Uniformization

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1 Introduction

The purpose of this project is to investigate a proof of the uniformization theorem due to Rafe Mazzeo and Michael Taylor, which consists basically in finding a conformal metric of constant curvature by solving the equation

$$K(x) = (K_0(x) - \Delta u(x))e^{-2u(x)}, \qquad (x \in M),$$

on a Riemann surface M.

Most of the paper is devoted to showing the existence of a Poincaré metric on some specific Riemann surfaces. Our treatment follows closely that in [2]. After providing the reader with the necessary mathematical background in §2, we construct complete metrics of constant negative curvature on smoothly bounded Riemann surfaces in §3. In §4, the question of the existence of a Poincaré metric on a general planar domain is studied. The results of §3 and §4 are then applied in §5 to establish the uniformization theorem for noncompact surfaces. Finally, we discuss briefly the uniformization theorem for compact surfaces in §6.

A knowledge of elementary Riemannian geometry is assumed. However, the required notions about Riemann surfaces are included below.

2 Preliminaries

Before starting the proof of the uniformization theorem, we must introduce some terminology and facts about Riemann surfaces and differential geometry. Most of the elementary definitions about Riemann surfaces introduced here can be found, for example, in [1]. For the remaining of this section, M is a smooth, connected, oriented two-dimensional manifold.

Definition 2.1. A *complex chart* on M is an ordered pair (U, φ), where U is an open subset of M and $\varphi : U \to V$ is a homeomorphism from U onto an open subset $V \subset \mathbb{C}$. Two complex charts (U_1, φ_1) and (U_2, φ_2) are said to be *compatible* if the *transiton map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a biholomorphism. A *complex atlas* on M is a collection of pairwise compatible complex charts $\mathscr{A} = \{(U_\alpha, \varphi_\alpha) \,:\, \alpha \in I\}$ such that $M = \bigcup_{\alpha \in I} U_\alpha$. Moreover, we say that two complex atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are *equivalent* if every chart of $\mathscr{A}_1$ is compatible with every chart of $\mathscr{A}_2$.

It is easily seen that the notion of equivalent complex atlases is an equivalence relation. Indeed, an atlas is trivially equivalent to itself. Moreover, since the inverse of a biholomorphic map is by definition a biholomorphic map, if a complex atlas (U_1, φ_1) is compatible with (U_2, φ_2), then (U_2, φ_2) is clearly compatible with (U_1, φ_1). Finally, if (U_1, φ_1) is compatible with (U_2, φ_2), and (U_2, φ_2) is compatible with (U_3, φ_3), then $\varphi_3 \circ \varphi_1^{-1} = \varphi_3 \circ \varphi_2^{-1} \circ \varphi_2 \circ \varphi_1^{-1}$ is a biholomorphism since it is the composition of two biholomorphic mappings.

Definition 2.2. A *complex structure* on M is an equivalence class of equivalent complex atlases on M.

If g is a Riemannian metric on M, then for any $u \in C^\infty(M)$, $e^{2u}g$ is also a metric on M. Two metrics g and h satisfying $h = e^{2u}g$ for some $u \in C^\infty(M)$ are said to be *conformally equivalent*. It is straightforward to verify that being conformally equivalent is an equivalence relation. The corresponding equivalence class is called the *conformal class* of g. It is a well-known fact that to any conformal class on M corresponds a unique complex structure on M, and reciprocally. In other words, the concepts of complex structure and of conformal class on a surface are equivalent. This fact motivates the following definition.

Definition 2.3. A *Riemann surface* is a smooth, connected, oriented two-dimensional manifolds endowed with a complex structure or with a conformal class.

These two equivalent definitions of a Riemann surfaces will be used interchangeably throughout this note.

Definition 2.4. Let M and N be two Riemann surfaces. A continuous map $f : M \to N$ is said to be *holomorphic* if for every pair of complex charts (U_1, φ_1) on M and (U_2, φ_2) on N, satisfying $f(U_1) \subset U_2$, the mapping $\varphi_2 \circ f \circ \varphi_1^{-1}$ is a holomorphism (in the usual sense of complex analysis). Moreover, f is *biholomorphic* if it is a bijection and if f and f^{-1} are both holomorphic.

We can now state the main result of this paper.

Theorem 2.1 (Uniformization Theorem)**.** *There is a biholomorphism between any simply connected Riemann surface and either $\mathbb{C}$, the Riemann sphere $\hat{\mathbb{C}}$ or the open unit disk D_1.*

Since it can be shown that the universal cover (or, more generally, any covering space) of a Riemann surface is also a Riemann surface, an equivalent statement of Theroem 2.1 is that any Riemann surface can be holomorphically covered by either $\mathbb{C}$, $\hat{\mathbb{C}}$ or D_1.

