An Introduction to the Morrey and Campanato Spaces

Abstract

Regularity theorems which allow one to conclude higher regularity of a function from a lower regularity and a differential equation play a central role in the theory of PDEs. One version of the Sobolev embedding theorem which states that $W^{p,k}(\mathbb{R}^n) \hookrightarrow C^{r,\beta}(\mathbb{R}^n)$ for $k-r-a = n/p$ is such an example. The Campanato spaces, an enhanced version of the Morrey spaces, extend the notion of functions of bounded mean oscillation and allow a full characterization $C^{0,\beta}(\mathbb{R}^n)$. The theory of Campanato spaces may come in useful when the Sobolev embedding theorem is not. The main results of this project are summarized in Theorems 0.0.12, 0.0.14 and 0.0.22.

Notation 0.0.1.

1. Given a set $\Omega \subset \mathbb{R}^n$ and an open ball $B_\rho(x)$, we denote $\Omega(x,\rho) := \Omega \cap B_\rho(x)$.

2. We will write the Lebesgue measure of a set E by $|E|$.

3. V_n denotes the volume of the unit ball $B_1^n(0)$ in $\mathbb{R}^n$: $V_n := |B_1^n(0)|$.

4. Given a function $f : \Omega \longrightarrow \mathbb{R}$, we denote the average of f over a set Ω by $(f)_\Omega := \frac{1}{|\Omega|}\int_\Omega f$

5. d denotes the diameter of a bounded domain $\Omega \subset \mathbb{R}^n$: $d := \text{diam}\Omega$.

Definition 0.0.2. *(Hölder continuous functions)* Let Ω be a domain in $\mathbb{R}^n$ and $k \geq 0, \beta > 0$. We denote by $C^{k,\beta}(\overline{\Omega})$ the subset of $C^k(\overline{\Omega})$ of functions f satisfying for every multiindex $|i| \leq k$.

$$H_{i,\beta}(f) := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^i f(x) - D^i f(y)|}{|x-y|^\beta} < \infty$$

The functions in the space $C^{0,\beta}(\overline{\Omega})$ are called Hölder continuous.

Definition 0.0.3. *(Morrey and Campanato spaces)* Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (open and connected), $1 \leq p \leq \infty$ and $\lambda \geq 0$.

1. The Morrey spaces, denoted $L_M^{p,\lambda}(\Omega)$, are the collection of all functions $f \in L^p(\Omega)$ such that

$$^M\|f\|_{p,\lambda} := \left(\sup_{\substack{x\in\Omega \\ 0<\rho<d}} \rho^{-\lambda} \int_{\Omega(x,\rho)} |f|^p \right)^{1/p} < \infty$$

2. The Campanato spaces, denoted $L_C^{p,\lambda}(\Omega)$, are the collection of all functions $f \in L^p(\Omega)$ such that

$$^C[f]_{p,\lambda} := \left(\sup_{\substack{x\in\Omega \\ 0<\rho<d}} \rho^{-\lambda} \int_{\Omega(x,\rho)} |f - (f)_{\Omega(x,\rho)}|^p \right)^{1/p} < \infty$$

Proposition 0.0.4.

1. *$^M\| \cdot \|_{p,\lambda}$ defines a norm on the Morrey spaces, making them into a normed vector space.*

2. *$^C[\cdot]_{p,\lambda}$ defines a seminorm on the Campanato spaces. These can be made into normed vector spaces by setting for every $f \in L_C^{p,\lambda}(\Omega)$, $^C\|f\|_{p,\lambda} = \|f\|_p +^C [f]_{p,\lambda}$*

Remark 0.0.5. $^C[f]_{p,\lambda} = 0$ if and only if $f\restriction_{\Omega(x,\rho)} = (f)_{\Omega(x,\rho)}$ a.e. for every $(x, \rho) \in \Omega \times (0, d)$, which implies that f is a.e. constant on Ω.

Theorem 0.0.6. *The Morrey and Campanato spaces are Banach.*

Proof We prove completeness in the Morrey space, the other being very similar. Let $\{u_n\}$ be a Cauchy sequence in $L_M^{p,\lambda}(\Omega)$. Recall $d := \text{diam}\Omega$. Choose $x \in \Omega$, then $\sup_{y\in\Omega} \text{dist}(x, y) < d$ because if we had $\sup_{y\in\Omega} \text{dist}(x, y) = d$, then because x is an interior point we could find a small ball $B_\delta(x) \subset \Omega$ and $t \in B_\delta(x)$ such that $\sup_{y\in\Omega} \text{dist}(x, y) > \text{diam}\Omega$, a contradiction. This means that there is $\rho^* < d$ such that $B_{\rho^*}(x) \supset \Omega$. This gives us the relation

$$\|u\|_p \leq \left(\left(\frac{d}{\rho^*} \right)^\lambda \int_{\Omega(x,\rho^*)} |u|^p \right)^{1/p} \leq d^{\frac{\lambda}{p}} \ ^M\|u\|_{p,\lambda}.$$

So in particular the sequence $\{u_n\}$ is Cauchy sequence in $L^p(\Omega)$, which is complete, so there exists $u \in L^p(\Omega)$ such that $\lim_{n\to\infty} \|u_j - u\|_p = 0$.

