Maximum Principles for Elliptic and Parabolic Operators

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1 Introduction

Maximum principles have been some of the most useful properties used to solve a wide range of problems in the study of partial differential equations over the years. Starting from the basic fact from calculus that if a function $f(x)$ satisfies $f'' > 0$ on an interval $[a, b]$, then it can only achieve its maximum on the boundary of that interval. For partial differential equations, the same idea allows to draw very useful conclusions from the properties of the solutions and the domain of a given problem. We will look over some results such as the Hopf Maximum Principle and its generalization, approximations and uniqueness of solution for elliptic operators. We will then consider how maximum principles are used in the study of parabolic operators, noting some of the similarities and differences with the elliptic operators. Some of the results will be presented in more detail, for others, only a sketch of the proof will be given.

2 Elliptic Operators

We begin by giving the definition of an second order differential elliptic operator.

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x_1, x_2, ..., x_n) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

where for all points $\mathbf{x}$ of a domain D, there exists a positive quantity $\mu(x)$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \geq \mu(x) \sum_{i=1}^{n} \zeta_i^2, \quad \text{for all n-tuples}(\zeta_1, ..., \zeta_n)$$

In terms of matrices, this condition means that the symmetric matrix of coefficients $\mathbf{A(x)}$ is positive definite. The operator is called uniformly elliptic if the same $\mu(x)$ holds for all points of the domain and there exists a constant μ_0 such that $\mu(x) \geq \mu_0$ for all x in D.

2.1 Maximum Principle

The first and most basic case one can consider in the study of elliptic operators is the Laplace operator Δ:

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_n^2}$$

If a function u has a local maximum on an interior point of a domain D, then at that point, it must hold that

$$\frac{\partial u}{\partial x_1} = 0,\ \frac{\partial u}{\partial x_2} = 0,\ ...,\ \frac{\partial u}{\partial x_n} = 0$$

and

$$\frac{\partial^2 u}{\partial x_1^2} \leq 0,\ \frac{\partial^2 u}{\partial x_2^2} \leq 0,\ ...,\ \frac{\partial^2 u}{\partial x_n^2} \leq 0$$

Then looking at the simple equation $\Delta u > 0$ on a given domain D, we can conclude right away that u cannot attain a maximum in the interior of D unless it is constant. The same result holds if u satisfies

$$\Delta(u) + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} + ... + b_n \frac{\partial u}{\partial x_n} > 0$$

where $b_1, ...b_n$ are any bounded functions in D. This result can be extended for inequalities which are not strict using the Mean Value Theorem. The proof of this is left to the reader or can be found in one of many books on this topic (namely [Protter and Weinberger]).

We now turn our attention to more general uniformly elliptic operators where the coefficients a_ij and b_i are uniformly bounded. It is easy to show that one can perform a change of coordinates by multiplying the matrix of coefficients by an orthogonal basis and that the operator remains elliptic and the quantity $\mu(x)$ is preserved. Moreover, for a given point in the domain, there exists a basis which transforms the operator into a Laplacian at that point. This will lead us to Hopf's Maximum principle. We know that if u has a relative maximum at a point p in the interior of D, then, at that point

$$\frac{\partial u}{\partial z_k} = 0 \text{ and } \frac{\partial^2 u}{\partial z_k^2} \leq 0, \qquad k = 1, 2, ..., n$$

for any coordinate system $z_1, ..., z_n$. We can now see that for L as above, there is a transformation changing into a Laplacian at p. So if u satisfies $L[u] > 0$ in D, we get a contradiction. A function satisfying $L[u] > 0$ cannot attain a maximum on the interior of the domain unless it is constant. The maximum principle also holds if the inequality is not strict and if a component $h(x)$ is added so long as $h(x) \leq 0$ and it is bounded.

$$(L + h)[u] = L[u] + h(x)u \geq 0$$

The proof by contradiction relies on constructing an auxiliary function $w = u + \epsilon z$ where z is defined on a ball contained in D such that the maximum of u is attained on the boundary of that ball. This auxiliary function is constructed in such a way that

$$L[w] = L[u] + \epsilon L[z] > 0$$

which gives a contradiction to the statement above.

2.2 Uniqueness for Boundary Value Problems

On a different note, we now look at the use of maximum principles for boundary value problems. For the simple case of the Poisson equation in two dimensions, we have

$$\Delta v \equiv f(x,y) \quad \text{in } D$$

with a boundary condition

$$v = g(s) \quad on \partial D$$

where $v(x,y)$ is twice differentiable in D and continuous on $D \cup \partial D$. If a solution exists, then it must be unique. Otherwise, suppose there is v_1, v_2 that satisfy the initial equation and agree on the boundary, we will look at $u = v_1 - v_2$. It must be that

$$\Delta u = 0 \text{ in } D \qquad u = 0 \text{ on } \partial D$$

By the maximum principle established earlier, u cannot achieve a maximum inside D. Deduce that the maximum is on the boundary where $u = 0$ so $u < 0$ in D or must be constant. Now since the same is true for $-u$, it must be that $u \equiv 0$ on the entire domain and the desired result is obtained.