In terms of conformal classes, the uniformization theorem can be stated as follows: Any simply connected Riemann surface admits in its conformal class a unique complete metric with constant curvature either -1, 0 or 1. To see how these two statements are related, suppose that φ is a biholomorphic map between the simply connected Riemann surface M and either $\mathbb{C}$, $\hat{\mathbb{C}}$ or D_1. It is well-known that $\mathbb{C}$ admits a metric of constant curvature 0, $\hat{\mathbb{C}}$ a metric of constant curvature 1 and D_1 a metric of constant curvatures -1. A constant curvature metric on M can then be defined by pushing down this metric through the biholomorphic map φ. Conversely, suppose that M is a simply connected Riemann surface that admits a metric g of constant curvature. By a basic result in Riemannian geometry, we know that any simply connected n-th dimensional manifold of constant curvature is isometric either to $\mathbb{R}^n$, an n-th dimensional ball or an n-th dimensional sphere, with flat, negative and positive curvature respectively. Since an isometry preserves the angles, it is in particular conformal. Hence M is biholomorphic to either $\mathbb{C}$, $\hat{\mathbb{C}}$ or D_1.

We will take as a fact the following basic result from Riemannian geometry.

Proposition 2.2. *Let g_0 be a Riemannian metric on M, and $K_0(x)$ be the corresponding Gauss curvature. If $g = e^{2u}g_0$ for some $u \in C^\infty(M)$, then*

$$K = (K_0 - \Delta u)e^{-2u}. \tag{2.1}$$

is the Gauss curvature of g.

In the following sections, we will mostly concentrate on the case of negative curvature. We will refer to a complete metric of constant Gauss curvature -1 in the conformal class of a Riemann surface M as a *Poincaré metric* on M. For example, the open unit disk D_1 admits a Poincaré metric defined componentwise by

$$\frac{4}{(1-(x_1^2+x_2^2))^2}\delta_{jk},$$

where $(x_1, x_2) \in D_1 \subset \mathbb{R}^2$ and δ_{jk} corresponds to a component of the euclidean metric on D_1.

It is clear from (2.1) that a Poincaré metric can be obtained by solving the partial differential equation

$$\Delta u - e^{2u} = K_0(x). \tag{2.2}$$

and by showing that the corresponding metric $e^{2u}g_0$ is complete. This is the strategy we will adopt here.

3 Bounded surfaces with smooth boundary

Let M be the interior of a compact, oriented, connected two-dimensional smooth manifold with boundary $\overline{M}$, endowed with a Riemannian metric g_0. The aim of this section is to show the existence of a Poincaré metric for M. This will be carried out in three steps. First, we obtain the existence of a unique solution $u_a \in C^\infty(\overline{M})$ to (2.2) satisfying $u_a|_{\partial\overline{M}} = a$, where a is any positive real number. In a second time, we consider the limit u_a as $a \to \infty$. We will see that this limit function is well-defined and is a solution to (2.2). Finally, the corresponding metric $e^{2u}g_0$ will be shown to be complete. The main result of this section is the following theorem.

Theorem 3.1. *If M is as above, then M admits a Poincaré metric.*

We start by studying the Dirichlet problem for the semilinear equation

$$\Delta u = f(x, u), \tag{3.1}$$

$$u|_{\partial M} = g, \tag{3.2}$$

under the hypothesis that

$$\frac{\partial f}{\partial u} \geq 0, \tag{3.3}$$

where g is a sufficiently smooth function on ∂M. Our approach to (3.1)-(3.2) follows closely that in §1 of Chap. 14 in [3].

Since (2.2) can be written in the form $\Delta u = f(x, u)$, where $f(x, u) = e^{2u} + K_0(x)$, and $\frac{\partial f}{\partial u} = 2e^{2u} > 0$, the existence of a unique $C^\infty(\overline{M})$ solution to (3.1)-(3.3) would give us the required function u_a discussed above.

Define

$$F(x,u) := \int_0^u f(x,s)ds \quad \text{and} \quad I(u) := \frac{1}{2}\|\nabla u\|^2_{L^2(M)} + \int_M F(x,u(x))dV(x).$$

It is clear from (3.3) that F is convex in u. So, I is strictly convex. Set

$$V := \left\{u \in H^1(M) \;:\; u = g \;\text{ on } \partial M\right\}.$$

If we are able to show that I has a unique minimum u on V, then for any $v \in C_0^\infty(M)$ and any $s \in \mathbb{R}$, the function $u + sv \in V$ minimizes the functional $I(u+sv)$ when $s = 0$. It follows that

$$\begin{aligned}
0 &= \frac{d}{ds} I(u+sv)|_{s=0} \\
&= \frac{d}{ds}\bigg|_{s=0} \left[\frac{1}{2}\int_M (\nabla u(x) + s\nabla v(x))^2 dV(x) + \int_M F(x, u(x)+sv(x))dV(x)\right] \\
&= \left[\int_M (\nabla u(x) + s\nabla v(x))\nabla v(x) dV(x) + \int_M f(x, u(x)+sv(x))v(x)dV(x)\right]_{s=0} \\
&= \int_M \nabla u(x)\nabla v(x)dV(x) + \int_M f(x,u(x))v(x)dV(x) \\
&= -\int_M (\Delta u(x))v(x)dV(x) + \int_M f(x,u(x))v(x)dV(x) \qquad (\textit{by Green formula}),
\end{aligned}$$

for all $v \in C_0^\infty(M)$. In other words, u is a weak solution of (3.1). So, in order to solve our Dirichlet problem, it is sufficient to show the existence of this minimum u. Before proceeding further, we make temporarily the following hypothesis on F:

$$\text{There exists a constant } K \text{ such that } \partial_u f(x,u) = 0 \text{ whenever } |u| > K. \tag{3.4}$$

A direct consequence of this additional restriction is that

$$\left|\frac{\partial F}{\partial u}(x,u)\right| \leq L, \qquad ((x,u) \in \overline{M} \times \mathbb{R}),$$

for some constant L. It follows from this inequality that

$$|F(x,u) - F(x,v)| \leq L|u-v|,$$

for $u, v \in \mathbb{R}$. Hence, it is easy to see that $I : V \to \mathbb{R}$ is Lipschitz continuous with respect to the norm topology on V.

