weakly in* Ω*, and*

4. $\mu_j(\Omega) \le A$ *for all* j*.*

Then $\{u_j\}$ *contains a subsequence, also denoted by* u_j*, and there exists* $u \in C(\overline{\Omega})$ *convex in* Ω *such that* u_j *converges to* u *uniformly on compact subsets of* Ω*, and* $Mu = \mu$*,* $u = g$ *in* $\partial\Omega$*.*

Proof. We have $u_j \in \mathcal{F}(\mu_j, g)$ and therefore u_j are uniformly bounded above as we have seen above. We now prove that u_j also uniformly bounded below in Ω. Let $\xi \in \partial\Omega$, $\varepsilon > 0$ and $a(x) = g(\xi) - \varepsilon - AP(x)$ be the affine function constructed in the proof of Theorem 8.2. Recall that $a(x) \le g(x)$ for $x \in \partial\Omega$, $P(\xi) = 0$, $P(x) \ge 0$ for $x \in \Omega$, and $A \ge 0$. Set $v_j(x) = u_j(x) - a(x)$. If $x \in \partial\Omega$, then $v_j(x) = g(x) - a(x) \ge 0$, and the v_j are convex in Ω. If $v_j(x) \ge 0$ for all $x \in \Omega$, then u_j is bounded below in Ω. If at some point $v_j(x) < 0$, then by the Aleksandrov maximum principle, Theorem 1.4.2, applied to v_j on the set $G = \{x \in \Omega : v_j(x) \le 0\}$, we obtain

$$\begin{aligned}(-v_j(x))^n &\le C_n \operatorname{dist}(x, \partial G)\Delta^{n-1} Mv_j(G)\\ &\le C_n \operatorname{dist}(x, \partial\Omega)\Delta^{n-1} Mv_j(\Omega)\\ &\le C_n \operatorname{dist}(x, \partial\Omega)\Delta^{n-1} A,\end{aligned}$$

with $\Delta = \operatorname{diam}(\Omega)$, and consequently $v_j(x) \ge -(C_n \operatorname{dist}(x, \partial\Omega)\Delta^{n-1}A)^{1/n}$, that is

$$u_j(x) \ge g(\xi) - \varepsilon - AP(x) - C \operatorname{dist}(x, \partial\Omega)^{1/n}, \tag{9}$$

which proves that u_j are uniformly bounded below in Ω. On the other hand, $u_j(x) \le w(x)$ with $\Delta w = 0$ in Ω and $w = g$ on $\partial\Omega$ by the maximum principle since u_j is weakly subharmonic from being convex. Now $\operatorname{dist}(x, \partial\Omega) \le |x - \xi|$ and from 9 we obtain

$$w(x) \ge u_j(x) \ge g(\xi) - \varepsilon - AP(x) - C|x - \xi|^{1/n}, \tag{10}$$

and therefore $u_j(x) \to g(\xi)$ as $x \to \xi$.

Therefore by Lemma 2.7 and Lemma 9, we get that u_j are locally uniformly Lipschitz in Ω and by Arzèla-Ascoli there exists a subsequence, denoted also u_j, and a convex function in Ω such that $u_j \to u$ uniformly on compact subsets of Ω. We also have from 10 that $u \in C(\overline{\Omega})$. The Lemma then follows from Lemma 3.4. □

We now state and prove the main result in this Section

Theorem 9.3. *If* $\Omega \subseteq \mathbb{R}^n$ *is open bounded and strictly convex,* μ *is a Borel measure in* Ω *with* $\mu(\Omega) < +\infty$*, and* $g \in C(\partial\Omega)$*, then there exists a unique convex function* $u \in C(\overline{\Omega})$ *generalized solution of the problem*

$$\begin{cases} \det D^2 u = \mu & \textit{in } \Omega \\ u = g & \textit{on } \partial\Omega \end{cases} \tag{11}$$

Proof. We start by observing that the uniqueness follows by the comparison principle, Theorem 7.1.

There exists a sequence of measure μ_j converging weakly to μ such that each u_j is a finite combination of delta masses with positive coefficients and $\mu_j(\Omega) \leq A$ for all j. If we solve the Dirichlet problem for each μ_j with data g, then the Theorem follows from Lemma 9.2. Therefore we assume from now on that

$$\mu = \sum_{i=1}^{N} a_i \delta_{x_i} \quad x_i \in \Omega, \quad a_i > 0.$$

We claim that

(a) $\mathcal{F}(\mu, g) \neq \emptyset$

(b) If $u, v \in \mathcal{F}(\mu, g)$ then $u \vee v := \max\{u, v\} \in \mathcal{F}(\mu, g)$

(c) $U \in \mathcal{F}(\mu, g)$, with defined by 8

Step 1 : proof of (a).