We show that $u \in L_M^{p,\lambda}(\Omega)$. By Minkowski inequality, for all $(x,\rho) \in \Omega \times (0,d)$,

$$\rho^{\frac{-\lambda}{p}} \left(\int_{\Omega(x,\rho)} |u|^p \right)^{1/p} \leq \rho^{\frac{-\lambda}{p}} \left(\int_{\Omega(x,\rho)} |u-u_n|^p \right)^{1/p} + \rho^{\frac{-\lambda}{p}} \left(\int_{\Omega(x,\rho)} |u_n|^p \right)^{1/p}.$$

Since the sequence is Cauchy in $L_M^{p,\lambda}(\Omega)$, the 2nd term on the RHS is uniformly bounded by, say, K. Then taking the limit when $n \to \infty$ the 1st term on the RHS $\to 0$ and we get

$$\rho^{\frac{-\lambda}{p}} \left(\int_{\Omega(x,\rho)} |u|^p \right)^{1/p} \leq K.$$

Taking the supremum over all $(x,\rho) \in \Omega \times (0,d)$ gives ${}^M\|u\|_{p,\lambda} \leq K$. Remains to show that $\lim_{n\to\infty} {}^M\|u_n - u\|_{p,\lambda} = 0$.

Let $\epsilon > 0$ be given. We have

$$\begin{aligned}\rho^{\frac{-\lambda}{p}} \left(\int_{\Omega(x,\rho)} |u-u_n|^p \right)^{1/p} &\leq \rho^{\frac{-\lambda}{p}} \left(\int_{\Omega(x,\rho)} |u-u_m|^p \right)^{1/p} + \rho^{\frac{-\lambda}{p}} \left(\int_{\Omega(x,\rho)} |u_m-u_n|^p \right)^{1/p} \\ &\leq \rho^{\frac{-\lambda}{p}} \left(\int_{\Omega(x,\rho)} |u-u_m|^p \right)^{1/p} +{}^M \|u_m - u_n\|_{p,\lambda}\end{aligned}$$

Choose n_o so that $n, m \geq n_o \Rightarrow \ {}^M\|u_m - u_n\|_{p,\lambda} < \epsilon$. Then taking the limit when $m \to \infty$ the 1st term on the RHS $\to 0$. Finally taking the supremum over all $(x,\rho) \in \Omega \times (0,d)$ gives ${}^M\|u - u_n\|_{p,\lambda} \leq \epsilon$. □

Definition 0.0.7. A family $\{E_r\}_{r>0}$ of Borel subsets of $\mathbb{R}^n$ is said to *shrink nicely* to $x \in \mathbb{R}^n$ if :

- $E_r \subset B_r(x)$ for all $r > 0$.
- There is a constant $\alpha > 0$, independent of r, such that $|E_r| > \alpha\, |B_r(x)| = \alpha V_n r^n$.

Remark 0.0.8. Given a bounded domain $\Omega \subset \mathbb{R}^n$, consider the family of sets $\{\Omega(x,\rho) : x \in \Omega, 0 < \rho < d\}$ $= \bigcup_{x\in\Omega} \{\Omega(x,\rho) : 0 < \rho < d\}$. Suppose that each family $\{\Omega(x,\rho) : 0 < \rho < d\}$ shrinks nicely, then we have strictly positive constants $\{\alpha_x\}_{x\in\Omega}$ such that $|\Omega(x,\rho)| > \alpha_x\, |B_\rho(x)|$ for all $x \in \Omega$, $\rho \in (0,d)$. If the α_x can be chosen so that $\alpha := \inf_{x\in\Omega} \alpha_x > 0$ then we shall say that the domain Ω is *type-α*, that is to say, there exists $\alpha > 0$ such that $|\Omega(x,\rho)|$ $> \alpha V_n \rho^n$ for all $(x,\rho) \in \Omega \times (0,d)$.

Definition 0.0.9. *For $f \in L_M^{p,\lambda}(\Omega)$, let*

$$^M\|f\|'_{p,n} = \left(\sup_{\substack{x\in\Omega \\ 0<\rho<d}} |\Omega(x,\rho)|^{\frac{-\lambda}{n}} \int_{\Omega(x,\rho)} |f|^p \right)^{1/p}$$

Lemma 0.0.10. *$^M\|\cdot\|'_{p,n}$ also defines a norm on the Morrey spaces. $^M\|\cdot\|'_{p,n}$ is finer than $^M\|\cdot\|_{p,n}$ and if Ω is type-α, then the two norms are equivalent.*

Proof Showing $^M\|\cdot\|'_{p,n}$ is a norm is straightforward. That $^M\|\cdot\|'_{p,n}$ is finer than $^M\|\cdot\|_{p,n}$ comes from the fact that $\frac{\rho^{-\lambda}}{V_n^{\frac{\lambda}{n}}} = \frac{1}{(V_n\rho^n)^{\frac{\lambda}{n}}} = \frac{1}{|B_\rho(x)|^{\frac{\lambda}{n}}} \leq \frac{1}{|\Omega(x,\rho)|^{\frac{\lambda}{n}}}$. If Ω is type-α, then there exists $\alpha > 0$ such that for all $(x,\rho) \in \Omega \times (0,d)$, $\alpha V_n \rho^n \leq |\Omega(x,\rho)|$, i.e. $(\alpha V_n)^{\frac{\lambda}{n}} |\Omega(x,\rho)|^{-\frac{\lambda}{n}} \leq \rho^{-\lambda}$. □

We recall a fundamental result that will be useful in the proof the subsequent theorem.