Moreover, I is bounded below. Indeed, since $F(x,u)$ is convex in u, there exists positive constants A and B such that

$$F(x,u) \geq -A|u| - B.$$

So,

$$I(u) \geq \frac{1}{2}\|\nabla u\|^2_{L^2} - A\|u\|_{L^1} - BV(M). \tag{3.5}$$

From the Poincaré inequality, we have

$$\|\nabla u\|_{L^2}^2 \geq C\|u\|_{L^2}^2 - D, \tag{3.6}$$

for some positive constants C and D. Combining (3.5), (3.6) and the fact that $V(M) < \infty$, we obtain the following inequality:

$$I(u) \geq A'\|\nabla u\|_{L^2}^2 + B'\|u\|_{L^2}^2 - C'\|u\|_{L^2} - D', \tag{3.7}$$

for some nonnegative constants A', B', C' and D'. If we take $\|u\|_{H^1} \to \infty$ in (3.7), then it is clear that $I(u) \to \infty$. Hence I is bounded from below on V.

Proposition 3.2. *Suppose that f satisfies (3.3) and (3.4). Then $I(u)$ has a unique minimum on V.*

Proof. Since $I(u)$ is bounded on V, there exists $\alpha \in \mathbb{R}$ such that $\alpha = \inf_{u \in V} I(u)$. Moreover, since $I(u) \to \infty$ as $\|u\|_{H^1} \to \infty$, we can take R large enough so that $K := V \cap B_R(0) \neq \emptyset$ and so that the inequality $I(u) \geq \alpha + 1$ holds whenever $\|u\|_{H^1} \geq R$. Define

$$K_\epsilon := \{u \in K \ : \ \alpha \leq I(u) \leq \alpha + \epsilon\},$$

for $\epsilon > 0$. We have that K_ϵ is a closed, convex subset of K, which is itself a closed, convex, bounded subset of $H^1(M)$. The sets K_ϵ are then weakly closed in the weakly compact set K. It follows that the intersection of any nesting sequence of K_ϵ is nonempty. So, $K_0 := \bigcap_{\epsilon > 0} K_\epsilon \neq \emptyset$. In other words, there exists $u \in K_0 \subset V$ minimising $I(v)$ on V. The uniqueness of u follows directly from the strict convexity of I. $\square$

In the proof of the uniformization theorem, we will only require that u equal a constant on ∂M. Our function g is then constant, and in particular $C^\infty(\partial M)$.

Proposition 3.3. *Let $u \in V$ be a solution of (3.1)-(3.2) in which f satisfies (3.4). If $g \in C^\infty(\partial M)$ then $u \in C^\infty(\overline{M})$.*

Proof. The additional restriction on f implies that $\Delta u = f(x, u) \in H^1(M)$. So, if $g \in H^{3/2}(\partial M)$, then $u \in H^2(M)$. The statement follows by applying this argument inductively. $\square$

Now, we would like relax the hypothesis (3.4). We need the two following propositions.

Proposition 3.4. *Let $u, v \in C^2(M) \cap C(\overline{M})$ be solutions of (3.1) that satisfy the Dirichlet boundary conditions $u = g$ and $v = h$ on ∂M respectively. If we assume furthermore hypothesis (3.3), then*

$$\sup_M (u - v) \leq \max\left\{\sup_{\partial M}(g - h), 0\right\},$$

and

$$\sup_M |u - v| \leq \sup_{\partial M} |g - h|.$$

Proof. Define $w := u - v$ and $O := \{x \in M \ : \ w(x) \geq 0\}$. Since $\partial_u f \geq 0$, we have

$$\lambda(x) := \frac{f(x,u) - f(x,v)}{u - v} \geq 0.$$

Therefore, w satisfies

$$\Delta w = \lambda(x) w \geq 0, \qquad w|_{\partial M} = g - h,$$

and then the maximum principle applies on O. So, we have

$$\sup_M (u - v) \leq \max\left\{\sup_O w, 0\right\} = \max\left\{\sup_{\partial O}(g - h), 0\right\} \leq \max\left\{\sup_{\partial M}(g - h), 0\right\},$$

which is the first statement. The second inequality is obtained by combining the first one with the inequality obtained by applying the same argument to the function $-w$ instead of w. $\square$

Proposition 3.5. *Let $\varphi(x) := f(x, 0) \in C^\infty(\overline{M})$. For $g \in C^\infty(\partial M)$, let $\Phi \in C^\infty(\overline{M})$ be the solution to*

$$\Delta \Phi = \varphi \quad \text{on } M, \qquad \Phi = g \quad \text{on } \partial M. \tag{3.8}$$