By Example 2.17, $M(|x - x_i|) = \omega_n \delta_{x_i}$, with ω_n the volume of the unit ball in $\mathbb{R}^n$. Let

$$f(x) = \frac{1}{\omega_n^{1/n}} \sum_{i=1}^{N} a_i^{1/n} |x - x_i|$$

and u be a solution to the Dirichlet problem

$$\begin{cases} \det D^2 u = 0 & \text{in } \Omega \\ u = g - f & \text{on } \partial\Omega \end{cases}$$

We claim that $v = u + f \in \mathcal{F}(\mu, g)$. Indeed, it is clear that $v \in C(\overline{\Omega})$, v is convex and $v = g$ on $\partial\Omega$. Let us calculate Mv. We have

$$Mv = M(u + f) \geq Mu + Mf \geq \frac{1}{\omega_n} \sum_{i=1}^{N} M\left(a_i^{1/n} |x - x_i|\right) = \sum_{i=1}^{N} a_i \delta_{x_i} = \mu.$$

Therefore $\mathcal{F}(\mu, g) \neq \emptyset$, and consequently U given by (1.6.1) is well defined.

Step 2 : proof of (b).

Let $\phi = u \vee v$ and

$$\Omega_0 = \{x \in \Omega : u(x) = v(x)\}$$
$$\Omega_1 = \{x \in \Omega : u(x) > v(x)\}$$
$$\Omega_2 = \{x \in \Omega : u(x) < v(x)\}.$$

If $E \subseteq \Omega_1$ then $M\phi(E) \geq Mu(E)$, and if $E \subseteq \Omega_2$, then $M\phi(E) \geq Mv(E)$. Also if $E \subseteq \Omega_0$ then $\partial u(E) \subseteq \partial\phi(E)$ and $\partial v(E) \subseteq \partial\phi(E)$. Given $E \subseteq \Omega$ a Borel set, write $E = E_0 \cup E_1 \cup E_2$ with $E_i \subseteq \Omega_i$. We have

$$\begin{aligned} M\phi(E) &= M\phi(E_0) + M\phi(E_1) + M\phi(E_2) \\ &\geq Mu(E_0) + Mu(E_1) + Mv(E_2) \\ &\geq \mu(E_0) + \mu(E_1) + \mu(E_2) \\ &= \mu(E). \end{aligned}$$

Step 3 : For each $y \in \Omega$ there exists a uniformly bounded sequence $v_m \in \mathcal{F}(\mu, g)$ converging uniformly on compact subsets of Ω to a function $w \in \mathcal{F}(\mu, g)$ such that $w(y) = U(y)$, where U is given by 8.

By Step 1, let $v_0 \in \mathcal{F}(\mu, g)$, then $v \leq W$ with W defined at the begging of this Section. Fix $y \in \Omega$, then by definition of U there exists a sequence $v_m \in \mathcal{F}(\mu, g)$ such that $v_m(y) \to U(y)$ as $m \to \infty$. Let $\overline{v}_m = v_0 \vee v_m$. By Step 2, $\overline{v}_m \in \mathcal{F}(\mu, g)$ and therefore $v_m(y) \leq \overline{v}_m(y) \leq U(y)$ and so $\overline{v}_m \to U(y)$. Notice that $|\overline{v}_m(x)| \leq C_1$ for all $x \in \Omega$. Therefore we may assume that the original sequence v_m is bounded above and below in Ω. Since v_m is convex in Ω, it follows from Lemma 1.1.6 that given $K \subseteq \Omega$ compact, v_m is Lipschitz in K with constant

$$C(K, m) = \sup \{|p| : p \in \partial v_m(K)\}.$$

We claim that $C(K, m)$ is bounded uniformly in m. Let $p \in \partial v_m(x_0)$ with $x_0 \in K$. Be Lemma 3.2.1, we get that $|p| \leq \frac{C_1}{\operatorname{dist}(K,\Omega)}$ and the claim follows. Therefore v_m are equicontinuous on K and bounded in Ω. By Arzèla-Ascoli there exists a subsequence v_{m_j} converging uniformly on compact subsets of Ω to a function w, and so $w(y) = U(y)$. By Lemma 3.3 we have that $w \in \mathcal{F}(\mu, g)$ and therefore $w \leq U$ in Ω.