Theorem 0.0.11. *(Lebesgue Differentiation Theorem) Supppose $f \in L^1_{loc}(\mathbb{R}^n)$. Then for Lebesgue-almost all $x \in \mathbb{R}^n$ $\lim_{r\to 0} \frac{1}{|E_r|} \int_{E_r} |f(y) - f(x)|dy = 0$ for every family $\{E_r\}_{r>0}$ that shrinks nicely to x.*

Theorem 0.0.12.

1. *For $1 \leq p < \infty$, $L_M^{p,0}(\Omega) = L^p(\Omega)$, i.e. $L_M^{p,0}(\Omega)$ and $L^p(\Omega)$ are continuously imbedded in each other.*
2. *For $1 \leq p < \infty$, $L_M^{p,n}(\Omega) \hookleftarrow L^\infty(\Omega)$. If further Ω is type-α (see Remark 0.0.8), then $L_M^{p,n}(\Omega) \hookrightarrow L^\infty(\Omega)$.*
3. *For $1 \leq p < \infty$, $\lambda > n$, $L_M^{p,\lambda}(\Omega) = \{0\}$.*
4. *For $1 \leq p \leq q < \infty$ and $\lambda, \mu \geq 0$ so that $\frac{\lambda - n}{p} \leq \frac{\mu - n}{q}$ then $L_M^{q,\mu}(\Omega) \hookrightarrow L_M^{p,\lambda}(\Omega)$.*

Proof

1. Straight from the definition of the Morrey norm we have $^M\|f\|_{p,0} = \|f\|_p$. So the identity map from $L_M^{p,0}(\Omega)$ into $L^p(\Omega)$ and vice-versa is continuous.
2. Let $f \in L^\infty(\Omega)$. Then for all $(x,\rho) \in \Omega \times (0,d)$ we have

$$\frac{1}{\rho^n} \int_{\Omega(x,\rho)} |f|^p \leq \frac{|\Omega(x,\rho)|}{\rho^n} \|f\|^p_\infty \leq \frac{|B_\rho(x)|}{\rho^n} \|f\|^p_\infty \leq V_n \|f\|^p_\infty.$$

where V_n is the volume of the unit ball in $\mathbb{R}^n$. From this we conclude that ${}^M\|f\|_{p,n} \le V_n^{1/p}\|f\|_\infty$ and the identity map from $L^\infty(\Omega)$ into $L_M^{p,n}(\Omega)$ is continuous. This map is also surjective when Ω is type-α. To see this, suppose that there exists $f \in L_M^{p,n}(\Omega) \setminus L^\infty(\Omega)$. Then $f \in L^p(\Omega)$. In particular $|f|^p \in L^1(\Omega)$ and so by the Lebesgue differentiation theorem, $\lim_{r\to 0} \frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} |f|^p = |f(x)|^p$ for almost all $x \in \Omega$. We also have $\|f\|_\infty = \infty$. So given $C > 0$ arbitrarily large, the set $\{x \in \Omega : |f(x)| > C^{1/p}\}$ has strictly positive measure and almost all points in that set are Lebesgue points. Thus we can find $(x,\rho) \in \Omega \times (0,d)$ such that $\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} |f|^p > C$. We conclude ${}^M\|f\|'_{p,n} = \infty$ and using the assumption that Ω is type-α, this is equivalent to ${}^M\|f\|_{p,n} = \infty$, but this is a contradiction. Finally, since the identity map from $L^p(\Omega)$ to $L_M^{p,0}(\Omega)$ is a continuous linear bijection, we conclude by the bounded inverse theorem that the identity map from $L_M^{p,n}(\Omega)$ to $L^\infty(\Omega)$ is continous.

3. For $\lambda > n$, say $\lambda = n + \epsilon$, then by the Lebesgue differentiation theorem $\lim_{\rho\to 0} \rho^{-n}\int_{\Omega(x,r)} |f|^p \le V_n|f(x)|^p$ for almost all $x \in \Omega$. From this we see that unless $|f(x)| = 0$ (almost everywhere) $\lim_{\rho\to 0} \rho^{-\epsilon}\rho^{-n}\int_{\Omega(x,r)} |f|^p = \infty$.

4. By Hölder's inequality with conjugate exponents $\frac{1}{p/q}$ and $\frac{1}{1-p/q}$, we have $\int_{\Omega(x,r)} |f|^p \le (\int_{\Omega(x,r)} 1)^{1-p/q}(\int_{\Omega(x,r)} |f|^q)^{p/q}$ $\le (V_n\rho^n)^{1-p/q}(\int_{\Omega(x,r)} |f|^q)^{p/q} \le V_n^{1-p/q}\rho^{n(1-p/q)+\mu p/q}(\rho^{-\mu}\int_{\Omega(x,r)} |f|^q)^{p/q}$. Now $\frac{\lambda-n}{p} \le \frac{\mu-n}{q} \Leftrightarrow \lambda \le n(1-p/q) + \mu p/q \Rightarrow \left(\frac{\rho}{d}\right)^{n(1-p/q)+\mu p/q} \le \left(\frac{\rho}{d}\right)^\lambda \Rightarrow \rho^{n(1-p/q)+\mu p/q} \le \rho^\lambda d^{n(1-p/q)+\mu p/q-\lambda}$. Putting everything together we have $(\rho^{-\lambda}\int_{\Omega(x,r)} |f|^p)^{1/p} \le C(\rho^{-\mu}\int_{\Omega(x,r)} |f|^q)^{1/q}$ and thus ${}^M\|f\|_{p,\lambda} \le C\,{}^M\|f\|_{q,\mu}$, where $C^p = V_n^{1-p/q}d^{n(1-p/q)+\mu p/q-\lambda}$ is a constant. It follows that the identity map from $L_M^{q,\mu}(\Omega)$ into $L_M^{p,\lambda}(\Omega)$ is continuous.