If f satisfies (3.3) and u is a solution to (3.1)-(3.2), then

$$\sup_M u \leq \sup_M \Phi + \left(\max\left\{\sup_M(-\Phi), 0\right\}\right),$$

and

$$\sup_M |u| \leq \sup_M 2|\Phi|.$$

Proof. By (3.3), we have

$$\lambda(x) := \frac{f(x,u) - f(x,0)}{u} \geq 0.$$

So,

$$\Delta(u - \Phi) = \lambda(x) u \geq 0$$

on $O^+ := \{x \in M \ : \ u(x) > 0\}$. By the maximum principle, we have

$$\sup_{O^+}(u - \Phi) = \sup_{\partial O^+}(u - \Phi) \leq \max\left\{\sup_M(-\Phi), 0\right\}. \tag{3.9}$$

The first part of the statement follows. Similarly, we have

$$\sup_{O^-}(\Phi - u) = \sup_{\partial O^-}(\Phi - u) \leq \max\left\{\sup_M \Phi, 0\right\}. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain the second part of the statement. $\square$

We can now get rid completely of the temporary restriction (3.4).

Theorem 3.6. *Let $f \in C^\infty(\overline{M} \times \mathbb{R})$ and satisfy (3.3). For any $g \in C^\infty(\partial M)$, there exists a unique solution $u \in C^\infty(\overline{M})$ to the Dirichlet problem (3.1)-(3.2).*

Proof. For each $j \in \mathbb{N}$, let $f_j(x,u) \in C^\infty(\overline{M} \times \mathbb{R})$ satisfy (3.3)-(3.4), and be such that $f_j(x,u) = f(x,u)$, for $|u| \leq j$. By the discussion above, for each j, there is a solution u_j of the equation

$$\Delta u_j = f_j(x, u_j), \quad u_j|_{\partial M} = g.$$

Since $f_j(x,0) = f(x,0) = \varphi(x)$ for all j, Proposition 3.5 implies that

$$\sup_M |u_j| \leq \sup_M 2|\Phi|,$$

where Φ is a solution of (3.8). The existence of the solution u follows by taking j large enough. Finally, the uniqueness of u is a direct consequence of the second inequality in Proposition 3.4. $\square$

Now, we start the construction of the Poincaré metric for M. For any $a \in (0, \infty)$, Theorem 3.6 gives the existence of a unique solution $u_a \in C^\infty(\overline{M})$ to (2.2) satisfying $u_a = a$ on ∂M. As explained at the beginning of the section, we want to take $a \to \infty$. We will need the following monotonicity result for u_a.

Lemma 3.7. *Let u_a, u_b be as defined above. If $a < b$, then $u_a \leq u_b$ on M.*

Proof. Deinfe $w := u_b - u_a$ and

$$\lambda_{ab} := \frac{e^{2u_b} - e^{2u_a}}{u_b - u_a} = \frac{1}{u_b - u_a} \int_{u_a}^{u_b} 2e^{2\sigma} d\sigma > 0. \tag{3.11}$$

We have

$$\Delta w = \Delta u_b - \Delta u_a = e^{2u_b} - e^{2u_a} = \lambda_{ab}(u_b - u_a) = \lambda_{ab} w. \tag{3.12}$$

Suppose that w attains its minimum at the point $p \in M$. If $w(p) < 0$, then (3.11) and (3.12) together imply that $\Delta w(p) < 0$, which contradicts the fact that p is a local minimum of w. Hence $w(p) \geq 0$ on M or w attains its minimum on the boundary. In both cases, we have $w \geq 0$ on M. $\square$

We will also make use of a bound for u_a uniform in a.

Lemma 3.8. *Let u_a be defined as above. There exists a locally bounded function $B : M \to \mathbb{R}$ independant of a such that*

$$e^{2u_a(x)} \leq B(x), \qquad (x \in M).$$

Proof. Let $x \in M$. Suppose first that $M \subset \mathbb{R}^2$. Set $\delta(x) := \mathrm{dist}(x, \partial M)$ et let $0 < \beta < \delta(x)$. It is well known that the disk $D_\beta(x)$ can be endowed with the following Poincaré metric:

$$g_{jk} = \frac{4\beta^2}{(\beta^2 - |y-x|^2)^2} \delta_{jk}, \qquad (y \in D_\beta(x)).$$

With

$$w := \frac{1}{2}\log\left(\frac{4\beta^2}{(\beta^2 - |y-x|^2)^2}\right), \tag{3.13}$$

we have $g_{jk} = e^{2w}\delta_{jk}$. Therefore, w is a solution of (2.2). Moreover, it is clear from (3.13) that $w(x) \to \infty$ as $x \to \partial D_\beta(x)$. It follows from Lemma 3.7 (applied to disks of radius large enough contained in $D_\beta(x)$) that

$$u_a \le w \quad \text{on} \quad D_\beta(x).$$

So,

$$e^{2u_a(x)} \le e^{2w(x)} = \frac{4}{\beta^2}.$$

By taking $\beta \to \delta(x)$ in the last equation, we get

$$e^{2u_a(x)} \le \frac{4}{\delta^2(x)} =: B(x),$$

which gives us the required locally bounded function.