Step 4 : $MU \geq \mu$ in Ω.

It is enough to prove that $MU(\{x_i\}) \geq a_i$ for $i = 1, \ldots, N$. We may assume $i = 1$. By Step 3, there exists a sequence $v_m \in \mathcal{F}(\mu, g)$, uniformly bounded, such that $v_m \to w \in \mathcal{F}(\mu, g)$ uniformly on compact of Ω as $m \to \infty$ with $w(x_1) = U(x_1)$. We have $Mw(\{x_1\}) \geq a_1$. If $p \in \partial w(x_1)$, then

$$w(x) \geq w(x_1) + p \cdot (x - x_1)$$

for all $x \in \Omega$ and hence, from the definition of U,

$$U(x) \geq U(x_1) + p \cdot (x - x_1)$$

for all $x \in \Omega$, i.e., $p \in \partial u(x_1)$. So $MU(\{x_1\}) \geq |\partial U(\{x_1\})| \geq |\partial w(\{x_1\})| \geq a_1$.

Step 5 : $MU \leq \mu$ in Ω.

We first prove that the measure MU is concentrated on the set $\{x_1, \ldots, x_N\}$. Let $x_0 \in \Omega$ with $x_0 \neq x_i$, $i = 1, \ldots, N$, and choose $r > 0$ so that $|x_i - x_0| > r$ for $i = 1, \ldots, N$ and $B_r(x_0) \subseteq \Omega$. Solve $Mv = 0$ in $B_r(x_0)$ with $v = U$ on $\partial B_r(x_0)$, and define the "lifting of U"

$$w(x) = \begin{cases} U(x) & x \in \Omega, |x - x_0| \geq r, \\ v(x) & |x - x_0| \leq r. \end{cases}$$

We claim that $w \in \mathcal{F}(\mu, g)$. In fact, w is convex, because by Step 4, $MU \geq \mu \geq 0 = Mv$ in $B_r(x_0)$, and then by the comparison principle Theorem 7.1, $v \geq U$ in $B_r(x_0)$. It is clear that $w \in C(\overline{\Omega})$. We verify that $Mw \geq \mu$ in Ω. Let $E \subseteq \Omega$ be a Borel set. We write

$$E = (E \cap B_r(x_0)) \cup (E \cap B_r(x_0)^c))$$

and so

$$Mw(E) = Mw(E \cap B_r(x_0)) + Mw(E \cap B_r(x_0)^c)).$$

Now notice that if $F \subseteq B_r(x_0)$, then $\partial w(F) = \partial v(F)$, and if $F \subseteq B_r(x_0)^c$, then $\partial w(F) = \partial U(F)$. Therefore

$$\begin{aligned} Mw(E) &= Mv(E \cap B_r(x_0)) + MU(E \cap B_r(x_0)^c)) \\ &= 0 + MU(E \cap B_r(x_0)^c)) \\ &\geq \mu(E \cap B_r(x_0)^c)) \\ &\geq \mu(E \cap \{x_1, \dots, x_N\}) \\ &= \mu(E), \end{aligned}$$

by (c) and the definition of μ. Hence we have $w \in \mathcal{F}(\mu, g)$ and by the definition of U $w \leq U$, and since $w = v \geq U$ in $B_r(x_0)$, we get $v = U$ in $B_r(x_0)$, and so $MU = Mv = 0$ in $B_r(x_0)$, where $B_r(x_0) \subseteq \Omega$ is any ball with $\overline{B_r(x_0)} \cap \{x_1, \dots, x_N\} = \emptyset$. Hence if $E \subseteq \Omega$ is a Borel set with $E \cap \{x_1, \dots, x_N\} = \emptyset$, then $MU(E) = 0$ by regularity of MU. Therefore MU is concentrated on the set $\{x_1, \dots, x_N\}$, that is

$$MU = \sum_{i=1}^{N} \lambda_i a_i \delta_{x_i},$$

with $\lambda_i \geq 1$, $i = 1, \dots, N$ since by Step 4 $MU \geq \mu$. We claim that $\lambda_i = 1$ for all $i = 1, \dots, N$. Suppose by contradiction that $\lambda_i > 1$ for some i. Without loss of generality, we may assume that $MU = \lambda a \delta_0$ with $\lambda > 1$ and in the ball $B_r(0)$. We have $|\partial U(\{0\})| = \lambda a > 0$. Since $\partial U(\{0\})$ is convex, there exists a ball $B_\varepsilon(p_0) \subseteq \partial U(\{0\})$. Then $U(x) \geq U(0) + p \cdot x$ for all $p \in B_\varepsilon(p_0)$ and $x \in \Omega$. Let $V(x) = U(x) - p_0 \cdot x$. Then $V(x) \geq V(0) + (p - p_0) \cdot x$ for all $x \in \Omega$ and $p \in B_\varepsilon(p_0)$. Given $x \in \Omega$ take $p - p_0 = \varepsilon x/|x|$ and so