$\square$

Remark 0.0.13. Theorem 1.1.11 suggests that for fixed $p \in [1,\infty)$ the Morrey spaces $\{L_M^{p,\lambda}(\Omega)\}_{\lambda\in[0,n]}$ provide a certain scaling of the spaces between $L^p(\Omega)$ and $L^\infty(\Omega)$. Also, taking $p = q$ in point 4, we have $L_M^{p,\mu}(\Omega) \hookrightarrow L_M^{p,\lambda}(\Omega)$ whenever $\mu \ge \lambda$, just like for finite L^p spaces.

Some, but not all, properties of the Morrey spaces also hold for the Campanato spaces:

Theorem 0.0.14.

1. *For* $1 \le p < \infty$, $L_C^{p,0}(\Omega) = L^p(\Omega)$.

2. *For* $1 \le p \le q < \infty$ *and* $\lambda, \mu \ge 0$ *so that* $\frac{\lambda-n}{p} \le \frac{\mu-n}{q}$ *then* $L_C^{q,\mu}(\Omega) \hookrightarrow L_C^{p,\lambda}(\Omega)$.

In order to state the next major results concerning the Morrey and Campanato spaces we must first develop useful tools. In what follows we will be assuming throughout that Ω is type-α.

Lemma 0.0.15. *$f \in L^{p,\lambda}_C(\Omega)$ if and only if $f \in L^p(\Omega)$ and*

$$|||f|||_{p,\lambda} := \left(\sup_{\substack{x\in\Omega \\ 0<\rho<d}} \rho^{-\lambda} \left(\inf_{c\in\mathbb{R}} \int_{\Omega(x,\rho)} |f-c|^p \right) \right)^{1/p} < \infty$$

.

PROOF Clearly $|||f|||_{p,\lambda} \leq {}^C[f]_{p,\lambda}$. This takes care of the 'only if' part of the statement. Now suppose that $f \in L^p(\Omega)$ and $|||f|||_{p,\lambda} < \infty$. By convexity of $t \longrightarrow |t|^p$ for $p \geq 1$, we have

$$\begin{aligned}
&\int_{\Omega(x,\rho)} |f-(f)_{\Omega(x,\rho)}|^p \\
&\leq 2^{p-1}\left(\int_{\Omega(x,\rho)} |f-c|^p + \int_{\Omega(x,\rho)} |c-(f)_{\Omega(x,\rho)}|^p\right) \\
&= 2^{p-1}\left(\int_{\Omega(x,\rho)} |f-c|^p + |\Omega(x,\rho)|\left|c - \frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} f\right|^p\right) \\
&= 2^{p-1}\left(\int_{\Omega(x,\rho)} |f-c|^p + |\Omega(x,\rho)|^{1-p}\left|\int_{\Omega(x,\rho)} (c-f)\right|^p\right) \\
&\leq 2^{p-1}\left(\int_{\Omega(x,\rho)} |f-c|^p + \left|\int_{\Omega(x,\rho)} (c-f)\right|^p\right) \\
&\leq 2^{p-1}\left(\int_{\Omega(x,\rho)} |f-c|^p + \int_{\Omega(x,\rho)} |c-f|^p\right) \\
&\leq 2^{p}\left(\int_{\Omega(x,\rho)} |f-c|^p\right)
\end{aligned}$$

Where $c \in \mathbb{R}$ is arbitrary and in the 2nd to last step we used the fact that $\|\cdot\|_1 \leq \|\cdot\|_p$ since we are in a finite measure space. Since $c \in \mathbb{R}$ is arbitrary we can take the infimum on the RHS and then take the supremum over all $(x,\rho) \in \Omega \times (0,d)$ on both side. So we have ${}^C[f]_{p,\lambda} \leq 2\,|||f|||_{p,\lambda}$. □

Corollary 0.0.16. *$|||\cdot|||_{p,\lambda} + \|\cdot\|_p$ defines yet another norm on $L^{p,\lambda}_C(\Omega)$ which is equivalent to ${}^C\|\cdot\|_{p,\lambda}$.*

Lemma 0.0.17. *Then there exists a constant $K = K(p, \alpha, n)$, where α is the constant concerning the regularity of Ω such that*

$$0 < r < s < d \Rightarrow \left|(f)_{\Omega(x,r)} - (f)_{\Omega(x,s)}\right| \leq K \left(\frac{r^\lambda + s^\lambda}{r^n}\right)^{1/p} \; {}^C[f]_{p,\lambda}$$

for all $f \in L_C^{p,\lambda}(\Omega)$ and $x \in \Omega$.