Now, we consider the general case. Using isothermal coordinates, we can find a neighbourhood $O_x \subset M$ of x conformal to the unit disk D_1 through some conformal map ψ_x. Moreover, O_x can be chosen so that ∂O_x is smooth and ψ_x can be extended to a diffeomorphism on $\overline{O_x}$. Let $e^{2w_x}g_0$ be the Poincaré metric on O_x obtained by pulling back the Poincaré metric on D_1. It is clear that $w_x(y) \to \infty$ as $y \to \partial O_x$. So, we can apply Lemma 3.7 in the same way as above to obtain the locally bounded function B. □

We can now consider the limit as $a \to \infty$ of u_a. By Lemma 3.8, $u_a(x) \to u(x)$ and $e^{2u(x)} \le B(x)$ for any $x \in M$. It follows from elliptic regularity that the derivatives of u_a are locally uniformly bounded. Therefore, $u_a \to u$ in $C^\infty_{\text{loc}}(M)$, from which we conclude that u solves (2.2).

In order for $e^{2u}g_0$ to be a Poincaré metric, it must be geodesically complete. So, our last step towards the proof of Theorem 3.1 is the following result.

Lemma 3.9. *If u is defined as above, then $g = e^{2u}g_0$ is complete on M.*

Proof. We proceed by contradiction. Suppose that g is not complete on M. Then, there exists a unit-speed geodesic $\gamma : [0, L) \to M$ with respect to g (where $L < \infty$), for which $\gamma(t)$ does not converge in M. Moreover, since $\gamma([0, L))$ has also finite length with repect to g_0, $\gamma(t)$ converges to a point $p \in \partial M$ as $t \to L$.

As in the proof of Lemma 3.8, we treat first the case where $M \subset \mathbb{R}^2$. We consider $\mathbb{R}^2$ as a subset of the Riemann sphere $\hat{\mathbb{C}}$. Let D_p be a disk in $\mathbb{R}^2\backslash M$ tangent to ∂M at p. In the Riemann sphere, the complement of D_p, i.e. $D_p^c = \hat{\mathbb{C}}\backslash D_p$, admits a Poincaré metric $h = e^{2w}g_0$ for some $w \in C^\infty(D_p^c)$. Since $M \subset D_p^c$ and $u(x) \to \infty$ as $x \to y \in \partial M$, we can apply Lemma 3.7 as in the proof of Lemma 3.8 to get

$$u \ge w \quad \text{on} \quad M, \tag{3.14}$$

from which it follows that $g \ge h$. However, since h is a Poincaré metric, it is in particular complete. This implies that the length of γ with respect to h is infinite. This together with (3.14) contradicts the fact that γ has finite length with respect to g.

We consider now the general case. We assume without loss of generality that $\overline{M}$ is contained inside a larger open Riemann Surface to which we extend smoothly the metric g_0. Let D be a small holomorphic disk containing p such that $D \cap \partial M$ cuts D into two connected components. Let $\omega : (0, T) \to D \backslash \overline{M}$ be a smooth curve such that $\omega(t) \to p$ transversally as $t \to T$. From the image of this curve, take a sequence of points $(p_j)_{j=1}^{\infty}$ converging to p. Define $D_j := D \backslash \{p_j\}$. It is elementary to get a conformal mapping between D_j and D^*, that is, the punctured unit disk missing the origin. It is easily verified that a Poincaré metric on D^* exists and is given by

$$g_{jk} = \left(r \log \frac{1}{r} \right)^{-2} \delta_{jk}. \tag{3.15}$$

So, we can obtain a Poincaré metric $e^{2v_j} g_0$ on D_j by pulling back (3.15). Moreover, we let $e^{2v} g_0$ denote the Poincaré metric obtained by pulling back (3.15) to $D \backslash \{p\}$.

Let O be a disk properly contained in D such that $p_j \in O$ for all j. Our goal is to show that

$$u \geq v - B \quad \text{on} \quad O \cap M, \tag{3.16}$$

for some constant B. Indeed, if (3.15) holds, then we can use the completeness of v to derive a contradiction following the same reasonning as for the case where M was a planar domain. Since v_j is bounded in O, for any j, there exists N_j such that

$$a \geq N_j \quad \Longrightarrow \quad u_a \geq u_j \quad \text{on} \quad \partial M \cap O. \tag{3.17}$$

Moreover, it is easy to see that the v_j's are uniformly bounded in O. Therefore, there exists $B > 0$ such that for all j,

$$u_1 \geq v_j - B \quad \text{on} \quad \partial O \cap M$$

It follows from Lemma 3.7 that for all j, and for all $a \geq 1$,

$$u_a \geq v_j - B \quad \text{on} \quad \partial O \cap M. \tag{3.18}$$

We can combine (3.17) and (3.18) to obtain

$$u_a \geq v_j - B \quad \text{on} \quad \partial \left(O \cap M \right),$$

for $a \geq \max\{1, N_j\}$ and for all j. By Proposition 3.4, it follows that

$$u_a \geq v_j - B \quad \text{on} \quad O \cap M,$$

for $a \geq \max\{1, N_j\}$ and for all j. Lemma 3.7 gives $u \geq v_j - B$ on $O \cap M$, for all j. Inequality (3.16) is then obtained by taking the limit $j \to \infty$. □

Theorem 3.1 now follows by endowing M with the Poincaré metric u constructed above.