$$V(x) \geq V(0) + \varepsilon |x|$$

for all $x \in \Omega$. Let α be a constant such that $V(0) - \alpha$ is negative and close to zero, and define $\overline{V}(x) = V(x) - \alpha$. We have $\overline{V}(0)$ is negative and small, and $\overline{V}(x) \geq \overline{V}(0) + \varepsilon |x|$ for all $x \in \Omega$. If $r = -\frac{\overline{V}(0)}{\varepsilon}$, the $\overline{V}(x) \geq \overline{V}(0) + \varepsilon |x| \geq 0$ for all $|x| \geq r$. Let

$$w(x) = \begin{cases} \overline{V}(x) & \text{if } \overline{V}(x) \geq 0 \\ \lambda^{-1/n} \overline{V}(x) & \text{if } \overline{V}(x) < 0. \end{cases}$$

Notice that since $\lambda > 1$, we have $\lambda^{-1/n} \overline{V}(x) > \overline{V}(x)$ on the set $\{x : \overline{V}(x) < 0\}$. Consequently the function w is convex in Ω. Also, on the set $\{x : \overline{V}(x) < 0\}$, we have $Mw = M(\lambda^{-1/n} \overline{V}) = \frac{1}{\lambda} M\overline{V} = \frac{1}{\lambda} MU = a\delta_0$. On the other hand $w = \overline{V}$ on the set $\{x : \overline{V}(x) \geq 0\}$, so $Mw = M\overline{V} = MU \geq \mu$ on the same set. Consequently $Mw \geq \mu$ in Ω. This means that $w \in \mathcal{F}(\mu, \overline{g})$, where $\overline{g}$ are the boundary values of $\overline{V}(x) = U(x) - p_0 \cdot x - \alpha$. By the definition of U,

$$\overline{V}(x) = U(x) - p_0 \cdot x - \alpha = \sup \{v(x) - p_0 \cdot x - \alpha : v \in \mathcal{F}(\mu, g)\}.$$

It is clear that $v'(x) = v(x) - p_0 \cdot x - \alpha \in \mathcal{F}(\mu, \overline{g})$ if and only if $v(x) \in \mathcal{F}(\mu, g)$. Therefore,

$$\overline{V(x)} = \sup \{v' : v' \in \mathcal{F}(\mu, \overline{g})\},$$

and since $w \in \mathcal{F}(\mu, \overline{g})$, we get that $w(x) \leq \overline{V}(x)$ for all $x \in \Omega$. In particular $w(0) \leq \overline{V}(0)$ and so $\lambda^{-1/n} \overline{V}(0) \leq \overline{V}(0)$, and since $\overline{V}(0) < 0$ we obtain $\lambda^{-1/n} \geq 1$, a contradiction since $\lambda > 1$. This completes the proof of Step 5 and the Theorem. $\square$

10 Return to viscosity solutions

In this Section we prove that viscosity solutions are generalized solutions.

We start with a comparison principle.

Lemma 10.1. *Suppose $f \in C(\Omega)$, $f \geq 0$, and $u \in C(\overline{\Omega})$ is a viscosity supersolution (resp. subsolution) to* $\det D^2 u = f$ *in Ω. Suppose $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical convex solution to* $\det D^2 v \geq g$ *(resp. $\leq g$) in Ω with $g \in C(\Omega)$. If $f < g$ (resp. $> g$) in Ω, then*

$$\min_{\overline{\Omega}}(u - v) = \min_{\partial\Omega}(u - v) \quad (\textit{resp.}\ \max_{\overline{\Omega}}(u - v) = \max_{\partial\Omega}(u - v)).$$

