MATH580 Final Project
The Harnack Inequality on Complete Riemannian Manifolds of Non-Negative Ricci Curvature

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December 16, 2012

Abstract

This report will deal with content of Yau's article, *Harmonic Functions on Complete Riemannian Manifolds*.

1 Introduction

We may classify open Riemann surfaces by properties of curvature by whether or not there exists a non-constant, bounded harmonic function on the surface. We will deal with higher-dimensional manifolds of non-negative Ricci curvature. Yau shows in his paper that for a complete Riemannian manifold M, with Ricci curvature bounded from below, we can find an upper bound on the absolute value of the gradient of f. In the case of harmonic functions, the constant involved in the bound is simply given by $\sqrt{(dimM-1)K}$, where -K is the lower bound for the Ricci curvature of M. The inequality that Yau proves in his article can be thought of as an infinitesimal form of the Harnack inequality. We approach this result by first showing a generalized maximal principle and then using gradient estimates to achieve an upper bound, as desired.

The Harnack inequality that we have seen in class deals with functions on R^n and states that for harmonic, non-negative function $u \in C^2(\Omega)$ such that $cl(B_R(y)) \subset \Omega$ with $R > 0$, and $x \in B_R(y)$:

$$u(x) \leq (\frac{R}{R-|x-y|})^n u(y) \tag{1}$$

What the inequality does is relate the value of u on some point in the ball centered at y with the value of $u(y)$. Now, we wish to establish an analogue of the Harnack inequality on a manifold M with Ricci curvature bounded from below. The main result will be in showing that for all harmonic functions f bounded from below, we have

$$|\nabla f(x)| \leq C(f(x) - inf_M f) \tag{2}$$

Using this, we can see that for $x, y \in V \subset M$,

$$\begin{aligned} f(x) - f(y) &\leq \int_0^\ell < \sigma'(t), \nabla f(t) > dt \\ &\leq \int_0^\ell |\sigma'(t)||\nabla f(t)| dt \\ &\leq \ell sup_V |\nabla f| \\ &\leq C diam(V)(sup_V f - inf_M f) \end{aligned} \tag{3}$$

Suppose f is non-negative on M, then the previous inequality gives us

$$sup_V f - inf_V f \leq C diam(V)(sup_V f) \tag{4}$$

and hence, $sup_V f \leq \frac{inf_V f}{1 - C diam(V)}$ whenever $diam(V) < 1/C$ which is clearly a form of Harnack's inequality.

Before we proceed, note that the Laplacian on a manifold with Riemannian metric $g_{i,j}$, is defined as

$$\Delta f = g^{i,j} D_i D_j f \tag{5}$$

where $g^{i,j}$ is the inverse of $g_{i,j}$ and D_i is the covariant derivative in the direction of the x_i coordinate.

2 A Generalized Maximal Principle

Lemma 1. *Let M be an n-dimensional Riemannian manifold. Let $x \in M$ be a point which can be joined to p by a minimal geodesic. Then if x is not on the cut locus of p, we have*

$$\begin{aligned} \Delta\gamma(x) &\leq K(x) \\ &= min_{0\leq k\leq \ell}\frac{n-1}{\ell-k} - \frac{1}{(\ell-k)^2}\int_0^\ell (t-k)^2 Ric(N)dt \end{aligned} \tag{6}$$

Proof. Let σ : $[0,\ell]\rightarrow$ M be the minimal geodesic of length $l = \gamma(x)$ from p to x, and J_i the unique Jacobi fields vanishing at $\sigma(0)$ such that $J_i(l) = E_i(l)$, where $E_1, E_2, \ldots, E_{n-1}, N = \sigma'$ form a parallel field along σ. Firstly, we need to show that,

$$0 = \partial_{J_i} < N, J_i > = < D_{J_i} N, J_i > - < D_{J_i} J_i, N > \tag{7}$$

$$\begin{aligned} \partial_N < N, J_i > &= \partial_t < N, J_i > \\ &= < D_N N, J_i > + < N, D_N J_i > \end{aligned} \tag{8}$$

Now, $< D_N N, J_i > = 0$, so

$$\begin{aligned} < D_N N, J_i > + < N, D_N J_i > &= < N, D_N J_i > \\ \partial_t < N, D_N, J_i > &= < D_N N, D_N J_i > + < N, D_N D_N J_i > \\ &= < N, D_N D_N J_i > \\ &= < R(J_i, N)N, N > \\ &= 0 \end{aligned} \tag{9}$$