4 Planar domains

For the the whole section, we suppose that $M \subset \mathbb{R}^2$ is a domain such that its complement, $\mathbb{R}^2 \backslash M$, contains at least two points. Moreover, we consider M as a smooth Riemannian manifold endowed with the Euclidean metric δ_{jk}. We want to prove the following.

Theorem 4.1. *If M is a planar domain such that its complement in $\mathbb{R}^2$ contains at least two points, then M admits a Poincaré metric.*

In order to establish this theorem, we consider an increasing sequence of smoothly bounded subsets Ω_ν of M converging to M. More precisely, $\Omega_k \subset\subset \Omega_{k+1}$ and $\Omega_k \to M$ as $k \to \infty$, in the sense that for any compact set $K \subset M$, there exists k such that $K \subset \Omega_k$. By the results from §3, for any k, Ω_k admits a conformal Poincaré metric $e^{2u_k}\delta_{ij}$, with $u_k|_{\partial\Omega_k} = \infty$. Our strategy will be to study the limit of u_k as $k \to \infty$. We start with the following lemma.

Lemma 4.2. *The planar domain $M = \mathbb{C}\setminus\{0,1\}$ has a Poincaré metric.*

Proof. We will first construct a metric of negative curvature $K \leq -1$ for M. We are looking for a metric of the form $e^{2w}\delta_{ij}$, where

$$e^w = A\frac{(1+|z|^a)^b}{|z|^c}\frac{(1+|z-1|^a)^b}{|z-1|^c},$$

with positive parameters A, a, b, c, which will be fixed later. By (2.1) with $K_0 \equiv 0$, we find that the Gauss curvatures for this metric is

$$K = -\frac{a^2 b}{A^2}\left[\frac{|z|^{a-2+2c}|z-1|^{2c}}{(1+|z|^a)^{2+2b}(1+|z-1|^a)^{2b}} + \frac{|z|^{2c}|z-1|^{a-2+2c}}{(1+|z|^a)^{2b}(1+|z-1|^a)^{2+2b}}\right].$$

This Gauss curvature is negative. Moreover, if we set $a = 1/3$, $b = 1/2$ and $c = 5/6$, it is straightforward to check that K is bounded away from zero. So, we can take $A > 0$ small enough so that the curvature satisfies the additional requirement that $K \leq -1$.

Let Ω_k and u_k be as above. By applying arguments similar to the ones used in the proof of Lemma 3.7, one can see that $u_{k+1} \leq u_k$ and $u_k \geq w$ on Ω_k for all k. Therefore, u_k converges to some solution $u \in C^\infty$ to (2.2), with $u \geq w$ on M. The same argument by contradiction as the one used in Lemma 3.9 shows us that it is enough to prove that the metric induced by w constructed above is complete. However, it is not. So, we will proceed differently.

Let $e^{2v}\delta_{ij}$ be the Poincaré metric on the punctured unit disk D^* (See (3.15) in the proof of Lemma 3.9). It is clear that v is bounded on $\partial D_{\frac{1}{2}} = \{z \in \mathbb{C} \,:\, |z| = 1/2\}$. So, since $u_k \to u$, there exists a nonnegative constant B such that

$$u_k \geq v - B \quad \text{on} \quad \partial D_{\frac{1}{2}}, \tag{4.1}$$

for all k. With the help of (2.1), one can check that the metric $e^{2(v-B)}\delta_{ij}$ has Gauss curvature less than or equal to one. We can then apply the argument of Lemma 3.7 in conjunction with (4.1) to obtain

$$u \geq v - B \quad \text{on} \quad \left\{z \in \mathbb{C} \,:\, 0 \leq |z| \leq \frac{1}{2}\right\}. \tag{4.2}$$

To show the completeness of $e^{2u}\delta_{ij}$ in a neighbourhood of 0, we can then proceed by contradiction, exactly as in the proof of Lemma 3.9, with (4.2) playing the same role as (3.16).

As for completeness near 1, we can apply the same argument, this time with a punctured unit disk centered around the point $z = 1$. It remains to consider completeness near ∞. It can be shown that formula (3.15), defining a Poincaré metric for D^*, defines also a Poincaré metric on $\{z \,:\, |z| > 1\}$. An analog argument to the one used above to prove completeness around 0 or 1 will then show completeness near ∞. ∎

We can now prove the main result of this section.

Proof of Theorem 4.1. From the proof of Lemma 4.2, it is clear that 0 and 1 can be replaced by any points p_1 and p_2 in the statement. So, let $p_1, p_2 \in \mathbb{R}^2 \backslash M$, and let $e^{2w}\delta_{ij}$ be the Poincaré metric on $\mathbb{R}^2 \backslash \{p_1, p_2\}$ obtained by Lemma 4.2. Applying Lemma 3.7 to the functions u_k and u_{k+1} inside of the domain Ω_k, we can conclude that $u_{k+1} \leq u_k$ for all k. Moreover, the argument of Lemma 3.7 applied to u_k and w gives

$$u_k \geq w, \tag{4.3}$$

for all k. We can then consider the limit u of u_k as $k \to \infty$. As in §2, we deduce from elliptic regularity that $u \in C^\infty$ and that u solves (2.2).