Proof. It follows directly from the definition of viscosity solutions. We consider only the case where u is a supersolution. Suppose by contradiction that $\min_{\overline{\Omega}}(u - v) < \min_{\partial\Omega}(u - v)$. Then there exists $x_0 \in \Omega$ such that $(u - v)(x_0) = \min_{\overline{\Omega}}(u - v)$, and so $u - v$ has a local minimum at x_0. Since u is a viscosity supersolution to $\det D^2 u = f$ in Ω we get $\det D^2 v(x_0) \leq f(x_0)$. But by assumption $g(x_0) \leq \det D^2 v(x_0)$ and so we have a contradiction. □

Proposition 10.2. *Let $f \in C(\overline{\Omega})$ with $f > 0$ in $\overline{\Omega}$. If u is a viscosity solution to* $\det D^2 u = f$ *in Ω, then u is a generalized solution to* $\det D^2 u = f$ *in Ω.*

Proof. We have $0 < \lambda \leq f(x) \leq \Lambda$ in $\overline{\Omega}$. Given $x_0 \in \Omega$ and $0 < \eta < \lambda/2$, there exists $\varepsilon > 0$ such that

$$f(x_0) - \eta < f(x) < f(x_0) + \eta$$

for all $x \in B_\varepsilon(x_0)$. Let $u_k \in C^\infty(\partial B_\varepsilon(x_0))$ be a sequence such that

$$\max_{\partial B_\varepsilon(x_0)} |u(x) - u_k(x)| \leq \frac{1}{k},$$

and v_k^+ and v_k^- the convex solutions to

$$\begin{aligned} \det D^2 v_k^\pm &= f(x_0) \pm \eta && \text{in } B_\varepsilon(x_0) \\ v_k^\pm &= u_k && \text{on } \partial B_\varepsilon(x_0). \end{aligned}$$

We have that $v_k^\pm \in C^2(B_\varepsilon(x_0)) \cap C(\overline{B_\varepsilon(x_0)})$ (see [3], Section 17.7) and

$$\begin{aligned} \det D^2 v_k^- < f(x) &< \det D^2 v_k^+ && \text{in } B_\varepsilon(x_0) \\ v_k^\pm &= u_k && \text{on } \partial B_\varepsilon(x_0). \end{aligned}$$

By Lemma 10.1 we get

$$v_k^+(x) - \frac{1}{k} \leq u(x) \leq v_k^-(x) + \frac{1}{k} \tag{12}$$

for $x \in \overline{B_\varepsilon(x_0)}$. By Theorem 9.3, let $v^\pm$ be the generalized solutions to

$$\begin{aligned} \det D^2 v^\pm &= f(x_0) \pm \eta && \text{in } B_\varepsilon(x_0) \\ v^\pm &= u && \text{on } \partial B_\varepsilon(x_0). \end{aligned}$$

Applying Theorem 7.1 we get that $|v^{\pm}(x) - v_k^{\pm}(x)| \leq 1/k$ and consequently letting $k \to \infty$ in (12) yields

$$v^+(x) \leq u(x) \leq v^-(x)$$

for $x \in \overline{B_\varepsilon(x_0)}$. From Lemma 5.1, we obtain

$$\partial v^-(B_\varepsilon(x_0)) \subseteq \partial u(B_\varepsilon(x_0)) \subseteq \partial v^+(B_\varepsilon(x_0)),$$

and consequently

$$|B_\varepsilon(x_0)|(f(x_0) - \eta) \leq |\partial u(B_\varepsilon(x_0))| = Mu(B_\varepsilon(x_0)) \leq |B_\varepsilon(x_0)|(f(x_0) + \eta). \tag{13}$$

Therefore if Q is a cube with diameter $\text{diam}(Q) < \varepsilon$, then

$$C_1|Q| \leq Mu(Q) \leq C_2|Q| \tag{14}$$

for some positive constant C_1, C_2. If $F \subseteq \Omega$ is a set of measure zero, then given $\delta > 0$ there exists a sequence of non overlapping cubes $Q_j \subseteq \Omega$ with $\text{diam}(Q_j) < \varepsilon$, $F \subseteq \cup Q_j$ and $\sum |Q_j| < \delta$. Then applying 14 we obtain $Mu(F) < C_2\delta$. That is Mu is absolutely continuous with respect to the Lebesgue measure and therefore there exists $h \in L^1_{\text{loc}}(\Omega)$ such that $Mu(E) = \int_E h(x)dx$. Dividing 13 by $|B_\varepsilon(x_0)|$ and letting $\varepsilon \to 0$ we get that $f(x_0) - \eta \leq h(x_0) \leq f(x_0) + \eta$ for almost all $x_0 \in \Omega$ and for all η sufficiently small. Hence Mu has density f and we are done. □

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