Proof

$$\left|(f)_{\Omega(x,r)} - (f)_{\Omega(x,s)}\right|^p \leq 2^{p-1}\left(\left|(f)_{\Omega(x,r)} - f\right|^p + \left|f - (f)_{\Omega(x,s)}\right|^p\right)$$

$$\Rightarrow \int_{\Omega(x,r)} \left|(f)_{\Omega(x,r)} - (f)_{\Omega(x,s)}\right|^p \leq 2^{p-1}\left(\int_{\Omega(x,r)} \left|(f)_{\Omega(x,r)} - f\right|^p + \int_{\Omega(x,r)} \left|f - (f)_{\Omega(x,s)}\right|^p\right)$$

$$\Rightarrow |\Omega(x,r)|\left|(f)_{\Omega(x,r)} - (f)_{\Omega(x,s)}\right|^p \leq 2^{p-1}\left(\int_{\Omega(x,r)} \left|(f)_{\Omega(x,r)} - f\right|^p + \int_{\Omega(x,s)} \left|f - (f)_{\Omega(x,s)}\right|^p\right)$$

$$\Rightarrow \alpha V_n r^n\left|(f)_{\Omega(x,r)} - (f)_{\Omega(x,s)}\right|^p \leq 2^{p-1}\left(r^\lambda + s^\lambda\right)^C [f]^p_{p,\lambda}$$

where we used the regularity condition in the last step. Now regroup terms. □

Lemma 0.0.18. *There exists a constant $K = K(p, \lambda, \alpha)$ such that*

$$\left|(f)_{\Omega(x,\rho)} - (f)_{\Omega(x,\frac{\rho}{2^k})}\right| \leq K \; {}^C[f]_{p,\lambda} \; \rho^{\frac{(\lambda-n)}{p}} \sum_{m=0}^{k-1} 2^{\frac{m(n-\lambda)}{p}}$$

whenever $f \in L_C^{p,\lambda}(\Omega)$ and $(x,\rho) \in \Omega \times (0,d)$ and $k \in \mathbb{N}$.

Proof Let $f \in L_C^{p,\lambda}(\Omega)$ and $(x,\rho) \in \Omega \times (0,d)$ be given. By Lemma 0.0.7 we have for every $m \in \mathbb{N}$ (taking $r = \frac{\rho}{2^{m+1}}$ and $s = \frac{\rho}{2^m}$):

$$\left|(f)_{\Omega(x,\frac{\rho}{2^{m+1}})} - (f)_{\Omega(x,\frac{\rho}{2^m})}\right| \leq K\left(\frac{(\frac{\rho}{2^{m+1}})^\lambda + (\frac{\rho}{2^m})^\lambda}{(\frac{\rho}{2^{m+1}})^n}\right)^{1/p} \; {}^C[f]_{p,\lambda} = K\rho^{\frac{\lambda-n}{p}} 2^{\frac{m(n-\lambda)}{p}} 2^{\frac{n}{p}}(1+2^{-\lambda})^{1/p} \; {}^C[f]_{p,\lambda}$$

$$\Rightarrow \left|(f)_{\Omega(x,\frac{\rho}{2^{m+1}})} - (f)_{\Omega(x,\frac{\rho}{2^m})}\right| \leq K'\rho^{\frac{\lambda-n}{p}} 2^{\frac{m(n-\lambda)}{p}} \; {}^C[f]_{p,\lambda}$$

Because $K' = K2^{\frac{n}{p}}(1+2^{-\lambda})^{1/p}$ is independent of m, we can sum over $m = 0, 1, 2, ...k$ and use the triangle inequality on the LHS to get the result. $\square$

Lemma 0.0.19. *Let $\lambda > n$. Then for all $f \in L_C^{p,\lambda}(\Omega)$ there exists a function F defined on $\overline{\Omega}$ that equals f a.e. in Ω and such that $F(x) = \lim_{\rho\to 0}(f)_{\Omega(x,\rho)}$ on $\overline{\Omega}$, the convergence being uniform.*

Proof The existence of F is just Lebesgue's differentiation theorem : $\lim_{\rho\to 0}(f)_{\Omega(x,\rho)} = f(x)$ a.e. in Ω. We have to show that the convergence is uniform. By Lemma 0.0.8 for any $n, q \in \mathbb{N}$ we have

$$\left|(f)_{\Omega(x,\frac{\rho}{2^n})} - (f)_{\Omega(x,\frac{\rho}{2^{n+q}})}\right| \leq K''\, {}^C[f]_{p,\lambda} \left(\frac{\rho}{2^n}\right)^{\frac{(\lambda-n)}{p}}$$

, where K'' is a constant independent of x and q. Here we see that the sequence

$$\{(f)_{\Omega(x,\frac{\rho}{2^n})}\}_{n=1}^{\infty}$$

is Cauchy uniformly with respect to x. So for each $x \in \overline{\Omega}$, let

$$F(x) := \lim_{n\to\infty}(f)_{\Omega(x,\frac{\rho}{2^n})}$$

By Lemma 0.0.18, we have

$$\left|(f)_{\Omega(x,\sigma)} - (f)_{\Omega(x,\frac{\sigma}{2^k})}\right| \leq K\, {}^C[f]_{p,\lambda}\, \sigma^{\frac{(\lambda-n)}{p}} \sum_{m=0}^{k-1} 2^{\frac{m(n-\lambda)}{p}}$$

Taking $n \to \infty$ we get

$$\left|(f)_{\Omega(x,\sigma)} - F(x)\right| \leq K'\, {}^C[f]_{p,\lambda}\, \sigma^{\frac{(\lambda-n)}{p}}$$

for some constant K'. This says that $(f)_{\Omega(x,\sigma)} \to F(x)$ uniformly as $\sigma \to 0$. $\square$

Lemma 0.0.20. *Let $0 \leq \lambda < n$. Then there exists a constant $K = K(\alpha, p, \lambda, n) > 0$ such that for all $f \in L_C^{p,\lambda}(\Omega)$ and $(x,\rho) \in \Omega \times (0,d)$ we have :*

$$|(f)_{\Omega(x,\rho)}| \leq |(f)_{\Omega}| + K\, {}^C[f]_{p,\lambda}\rho^{\frac{(\lambda-n)}{p}}$$