The more general boundary value problem for an n-dimensional domain is defined in a slightly more complicated way. We will consider the equation

$$(L + h)[v] = f \qquad inD$$

with boundary conditions

$$\begin{cases} \frac{\partial v}{\partial \nu} + \alpha(x) & v = g_1 \quad \text{on } \Gamma_1 \\ & v = g_2 \quad \text{on } \Gamma_2 \end{cases}$$

where $\partial / \partial \nu$ is the derivative with respect to an outward pointing vector at each point of Γ_1. Γ_1 and Γ_2 constitute the boundary of the domain D and $\alpha \geq 0$

Using the same approach as before to show uniqueness, we apply the maximum principle to $u = v_1 - v_2$. If this function is positive anywhere, it must have a positive maximum which has to be attained on the boundary and more specifically on Γ_1. A result not mentioned previously states that at the point of the boundary where the maximum is achieved, the outward pointing derivative must be positive unless the function is constant. This would contradict the condition that

$$\frac{\partial u}{\partial \nu} + \alpha(x) v = 0 \quad \text{on } \Gamma_1$$

So either $u \leq 0$ or it is constant. Since the same holds for $-u$, it must be equal to 0 unless Γ_2 is empty and $h \equiv \alpha \equiv 0$ in which case any constant will satisfy the conditions.

2.3 Generalizations

Those various results rely on the condition that $h(x) \leq 0$ and is bounded. There are simple examples that illustrate that this condition cannot be removed. But it can be loosened and compensated in different ways. Given a uniformly elliptic operator L and a function u that satisfies

$$(L + h)[u] \geq 0 \text{ in D}$$

we assume that it is possible to define

$$v(x) = \frac{u(x)}{w(x)}$$

where $w(x)$ is a positive function on $D \cup \partial D$. Then if $(L + h)[w] \leq 0$, we can apply the maximum principle to the function $v(x)$. From this, we deduce that the uniqueness (and other) results will hold when the condition on $h(x)$ is removed so long as a function $w(x)$ with the properties stated above exists. Although it may be hard to find, Protter and Weinberger give an explicit formula in the case when "the domain D is contained in a sufficiently narrow slab" $a < x_1 < b$ and $h(x)$ is bounded.In that case, $w(x) = 1 - \beta e^{\alpha(x_1 - a)}$ For those who don't like the notion of 'narrow enough', other results exist. In fact, rather than putting somewhat ambiguous restrictions on the domain, it is possible to restrict the operator. In his paper on maximum principles, Danet considers the one-dimensional operator $L[u] \equiv u'' + h(x)u \geq 0$ in a bounded domain. Then if

$$\sup_{\Omega} h(x) < \frac{\pi^2}{(diam\Omega)^2}$$

the same conclusion holds for the function $v = u/w_\epsilon$ where w_ϵ is defined to be

$$w_\epsilon(x) = cos(\frac{\pi(2x - diam\Omega)}{2(diam\Omega + \epsilon)})cosh(\epsilon x), \quad \text{for } \epsilon > 0 \text{ small}$$

This result has an equivalent formulation in higher dimensions but was presented here mostly to show that there are different ways of restricting the differential inequality in order to get maximum principles for the functions that satisfy it.

2.4 Approximation of Solutions

We are now in good shape to get back to the boundary value problem and look at how the maximum principles can help find bounds on a solution which might be complicated to find explicitly. Let u be a solution of

$$(L + h)[u] = f \qquad \text{in } D$$

with boundary conditions

$$\begin{cases} \frac{\partial u}{\partial \nu} + \alpha(x) & u = g_1 \quad \text{on } \Gamma_1 \\ & u = g_2 \quad \text{on } \Gamma_2 \end{cases}$$

We assume L is uniformly elliptic in D, $\Gamma_1 \cup \Gamma_2$ form the boundary ∂D and $\partial u / \partial \nu$ is an outward pointing derivative. Moreover, we suppose that L, h and D are such that there exists a positive function $w(x)$ with properties:

$$(L + h)[w] \leq 0 \text{ in D} \qquad \text{and} \qquad \frac{\partial w}{\partial \nu} + \alpha(x) w \geq 0 \text{ on } \Gamma_1$$

If we can find a function z_1 which satisfies

$$(L + h)[z_1] \leq f \qquad \text{in } D$$

$$\begin{cases} \frac{\partial z_1}{\partial \nu} + \alpha(x) & z_1 = g_1 \quad \text{on } \Gamma_1 \\ & z_1 = g_2 \quad \text{on } \Gamma_2 \end{cases}$$

then we can apply the maximum principle to the function $v = (u - z_1)/w$ to conclude that z_1 is an upper bound for u in the domain unless $\Gamma_2 = \emptyset$ and the two conditions on $w(x)$ are identically 0. With the same argument but the opposite inequalities and $v = (z_2 - u)/w$, it is possible to obtain a lower bound for the solution.