It remains to show that the metric g corresponding to u is complete. As in the proof of Lemma 3.9, we suppose that it is not, and that there exists then a unit-speed geodesic $\gamma : [0, L) \to M$ with respec to g such that $\gamma(t) \to p$ as $t \to L$, where either $p \in \partial M$ or $p = \infty$. If $p \neq \infty$, then let w be such that $e^{2w}\delta_{ij}$ is the Poincaré metric on $\mathbb{R}^2 \backslash \{p, p_2\}$, where p_2 is any other point not in M. By (4.3) we have that $u \geq w$. As argued before, the completeness of $e^{2w}\delta_{ij}$ leads to a contradiction. On the other hand, if $p = \infty$, then γ has necessarily infinite length with respect to $e^{2w}\delta_{ij}$ (for any choice of $p_1, p_2 \notin M$), which leads to the same contradiction. Thus g is complete. □

5 Noncompact surfaces

In this section, we establish the uniformization theorem in the noncompact case. In addition to the results proved in the two preceeding sections, we make use of two classical theorems from complex analysis, namely the Schwarz Lemma and a theorem about normal families due to Koebe. For the sake of completeness, the statements of those results are included below.

Lemma 5.1 (Schwarz lemma)**.** *Let f be a holomorphic map from the complex open unit disk to itself such that $f(0) = 0$. Then*

$$|f(z)| \leq |z| \quad \text{for all} \quad z, \qquad \text{and} \quad |f'(0)| \leq 1. \tag{5.1}$$

Moreover, we have strict inequalities in (5.1) unless $f(z) = e^{i\gamma}z$, $\gamma \in \mathbb{R}$.

This result is elementary and a proof of it can be found in almost any introductory complex analysis book.

Until the end, S will denote the familly of one-to-one holomorphic maps $f : D_1 \to \mathbb{C}$ satisfying $f(0) = 0$ and $f'(0) = 1$.

Theorem 5.2 (Koebe)**.** *The set S is compact in the space of holomorphic functions $f : D_1 \to \mathbb{C}$.*

For a curvature proof of this theorem, the reader can consult §5 of [2]. We now have all the ingredients required to prove the main result of this paper.

Theorem 5.3. *If M is a noncompact, simply connected Riemann surface, then M is biholomorphically equivalent to either D_1 or $\mathbb{C}$.*

Proof. As in the previous section, we consider a sequence of relatively compact sets increasing to M. More precisely, let $\{\Omega_k\}$ be a sequence of subsets of M such that $\Omega_k \subset\subset \Omega_{k+1}$, $\overline{\Omega_k}$ is compact with smooth boundary, and $\Omega_k \to M$ (in the same sense as in §4). Let $p \in \Omega_0$. By Theorem 3.1, there exists a biholomorphism

$$\psi_k : \Omega_k \to D_1,$$

for each k. Moreover, since for any point $x \in D_1$, there exists a biholomorphism $f_x : D_1 \to D_1$ sending any point x to 0, we may assume without loss of generality that $\psi_k(p) = 0$. Since $D\psi_k(p)$ is a linear map from $T_pM \simeq \mathbb{C}$ to $\mathbb{C}$, there exists a unique $0 \neq a_k \in \mathbb{C}$ such that

$$D\psi_k(p) = a_k D\psi_0(p).$$

By Schwarz lemma, we have

$$|a_k| > |a_{k+1}|, \quad \text{for all} \quad k. \tag{5.2}$$

Indeed, if we set $f := \psi_{k+1} \circ \psi_k^{-1}$, then f is a well-defined mapping from D_1 to itself such that $f(0) = 0$. Therefore, Lemma 5.1 applies and gives

$$\frac{|a_{k+1}|}{|a_k|} = |D\psi_{k+1}(p)| \frac{1}{|D\psi_k(p)|} = |D\psi_{k+1}(p)| \left|D\left(\psi_k^{-1}(0)\right)\right| = |f'(0)| \leq 1. \tag{5.3}$$

Moreover, since f is not bijective (due to the fact that Ω_k is properly contained in Ω_{k+1}), the second part of Lemma 5.1 implies that the inequality in (5.3) is strict.