Proof Let $f \in L_C^{p,\lambda}(\Omega)$ and $\rho \in (0,d)$ be given. Choose $k \in \mathbb{N}$ so that $\frac{d}{2^{k+1}} \le \rho < \frac{d}{2^k}$. We have

$$|(f)_{\Omega(x,\rho)}| \le |(f)_\Omega| + |(f)_\Omega - (f)_{\Omega(x,\frac{d}{2^k})}| + |(f)_{\Omega(x,\frac{d}{2^k})} - (f)_{\Omega(x,\rho)}|$$

To show the result, we must bound appropriately the 2nd and 3rd terms on the RHS. Since in fact $(f)_\Omega = (f)_{\Omega(x,d)}$, we can bound the 2nd term on the RHS using Lemma 0.0.8 We have

$$|(f)_\Omega - (f)_{\Omega(x,\frac{d}{2^k})}| \le K_2 \; {}^C[f]_{p,\lambda} \; d^{\frac{(\lambda-n)}{p}} \sum_{m=0}^{k-1} 2^{\frac{m(n-\lambda)}{p}}$$

$$= K_2 \; {}^C[f]_{p,\lambda} \; d^{\frac{(\lambda-n)}{p}} \frac{2^{\frac{k(n-\lambda)}{p}} - 1}{2^{\frac{(n-\lambda)}{p}} - 1}$$

$$\le K_2 \; {}^C[f]_{p,\lambda} \rho^{\frac{(\lambda-n)}{p}} 2^{(k+1)\frac{(\lambda-n)}{p}} \frac{2^{\frac{k(n-\lambda)}{p}} - 1}{2^{\frac{(n-\lambda)}{p}} - 1}$$

$$\le K_2 \; {}^C[f]_{p,\lambda} \; \rho^{\frac{(\lambda-n)}{p}} 2^{\frac{k(\lambda-n)}{p}} \frac{2^{\frac{k(n-\lambda)}{p}} - 1}{2^{\frac{(n-\lambda)}{p}} - 1}$$

$$= K_2 \; {}^C[f]_{p,\lambda} \; \rho^{\frac{(\lambda-n)}{p}} \frac{1}{2^{\frac{(n-\lambda)}{p}} - 1} \left(1 - \frac{1}{2^{\frac{k(n-\lambda)}{p}}}\right)$$

Although the last term in brackets depends on k, it is bounded above by 1.

We bound the 3rd term on the RHS using Lemma 0.0.7

$$|(f)_{\Omega(x,\frac{d}{2^k})} - (f)_{\Omega(x,\rho)}| \le K_3 \left(\frac{\rho^\lambda + \left(\frac{d}{2^n}\right)^\lambda}{\rho^n}\right)^{1/p} \; {}^C[f]_{p,\lambda}$$

$$\le K_3 \left(\frac{\rho^\lambda + (2\rho)^\lambda}{\rho^n}\right)^{1/p} \; {}^C[f]_{p,\lambda}$$

$$\le K_3 \left(\rho^{\frac{\lambda-n}{p}} (2^\lambda + 1)^{1/p}\right) \; {}^C[f]_{p,\lambda}$$

$$\le K_3' \left(\rho^{\frac{\lambda-n}{p}}\right) \; {}^C[f]_{p,\lambda}$$

Now take $K = K_3' + K_2 \frac{1}{2^{\frac{(n-\lambda)}{p}} - 1}$. □

Lemma 0.0.21. *There exists a constant $K = K(\alpha, n, \lambda)$ such that for all $f \in L_C^{1,\lambda}(\Omega)$ and all $x, y \in \overline{\Omega}$,*

$$\rho = 2|x-y| \Rightarrow |(f)_{\Omega(x,\rho)} - (f)_{\Omega(y,\rho)}| \leq K\ {}^C[f]_{1,\lambda}|x-y|^{\lambda-n}$$

Proof Let $f \in L_C^{1,\lambda}(\Omega)$ and $x, y \in \overline{\Omega}$ be given. $\rho = 2|x-y| \Rightarrow \Omega(x,\rho) \cap \Omega(y,\rho)$ contains both $\Omega(x,\rho)$ and $\Omega(y,\rho)$.

$$|(f)_{\Omega(x,\rho)} - (f)_{\Omega(y,\rho)}| \leq |(f)_{\Omega(x,\rho)} - u| + |u - (f)_{\Omega(y,\rho)}|$$

$$\Rightarrow \int\limits_{\Omega(x,\rho)\cap\Omega(y,\rho)} |(f)_{\Omega(x,\rho)} - (f)_{\Omega(y,\rho)}| \leq \int\limits_{\Omega(x,\rho)\cap\Omega(y,\rho)} |(f)_{\Omega(x,\rho)} - u| + \int\limits_{\Omega(x,\rho)\cap\Omega(y,\rho)} |u - (f)_{\Omega(y,\rho)}|$$

.

$$\Rightarrow |(f)_{\Omega(x,\rho)} - (f)_{\Omega(y,\rho)}| \Big|\Omega(x,\rho) \cap \Omega(y,\rho)\Big| \leq \int\limits_{\Omega(x,\rho)} |(f)_{\Omega(x,\rho)} - u| + \int\limits_{\Omega(y,\rho)} |u - (f)_{\Omega(y,\rho)}|$$

$$\Rightarrow |(f)_{\Omega(x,\rho)} - (f)_{\Omega(y,\rho)}| \Big|\Omega(x,\rho) \cap \Omega(y,\rho)\Big| \leq 2\rho^{\lambda}\ {}^C[f]_{1,\lambda}$$