Since $< J_i(0), N > = 0$ and $< J_i(l), N > = 0$ since $J_i(l) = E_i$ which is orthogonal to N. Thus, $< N, J_i >$ does not change with respect to N and its second derivative is 0. This means that $< N, J_i >$ is identically 0 on $[0, \ell]$. A straightforward computation then yields:

$$
\begin{aligned}
&\int_0^\ell \Sigma_{i=1}^{n-1} < D_N J_i, D_N J_i > - < R(N, J_i)J_i, N > dt \\
&= \int_0^\ell \Sigma_{i=1}^{n-1} < D_N J_i, D_N J_i > + < R(N, J_i)N, J_i > dt \\
&= \int_0^\ell \Sigma_{i=1}^{n-1} < D_N J_i, D_N J_i > + < D_N D_N J_i, J_i > dt \\
&= \int_0^\ell \Sigma_{i=1}^{n-1} \partial_N < D_N J_i, J_i > dt \\
&= \int_0^\ell \Sigma_{i=1}^{n-1} \partial_t < D_{J_i} N, J_i > dt \\
&= -\int_0^\ell \Sigma_{i=1}^{n-1} \partial_t < D_{J_i} J_i, N > dt \\
&= -\Sigma_{i=1}^{n-1} < D_{E_i} E_i, N > \\
&= -\Sigma_{i=1}^{n-1} \partial_{D_{E_i} E_i} \gamma \\
&= \Delta\gamma
\end{aligned}
\tag{10}
$$

and hence, we have

$$
\Delta\gamma(x) = \int_0^\ell \Sigma_{i=1}^{n-1} < D_N J_i, D_N J_i > - < R(N, J_i)J_i, N > dt \tag{11}
$$

Now let $f(t)$ be any piecewise smooth function defined on $[0, \ell]$ such that $f(0) = 0$ and $f(\ell) = 1$. Using the fundamental inequality of the index form we have that

$$
\begin{aligned}
&\int_0^\ell \Sigma_{i=1}^{n-1} < D_N J_i, D_N J_i > - < R(N, J_i)J_i, N > dt \\
&\leq \int_0^\ell \Sigma_{i=1}^{n-1} < D_N f(t)E_i, D_N f(t)E_i > - < R(N, f(t)E_i)f(t)E_i, N > dt
\end{aligned}
\tag{12}
$$

For $0 \leq k \leq \ell$, we define f to be zero for $0 \leq t \leq k$ and $(t - \ell)/(\ell - k)$ for $t \geq k$.

Then, we have that:

$$\begin{aligned}\Delta\gamma &\leq \int_0^\ell \Sigma_{i=1}^{n-1} < D_N f(t)E_i, D_N f(t)E_i > - < R(N, f(t)E_i)f(t)E_i, N > dt \\ &\leq \int_0^\ell (\Sigma_{i=0}^{n-1}(f'(t))^2) - (f(t))^2 Ric(N)dt \\ &= \int_0^\ell (n-1)(f'(t))^2 - (f(t))^2 Ric(N)dt \\ &= (1/(\ell-k)^2)(\ell-k)(n-1) - (1/(\ell-k)^2)\int_0^\ell (t-k)^2 Ric(N)dt \\ &= (n-1)/(\ell-k) - (1/(\ell-k)^2)\int_0^\ell (t-k)^2 Ric(N)dt\end{aligned} \tag{13}$$

And since this inequality holds for all $k \in [0, \ell]$, it must also hold for the minimum. Thus,
$\Delta\gamma \leq min_{0\leq k\leq \ell}(n-1)/(\ell-k) - (1/(\ell-k)^2)\int_0^\ell (t-k)^2 Ric(N)dt$
and we have shown the result for Lemma 1. □

Corollary 1. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Then, $\Delta\gamma$ is bounded uniformly from above when $\gamma \geq 1$*