There are more interesting results and applications of various forms of maximum principles for elliptic operators but the time is getting late and I have another final in the morning.

3 Parabolic Operators

Once more, we begin by giving a formal definition of a formal operator: the operator

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x_1, x_2, ..., x_n, t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} - \frac{\partial}{\partial t}$$

is said to be parabolic if for fixed t, the operator consistent of the first sum is an elliptic operator. It is said to be uniformly parabolic if the definition of ellipticity holds uniformly for all points(x, t) of a region E_T. We will now briefly discuss some results analogous to those of the previous section noting the key similarities and differences.

3.1 One-Dimensional Parabolic Operator

We will consider the one-dimensional case which already provides enough information to extract important similarities and differences with the case of elliptic operators. Suppose $u(x, t)$ satisfies

$$L[u] \equiv \frac{\partial^u}{\partial x^2} - \frac{\partial u}{\partial t} > 0 \text{ in D}$$

If u were to achieve a local maximum at any point in D, we know that at that point, it must be that

$$\frac{\partial^u}{\partial x^2} \leq 0, \qquad \text{and } \frac{\partial u}{\partial t} = 0$$

which contradicts the assumption. So far, the result is the same as before, if the strict inequality is respected, then u must have a maximum on the boundary of the domain or be constant everywhere. The conclusion differs however if the inequality is not strict. In the case $L[u] \geq 0$ we get a stronger result for parabolic operators than we had for elliptic ones. In fact, simply by looking at a rectangular domain $E = \{0 < x < l, 0 < t < T\}$ together with its boundary, it's possible to show the maximum of u must be achieved on the boundary but not at the final time $t = T$. The proof goes by contradiction with the use of a familiar auxiliary function $w(x) > 0$: if M is the maximum of u on $\partial E \backslash \{t = T\}$ and $M_1 > M$ is the maximum on $E \cup \partial E$ at the point (x_0, t_0)then

$$w(x) = \frac{M_1 - M}{2l^2}(x - x_0)^2$$

and we can apply the maximum principle to $v = u(x,t) + w(x,t)$ to see that (x_0, t_0) cannot be on th interior of the domain. If that point is at $t = T$ however, then a contradiction arise from the fact that $\partial v / \partial t$ must be strictly negative implying the value of v must have been greater at a previous time! We skip over a few intermediate results to get to the theorem which explicitly formulates the consequence of this additional condition. Let E be a domain and the inequality

$$L[u] \equiv a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \geq 0$$

holds with a and b bounded, L uniformly parabolic. If u achieves its maximum value m at an interior point (x_1, t_1) then it must be that $u(x,t) = M$ for every point which can be connected to (x_1, t_1) by vertical and/or horizontal line segments contained in the domain. In other words, given an interior point at which the function is maximal, it is possible to identify a region on which it must be constant.

3.2 Uniqueness for Boundary Value Problems

For an n-dimensional parabolic operator, we pose the boundary value problem in a similar way to what was seen previously: let v be a solution to

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x_1, x_2, ..., x_n, t)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial v}{\partial x_i} + h(x,t)v - \frac{\partial v}{\partial t} = f(x,t)$$

in a domain $E = D \times [0, T]$ with the coefficients of L bounded. The boundary conditions

$$v(x, 0) = g_1(x) \text{ in D} \qquad \text{and} \qquad \alpha(x,t)v(x,t) + \beta(x,t)\frac{\partial v}{\partial \nu} = g_2(x,t) \text{ on } \Gamma$$

where $\alpha \geq 0$, $\beta \geq 0$ on Γ and they are not both 0 at the same time, $\partial v/\partial \nu$ is an outward and upward pointing derivative and $h(x,t)$ is bounded above. Then the solution can be shown to be unique by assuming the existence of distinct solutions and applying the maximum principle to $w = v_1 - v_2$. As for the elliptic operators, the doing so for both w and $-w$ yields uniqueness of the solution in the domain if it exists at all.

Although there is a lot more to say, we shall proceed to the closing statement.

4 Conclusion

Overall, it is important to understand the power of the various forms of the maximum principles in the study of PDEs. Using relatively simple and only few technical tools, one can achieve a wide variety of results concerning the behaviour of solutions or properties of different differential operators be they elliptic, parabolic or even hyperbolic. Many of the proofs use similar approaches and the results obtained extend well to more general cases for the most part. Once acquainted with the 'tricks', so many different things become easy to prove: be it results concerning boundary problems, eigenvalues, growth in unbounded domains, etc. Only the limited amount of time forces me to stop here and not elaborate on more applications of maximum principles related to Harnack's inequalities, non-linear operators or other cool problems.

For further reading, consult 'Maximum Principles in Differential Equations' by Murray Protter and Hans Weinberger or 'The Maximum Principle' by Patrizia Pucci and James Serrin as well as the paper by Cristian-Paul Danet entitled 'The Classical Maximum Principle. Some of its Extensions and Applications'.