Since $\Omega(x, \frac{\rho}{2}) \subset \Omega(x,\rho) \cap \Omega(y,\rho)$ and $|\Omega(x, \frac{\rho}{2})| \geq \alpha V_n \left(\frac{\rho}{2}\right)^n$ by regularity of Ω, we get

$$|(f)_{\Omega(x,\rho)} - (f)_{\Omega(y,\rho)}| \leq K\rho^{\lambda-n}\ {}^C[f]_{1,\lambda}$$

for some constant K. □

Theorem 0.0.22. *Let $1 \leq p < \infty$*

1. *For $\lambda \in [0, n)$, $L_C^{p,\lambda}(\Omega) = L_M^{p,\lambda}(\Omega)$*
2. *For $\lambda \in (n, n+p]$, $L_C^{p,\lambda}(\Omega) = C^{0,\beta}(\overline{\Omega})$, where $\beta = \frac{\lambda - n}{p}$.*
3. *For $\lambda > n + p$ or $\beta > 1$, the spaces $L_C^{p,\lambda}(\Omega)$ and $C^{0,\beta}(\overline{\Omega})$ contain only constant functions.*
4. *For $\lambda = n, L^{\infty}(\Omega) \hookrightarrow L_C^{p,n}(\Omega)$, but $L\infty(\Omega) \neq L_C^{p,n}(\Omega)$.*

Proof

1. Let $\lambda \in [0,n)$ and $f \in L_M^{p,\lambda}(\Omega)$. By Corollary 0.0.6,

$$^C\|f\|_{p,\lambda} \leq 2\left(\|f\|_p + |||f|||_{p,\lambda}\right)$$

$$= 2\left(\|f\|_p + \left(\sup_{\substack{x\in\Omega \\ 0<\rho<d}} \rho^{-\lambda}\left(\inf_{c\in\mathbb{R}} \int_{\Omega(x,\rho)} |f-c|^p\right)\right)^{1/p}\right)$$

$$\leq 2\left(\|f\|_p +^M \|f\|_{p,\lambda}\right) \leq K^M\|f\|_{p,\lambda}$$

where K is some constant; the last inequality is derived in the proof of Theorem 0.0.6. Hence $f \in L_C^{p,\lambda}(\Omega)$ and the identity operator from $L_M^{p,\lambda}(\Omega)$ into $L_C^{p,\lambda}(\Omega)$ is continuous.

Conversely, suppose $f \in L_C^{p,\lambda}(\Omega)$. We have :

$$\int_{\Omega(x,\rho)} |f|^p \leq 2^{p-1}\left(\int_{\Omega(x,\rho)} |f-(f)_{\Omega(x,\rho)}|^p + \int_{\Omega(x,\rho)} |(f)_{\Omega(x,\rho)}|^p\right)$$

The 1st term on the RHS is bounded above by $\rho^\lambda \, {}^C[f]_{p,\lambda}^p$ and to bound the 2nd term on the RHS we have by Lemma 0.0.20

$$|(f)_{\Omega(x,\rho)}|^p \leq \left(|(f)_\Omega| + K\, {}^C[f]_{p,\lambda}\, \rho^{\frac{(\lambda-n)}{p}}\right)^p \leq 2^{p-1}\left(|(f)_\Omega|^p + K^p\, {}^C[f]_{p,\lambda}^p\, \rho^{\lambda-n}\right)$$

Therefore,

$$\int_{\Omega(x,\rho)} |f|^p \leq 2^{p-1}\left(\rho^\lambda\, {}^C[f]_{p,\lambda}^p + |\Omega(x,\rho)|2^{p-1}\left(|(f)_\Omega|^p + K^p\, {}^C[f]_{p,\lambda}^p\, \rho^{\lambda-n}\right)\right)$$

$$= 2^{p-1}\left(\rho^\lambda\, {}^C[f]_{p,\lambda}^p + |\Omega(x,\rho)|2^{p-1}|(f)_\Omega|^p + |\Omega(x,\rho)|2^{p-1}K^p\, {}^C[f]_{p,\lambda}^p\, \rho^{\lambda-n}\right)$$

$$\leq 2^{p-1}\left(\rho^\lambda\, {}^C[f]_{p,\lambda}^p + 2^{p-1}\frac{|\Omega(x,\rho)|}{|\Omega|}\|f\|_p^p + V_n\rho^n 2^{p-1}K^p\, {}^C[f]_{p,\lambda}^p\, \rho^{\lambda-n}\right)$$

$$\leq K'\left(\rho^\lambda\, {}^C[f]_{p,\lambda}^p + \rho^n\|f\|_p^p\right)$$

for some constant K'. Multiplying both sides by $\rho^{-\lambda}$, taking the supremum and using the fact that $n-\lambda > 0$ gives us ${}^M\|f\|_{p,\lambda} \leq K''\, {}^C\|f\|_{p,\lambda}$.