Proof. Suppose the Ricci curvature is bounded below by a constant R and $\gamma(x) \geq 1$, then

$$\begin{aligned}K(x) &\leq min_{0\leq k\leq \ell-1}(n-1)/(\ell-k) - (1/(\ell-k)^2)\int_0^\ell (t-k)^2 R dt \\ &= min_{0\leq k\leq \ell-1}(n-1)/(\ell-k) - (\ell-k)R/3 \\ &= (n-1) - R/3\end{aligned} \tag{14}$$

Thus, by Lemma 1, $\Delta\gamma(x) \leq (n-1) - R/3, \forall x$ such that $\gamma(x) \geq 1$. □

Let us consider C^2 function f defined on M and the graph $\Gamma = \{(f(x), x) : x \in M\}$. Γ is a closed submanifold of the product space R×M with R being the real line and M the manifold. The metric defined on Γ is the product metric.

Now, let us fix a point $p \in$ M. Let us define $p_k = (k, p)$, where $k \in \mathbb{N}$ and let g_k be a segment of a geodesic from p_k to the graph Γ and so the length of g_k represents the shortest distance curve joining p_k and the graph Γ. Let $(f(q_k), q_k)$ be another end point of g_k and let $\pi(g_k)$ be the projection of g_k on M with end points p and q_k. Since g_k was a minimal geodesic, its projection $\pi(g_k)$ is also a minimal geodesic, since we have chosen the metric to be the product metric.

Claim 1. *The point q_k is not a conjugate point of p along $\pi(g_k)$.*

Proof. If q_k were a conjugate point of p along the projected geodesic $\pi(g_k)$, then the point $(f(q_k), q_k)$ would also be a conjugate point of p along g_k. Representing the arclength of g_k by l_k, for any $\epsilon > 0$ chosen sufficiently small, there exists a sphere centered at $g_k(l_k + \epsilon)$ with radius ϵ which touches the graph Γ at a single point, $g_k(l_k)$. If we assumed that $(f(q_k), q_k)$ were a conjugate point of p_k along g_k, then in a neighbourhood of g_k there would exist another geodesic, say g'_k from p_k to $g_k(l_k + \epsilon)$ of length $< l_k + \epsilon$. However, now if you let q'_k be the point of intersection of Γ and g'_k, then the distance from p_k to q'_k is $< l_k$, contradicting the fact that g_k was a minimal geodesic segment. □

Call the neighbourhood of the geodesic $\pi(g_k)$ N_k such that $\pi(g_k)$ is the unique minimal geodesic in N_k which joins p and g_k. Let γ_k represent the distance function on N_k that is smooth at x, since p and q_k are not conjugate points. Let us, for now, shift f such that $f(p) = 0$ and define:

$$F_k(x) = k - \sqrt{l_k^2 - \gamma_k^2} \tag{15}$$

The squared distance from $(f(p), p)$ to (k, p) is given by k^2 since $f(p) = 0$. Then, we have that

$$(k - f(q_k))^2 + \gamma_k(q_k)^2 \leq k^2 \tag{16}$$

Furthermore, again since $(q_k, f(q_k))$ is the closest point from p_k to Γ, for any $x \in M$.

$$l_k^2 \leq (k - f(x))^2 + \gamma_k(x)^2 \tag{17}$$

By the latter inequality and rearranging its terms, we see that $f(x) \leq F_k(x)$ and at $x = q_k$, we have that $f(q_k) = F_k(q_k)$ since

$$\begin{aligned} F_k(q_k) &= k - \sqrt{l_k^2 - \gamma_k(q_k)^2} \\ &\leq k - \sqrt{(k - f(q_k)^2} \\ &= f(q_k) \end{aligned} \tag{18}$$

Thus, q_k is a local minimum for the function $F_k(x) - f(x)$ since $0 \leq F_k(x) - f(x)$ $\forall x$ and equality holds at $x = q_k$. Thus, the Laplacian at $F_k(x) - f(x)$ at $x = q_k$ should be greater than or equal to 0 since it is a local minimum.

$$\begin{aligned} \Delta(F_k(x) - f(x)) &\geq 0 \\ \Delta f(q_k) &\leq \Delta F_k(q_k) \\ &= \Delta(k - \sqrt{l_k^2 - \gamma_k^2}) \\ &= \frac{\Delta\gamma_k^2(q_k)}{2(k - f(q_k))} + \frac{\nabla\gamma^2\nabla\gamma^2(q_k)}{4(k - f(q_k)^3)} \\ &\leq \frac{\gamma_k(q_k)\Delta\gamma_k(q_k)}{k - f(q_k)} + \frac{1}{k - f(q_k)} + \frac{(\gamma_k(q_k)^2}{(k - f(q_k)^3} \end{aligned} \tag{19}$$