Let $R_k = |a_k|^{-1}$ and define $\varphi_k : \Omega_k \to D_{R_k}$ by $\varphi_k(x) := a_k^{-1}\psi_k(x)$. By (5.2), we have $R_0 < R_1 < \cdots$. Our next goal is to find a subsequence of $\{\varphi_k\}$ that converges to a one-to-one map $M \to \mathbb{C}$. In order to find such a subsequence, we will use a diagonalization argument in combination with the theorem due to Koebe stated above. First, we define $\Phi_{kl} := \varphi_k \circ \psi_l^{-1} : D_1 \to D_{R_k}$. It is clear that Φ_{kl} is one-to-one and satisfies $\Phi_{kl}(0) = 0$. Moreover, we have

$$\Phi'_{kl}(0) = a_k^{-1} D\psi_k(p)(D\psi_0(p))^{-1} = a_k^{-1}(a_k D\psi_0(p))(D\psi_0(p))^{-1} = 1.$$

Then for any fixed l, the sequence $(\Phi_{kl})_{k=0}^{\infty}$ is in S, and Theroem 5.2 applies. So, for $l = 0$, there exists a subsequence $(\Phi_{k_i 0})_{i=0}^{\infty}$ converging to a one-to-one holomorphic function $\Phi_0 : D_1 \to \mathbb{C}$. It follows that the subsequence $(\varphi_{k_i})_{i=1}^{\infty}$ converges to the one-to-one holomorphic map $\varphi^{(0)} := \Phi_0 \circ \psi_0 : \Omega_0 \to \mathbb{C}$. Applying Theroem 5.2 again, this time to the sequence $(\Phi_{k_{1i}1})_{i=0}^{\infty}$, we get, by the same reasoning as above, a subsequence $(\varphi_{k_{2i}})_{i=1}^{\infty}$ of $(\varphi_{k_{1i}})_{i=1}^{\infty}$ converging to a one-to-one holomorphic map $\varphi^{(1)} : \Omega_1 \to \mathbb{C}$. Repeating this process again, we find subsequences

$$(\varphi_{k_{0i}})_{i=1}^{\infty} \supset (\varphi_{k_{1i}})_{i=1}^{\infty} \supset (\varphi_{k_{2i}})_{i=1}^{\infty} \supset (\varphi_{k_{3i}})_{i=1}^{\infty} \supset \cdots,$$

such that for any m, $\varphi_{k_{mi}}$ converges to a one-to-one holomorphic map $\varphi^{(m)} : \Omega_m \to \mathbb{C}$ as $i \to \infty$. Therefore, the subsequence $(\varphi_{k_{ii}})_{i=1}^{\infty}$ converges to a one-to-one holomorphic map $\varphi : M \to \mathbb{C}$.

If $\varphi(M) = \mathbb{C}$, then φ is the required biholomorphism between M and $\mathbb{C}$. Else, since M is simply connected, $\varphi(M) \neq \mathbb{C}$ is also simply connected, which implies that $\mathbb{C}\backslash\varphi(M)$ contains at least two points. Theorem 4.1 then applies to yield a biholomorphic map between $\varphi(M)$ and D_1. Composing this mapping with φ, we obtain the required biholomorphism between M and D_1. $\square$

6 Compact surfaces

In this final section, we state the uniformization theorem in the case where M is a compact Riemann surface. Since a complete proof of this result would require considerable further analysis, which do not fall within the scope of this paper, we only give a sketch of the proof.

Theorem 6.1. *Let M be a compact Riemann surface, with Euler characteristic $\chi(M)$.*

1) If $\chi(M) = 2$, then there is a biholomorphism between M and $\hat{\mathbb{C}}$.

2) If $\chi(M) = 0$, then there is a biholomorphism between M and a flat torus.

3) If $\chi(M) < 0$, then M is holomorphically covered by D_1.

Sketch of proof. Cases 2 and 3 can be established by solving (2.1). In Case 2, the curvature equations reads

$$\Delta u = K_0(x). \tag{6.1}$$

As a consequence of the Green's formula, a necessary condition for (6.1) to hold is that

$$\int_M K_0(x) dV(x) = 0. \tag{6.2}$$

Moreover, this condition can be shown to be sufficient for the existence of a solution to (6.1). By the Gauss-Bonnet formula, (6.2) is satisfied if and only if $\chi(M) = 0$. Therefore, the metric on M is conformally equivalent to a flat metric. It follows that the universal cover of M is isometric to $\mathbb{C}$. Hence M is biholomorphically equivalent to $\mathbb{C}/\Lambda$, where Λ is a discrete group of holomorphic automorphisms acting on $\mathbb{C}$ without fixed points. Such automorphisms can only be translations. Thus $\mathbb{C}/\Lambda$ is a torus.

In Case 3, the idea is to look for a function u that minimizes the functional

$$F(u) = \int_M \left(\frac{1}{2} |\nabla u|^2 + K_0(x) u \right) dV(x)$$

over the set

$$S = \left\{ u \in H^1(M) \ : \ \int_M K(x) e^{2u} dV(x) = 2\pi \chi(M) \right\}.$$

This minimum solves (2.1), and the metric on M is then conformal to a metric of negative curvature K. For a complete proof, the reader is referred to §2 of Chap. 14 in [3].

A proof of Case 1 requires tools that were not discussed in this note. For a complete treatment of this case, which makes use of the Riemann-Roch theorem, the reader can consult §9 of Chap. 10 in [3]. $\square$

References

[1] O. Forster, *Lectures on Riemann Surfaces*, Springer-Verlag, New York, 1981.

[2] R. Mazzeo and M. Taylor, *Curvature and Uniformization*, Israel Journal of Mathematics, 130, 323-346, 2002.

[3] M. E. Taylor, *Partial Differential Equations III: Nonlinear Equations*, 2nd Edition, Springer, 715 pages, 2011.