2. Let $\lambda > n$ and $\alpha = (\lambda - n)p^{-1}$, $f \in C^{0,\alpha}(\overline{\Omega})$.Then

$$\int\limits_{\Omega(x,r)} |f - (f)_{\Omega(x,r)}|^p$$

$$= \int\limits_{\Omega(x,r)} |\Omega(x,r)|^{-p} \left(\left| \int\limits_{\Omega(x,r)} (f(y) - f(t)dt \right|^p \right) dy$$

$$\leq \int\limits_{\Omega(x,r)} |\Omega(x,r)|^{-1} \left(\int\limits_{\Omega(x,r)} |f(y) - f(t)|^p dt \right) dy$$

where we have used precisely the following basic relation for finite L^p spaces: $\|f\|_1 \leq \|f\|_p |\Omega(x,r)|^{1-1/p}$.

$$\leq \int\limits_{\Omega(x,r)} |\Omega(x,r)|^{-1} \left(\int\limits_{\Omega(x,r)} \frac{|f(y) - f(t)|^p}{|y-t|^{\alpha p}} |y-t|^{\alpha p} dt \right) dy$$

$$\leq \int\limits_{\Omega(x,r)} |\Omega(x,r)|^{-1} \left(\int\limits_{\Omega(x,r)} H^p_{0,\alpha} r^{\alpha p} dt \right) dy$$

$$= |\Omega(x,r)| H^p_{0,\alpha} r^{\alpha p}$$

$$\Rightarrow r^{-\lambda} \int\limits_{\Omega(x,r)} |f - (f)_{\Omega(x,r)}|^p \leq V_n r^n H^p_{0,\alpha} r^{\alpha p - \lambda}$$

$$\Rightarrow^C [f]_{p,\lambda} \leq V_n H^p_{0,\alpha}$$

Also $\|f\|_p \leq \|f\|_\infty |\Omega|^{1/p} = \|f\|_{C(\overline{\Omega})} |\Omega|^{1/p}$.

So taking $K = \max\{V_n, |\Omega|^{1/p}\}$ (for example) we get ${}^C\|f\|_{p,\lambda} \leq K \|f\|_{C^{0,\beta}(\overline{\Omega})}$.

For the converse, by part 4 of Theorem 0.0.12, $L^{p,\lambda}_C(\Omega) \hookrightarrow L^{1,\beta+n}_C(\Omega)$ whenever $\beta \leq \frac{\lambda - n}{p}$, so it is enough to show that $L^{1,\beta+n}_C(\Omega) \hookrightarrow C^{0,\beta}$ for $\beta \in (0,1]$. Let $f \in L^{1,\beta+n}_C(\Omega)$. By Lemma 0.0.19 $F(x) = \lim\limits_{r \to 0} (f)_{\Omega(x,r)}$ and $F = f$ a.e. in (Ω) Let $x, y \in \Omega, r = 2|x-y|$. We have

$$|F(x) - F(y)| \leq |F(x) - (f)_{\Omega(x,r)}| + |(f)_{\Omega(x,r)} - (f)_{\Omega(y,r)}| + |(f)_{\Omega(y,r)} - F(y)|$$

By the last relation in the proof of Lemma 0.0.19, we have

$$|(f)_{\Omega(x,r)} - F(x)| \leq K_1 \,{}^{C}[f]_{1,\beta+n}\; r^{\beta}$$

and

$$|(f)_{\Omega(y,r)} - F(y)| \leq K_1 \,{}^{C}[f]_{1,\beta+n}\; r^{\beta}$$

and by Lemma 0.0.21 we get

$$|(f)_{\Omega(x,r)} - (f)_{\Omega(y,r)}| \leq K_2 \,{}^{C}[f]_{1,\beta+n}\; (r/2)^{\beta}$$

Putting everything together, we have

$$\frac{|F(x) - F(y)|}{|x-y|^{\beta}} \leq K^{C}[f]_{1,\beta+n}$$

$$\Rightarrow H_{0,\beta}(f) \leq K^{C}[f]_{1,\beta+n}$$

3. The result is well known for $C^{0,\beta}(\overline{\Omega})$.

4. Let $f \in L^{\infty}(\Omega)$ and $(x,\rho) \in \Omega \times (0,d)$. We have:

$$\int_{\Omega(x,\rho)} |f - (f)_{\Omega(x,\rho)}|^p$$

$$\leq 2^{p-1}\left(\int_{\Omega(x,\rho)} |f|^p + |\Omega(x,\rho)||(f)_{\Omega(x,\rho)}|^p\right)$$

$$\leq 2^p|\Omega(x,\rho)|\,(\|f^p\|_{\infty} + \|f\|_{\infty}^p)$$

$$\leq V_n\rho^n 2^p\|f\|_{\infty}^p$$

$$\Rightarrow^C [f]_{p,N} \leq K\|f\|_\infty$$

So this gives us $L_\infty(\Omega) \hookrightarrow L_C^{p,n}(\Omega)$.

For $n = 1$, say $\Omega = (0,1)$, then $\log|x|$ is a typical example of a function that doesn't belong to $L_\infty(0,1)$, but that does belong to $L_C^{p,n}(0,1)$. This can be generalized to higher dimensions and more arbitrary domains.

$\square$

References

[1] KUFNER, JOHN, FUČIK, *Function Spaces*, Noordhoff International Publishing, Leyden, (1977), 209 - 224

[2] PEETRE, *On the Theory of* $\mathcal{L}_{p,\lambda}$ *Spaces*, Journal of Functional Analysis 4, 71 - 87 (1969)

[3] RUPFLIN, *What is a . . . Morrey Space?, What is a . . . Campanato Space?*, (2008)

[4] ADAMS, XIAO, *Morrey spaces in Harmonic Analysis*, Ark. Mat., 50 (2012), 201 - 230