By the result of Lemma 1, we can replace $\Delta\gamma_k(q_k)$ by $K(q_k)$ with an inequality to get:

$$\Delta f(q_k) \leq \frac{\gamma_k(q_k)K(q_k)}{k - f(q_k)} + \frac{1}{k - f(q_k)} + \frac{(\gamma_k(q_k)^2}{(k - f(q_k)^3} \tag{20}$$

We also have that since $f(p) = 0$, $f(q_k) \geq 0 = f(p)$. A straightforward computation then gives that

$$\Delta f(q_k) \leq (K(q_n) + \frac{1}{\gamma(q_k)})(\frac{2f(q_k)}{\gamma(q_k)(1 - f(q_k)/k)} + \frac{1}{\gamma(q_k)}(\frac{2f(q_k)}{\gamma(q_k)(1 - f(q_k)/k))})^3 \tag{21}$$

Suppose function f satisfies,

$$limsup_{\gamma(x)\to\infty} \frac{f(x) - f(p)}{\gamma(x)} < \infty \tag{22}$$

Then, we may find an appropriate scale factor a such that

$$sup_{\gamma(x)\geq\delta} \frac{a(f(x) - f(p))}{\gamma(x)} < \epsilon \tag{23}$$

where $\delta = liminf_{k\to\infty}\gamma(q_k)$ which exists unless $\gamma(q_k) = 0\forall k$. Letting $d_k = f(q_k)/\gamma(q_k)$, choosing a small enough and taking $\epsilon \leq 1/6$, we get:

$$\begin{aligned}
\Delta af(q_k) &\leq (K(q_n) + \frac{1}{\gamma(q_k)})(\frac{2af(q_k)}{\gamma(q_k)(1 - af(q_k)/k)}) + \frac{1}{\gamma(q_k)}(\frac{2af(q_k)}{\gamma(q_k)(1 - af(q_k)/k))})^3 \\
\Delta f(q_k) &= (K(q_n) + \frac{1}{\gamma(q_k)})(\frac{2f(q_k)}{\gamma(q_k)(1 - af(q_k)/k)}) + \frac{1}{\gamma(q_k)}(\frac{2f(q_k)}{\gamma(q_k)(1 - af(q_k)/k))})^3 \\
&\leq (K + 1/\gamma)(\frac{2}{1 - af})d_k + \frac{a^2}{\gamma}d_k\frac{f^2}{\gamma^2}(\frac{2}{1 - af})^3 \\
&\leq (K + 1/\gamma + \frac{4}{\gamma}\epsilon^2(1 - \epsilon)^{-2})(2d_k(1 - \epsilon)^{-1}) \\
&\leq (K + \frac{1 + \epsilon}{\gamma})(\frac{2d_k}{1 - \epsilon})
\end{aligned} \tag{24}$$

Observe that since q_k is a local minimum of the function $F_k(x) - f(x)$, $\nabla f(q_k) = \nabla F_k(q_k)$, we get

$$\begin{aligned}
|\nabla f(q_k)| = |\nabla F_k(q_k)| &= \frac{\gamma(q_k)}{a(k - af(q_k)} \\
&\leq 2d_k \frac{1}{a(1 - af(q_k)/k)} \\
&\leq \frac{2d_k}{a(1 - \epsilon)}
\end{aligned} \tag{25}$$

Claim 2. *For all $x \in M$, for all $\delta > 0$, there exists a point q_k in our sequence such that $f(q_k) \geq f(x) - \delta$.*

Proof. Let us assume that there exists $\delta > 0$ such that

$$f(x) > limsup_{k\to\infty} f(q_k) + \delta \tag{26}$$

Then, for any k sufficiently large,

$$\begin{aligned}
\delta(2k - f(x) - f(q_k)) + \gamma(q_k)^2 - \gamma(x)^2 > 0 \\
(f(x) - f(q_k))(2k - f(x) - f(q_k)) + \gamma(q_k)^2 - \gamma(x)^2 > 0 \\
2f(x)k - f(x)^2 - f(x)f(q_k) - 2f(q_k)k + f(x)f(q_k) + f(q_k)^2 + \gamma(q_k)^2 - \gamma(x)^2 > 0 \\
(k - f(x))^2 + \gamma(x)^2 < (k - f(q_k))^2 + \gamma(q_k)^2
\end{aligned} \tag{27}$$

This, however, contradicts the choice of q_k as having minimized the quantity on the right-hand side which equals to l_k^2. □

If there exist infinitely many values of $\gamma(q_k)$ such that $\gamma(q_k) \leq 1$, then on this compact set, f must attain its supremum at one of the limit points of the q_k. Thus, we have proven the following theorem.

Theorem 1. *Let f be a C^2-function defined on a complete Riemannian manifold. Suppose*

$$limsup_{\gamma(x)\to\infty} \frac{f(x) - f(p)}{\gamma(x)} < \infty \tag{28}$$

Then either f attains its supremum at some point with $\gamma(x) \leq 1$ or, for all $\epsilon \in (0,1)$, we can find a sequence $\{q_k\}$ in M such that $\gamma(q_k) \geq 1$ and the following hold, where $d_k = (f(q_k) - f(p))/\gamma(q_k)$.

$$\begin{aligned}
lim_{k\to\infty} f(q_k) &= supf \\
|\triangle f(q_k)| &\leq \frac{2d_k}{a(1-\epsilon)} \\
\Delta f(q_k) &\leq (K(q_k) + \frac{1+\epsilon}{\gamma(q_k)})\frac{2d_k}{1-\epsilon}
\end{aligned} \tag{29}$$

Corollary 2. *Suppose in Theorem 1*

$$\begin{aligned}
limsup_{\gamma(x)\to\infty} \frac{f(x) - f(p)}{\gamma(x)} \leq 0 \\
limsup_{\gamma(x)\to\infty, f(x)\geq f(p)} \frac{K(x)(f(x) - f(p))}{\gamma(x)} = 0
\end{aligned} \tag{30}$$

Then, there are points $\{q_k\} \subset M$ such that $lim_{k\to\infty} f(q_k) = supf$, $lim_{k\to\infty} \nabla f(q_k) = 0$ and $limsup_{k\to\infty} \Delta f(q_k) \leq 0$.

Proof. Suppose $limsup_{k\to\infty}\gamma(q_k) = \infty$. Then, Corollary 2 follows by taking the limsup of both sides of the inequalities in Theorem 1. Otherwise, $f(x)$ attains its supremum at some point q'. In this case, we may simply choose $p = q'$ and the inequalites hold. □

Let $X : M \to R^N$ be a proper isometric immersion of a complete Riemannian manifold into some Euclidean space of dimension N. Let the distance function be given simply by the Euclidean distance function restricted to M that is, $\gamma(x) = ||X(x) - X(p)||$.

Claim 3. *Let H represent the mean curvature vector of M, then*

$$\begin{aligned} |\nabla\gamma| &\leq 1 \\ \gamma\Delta\gamma + |\nabla\gamma|^2 &= n + n < H, X(x) - X(p) > \end{aligned} \tag{31}$$

Proof. Using the fact that $Hn = \sum_{i,j} g^{i,j}\Delta_i\Delta_j X$ and that $< D_iX(x), D_jX(x) >$ is the first fundamental form and thus given by $g_{i,j}$, where the product of $\sum_{i,j} g^{i,j}g_{i,j} = trace(I_n)$, we have

$$\begin{aligned} \Delta(\gamma^2) &= 2\gamma\Delta(\gamma) + 2(\nabla\gamma)^2 \\ &= 2\gamma\Delta(\gamma) + 2(\nabla\gamma)^2 \\ &= \Delta(< X(x) - X(p), X(x) - X(p) >) \\ &= \sum_{i,j} g^{i,j} D_iD_j(< X(x) - X(p), X(x) - X(p) >) \\ &= 2\sum_{i,j} g^{i,j} D_i < D_jX(x), X(x) - X(p) > \\ &= 2\sum_{i,j} g^{i,j}(< D_iD_jX(x), X(x) - X(p) > + < D_jX(x), D_iX(x) >) \\ &= 2\sum_{i,j} g^{i,j} < D_iD_jX(x), X(x) - X(p) > + 2g_{i,j} \\ &= 2 < Hn, X(x) - X(p) > + 2n \end{aligned} \tag{32}$$

□

Now, let f be any $C^2 - function$ which is bounded from above on M. Let us define, for all $k > 0$,

$$g(x) = \frac{f(x) - f(p) + 1}{[log(\gamma^2(x) + 2)]^k} \tag{33}$$

It is clear that $g(p) = 1/(log2)^k$ and that the limit of g(x) as $\gamma(x) \to \infty$ is 0. Thus, g must attain its supremum at some point $x_k \in M$. So,

$$\begin{aligned} g(x_k) &= supg(x) \\ \nabla g(x_k) &= 0 \\ \Delta g(x_k) &\leq 0 \end{aligned} \tag{34}$$

A straightforward computation, that can be found in Yau's original article, shows the above.

Theorem 2. *Let M be a complete, properly immersed submanifold of some Euclidean space. Let f be any C^2-function which is bounded above on M. Then there exists a sequence of points $\{x_k\}$ in M such that*

$$\begin{aligned} &limsup_{k\to\infty} f(x_k) = supf \\ |\nabla f| &\leq \frac{2(f(x_k)-f(p)+1)\gamma(x_k)}{k(\gamma^2(x_k)+2)log(\gamma^2(x_k)+2)} \\ \Delta f(x_k) &\leq \frac{2n(f(x_k)-f(p)+1)}{k(\gamma^2(x_k)+2)log(\gamma^2(x_k)+2)}[1+ < H, X(x_k)-X(p) >] \\ &- \frac{4\gamma^2(f(x_k)-f(p)+1)}{k(\gamma^2(x_k)+2)log(\gamma^2(x_k)+2)} \end{aligned} \tag{35}$$

3 Gradient Estimates of Partial Differential Equations on a Riemannian Manifold

In this section, we will state results from Yau's article which were obtained by applying Corollary 2 on the function

$$g(x) = -\frac{f(x)+b-inf f}{\sqrt{|\nabla f|^2+a}} \tag{36}$$

where a and b are small positive constants. This allows us to obtain a negative upper bound, -C, on the supremum of g which then yields the following theorems.

Theorem 3. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below by a constant -K. Let f be a C^2-function bounded from below on M such that, for positive constants C_i,*

$$\begin{aligned} |\Delta f| &\leq C_i|\triangle f| + C_2(f-inf f)+C_3 \\ |\nabla(\Delta f)| &\leq C_4\sqrt{\sum_{i,j} f_{i,j}^2} + C_5|\nabla f| + C_6(f-inf f)+C_7 \end{aligned} \tag{37}$$

Then, for any constant b such that b+inf f > 0, there exists a constant C that depends only on b, K, dimM, and the C_i such that

$$|\nabla f| \leq C(f-inf f) \tag{38}$$

From the above theorem, we see that for harmonic functions, the C_i are all zero and we have the following theorem. In particular, this gives us the result we discussed in the introduction.

Theorem 4. *Let M be an n-dimensional complete Riemannian manifold with Ricci curvature bounded from below by -K. Let f be a harmonic function bounded from below on M. Then,*

$$|\Delta f| \leq \sqrt{(n-1)K}(f-inf f) \tag{39}$$

Corollary 3. *Let M be a complete Riemannian manifold with non-negative Ricci curvature. Then every positive harmonic function on M is a constant.*

Corollary 4. *Let B be the open unit ball in an n-dimensional Euclidean space. Let f be a harmonic function which is bounded from below on B. Then there is a constant C, depending only on n, such*

$$|\nabla f|(x) \leq \frac{C(f(x) - inf f)}{1 - |x|^2} \tag{40}$$

Corollary 5. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let H be a smooth function defied on M such that both —H— and $|\nabla H|$ are uniformly bounded. If the equation $\Delta f = Hf$ has a smooth positive solution, then, for any sequence of domains $\{D_i\}$ of M with smooth boundaries $\{\partial D_i\}$ and $lim_{i\to\infty}(Vol(D_i)/Vol(\partial D_i)) = \infty$, we have*

$$lim_{i\to\infty} \frac{1}{Vol(D_i)} \int_{D_i} H \geq 0 \tag{41}$$