# Hodge Theory

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**Hodge theory is an important perspective on the study of differential forms on a smooth manifold. The theory was named after British mathematician William Hodge and it has applications on Riemannian manifolds, Kahler manifolds and algebraic geometry of complex projective varieties. This expository paper is aimed as an introduction to basic ideas of Hodge theory. In particular we will prove Hodge theorem and study its consequences.**

## Introduction

The connection between the differential forms which represents infinitesimal volume elements on a manifold and global geometry of the manifold has been elusive for centuries. Hodge theory is a bridge that connects the two seemingly unrelated realms. It says that we can get information about the cohomology groups, which gives global information about the manifold, from special type of differential forms. We study both of the subjects and explore their connections using Hodge theory.

## Differential forms and Integration

In this section we will introduce differential k forms. It is convienent to think of them as k-dimensional volume. Hence in differential geometry differential forms are often used to integrate over manifolds. Integration of forms are volumes, so they should not depend on the parametrization thus there is natural restrictions on what constitutes differential forms.

In order to show what are differential forms it is necassary to show few things from linear algebra. Let T be a k-tensor set of all k tensors on a smooth manifold $M$ is denoted as $J^k(M)$.

**Definition 1.** A *tensor product* between $T \in J^k(M)$ and $P \in J^r(M)$ is denoted $T \otimes P \in J^{k+r}(M)$ satisfying

$$T \otimes P(x_1, \cdots, x_k, x_{k+1}, \cdots, x_{k+r}) = T(x_1, \cdots, x_k, x_{k+1}, \cdots, x_{k+r})$$

This tensor product is multilinear and associative. Notice that the vector space $J^1(M)$ is same as $T^*\mathcal{M}$. In a similar fashion for 1 covariant tensors $T : T^*M \to \mathbb{R}$ which is equivalent to $T \in T\mathcal{M}$, so $J_1(M) = TM$. Thus any mixed tensor of type (m,n) which is also a vector space denoted by $J^m_n(M) = \bigotimes^m T^*M \bigotimes^n TM$. Hence in local coordinates $x^i$ , these tensors can be written as

$$T = T^{j_1, \cdots, j_n}_{i_1, \cdots, i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_n} \qquad T \in J^m_n(M)$$

Given that we now know tensor products we are in position to define differential forms. Simply stated differential forms are tensors that satisfy the property $\text{Alt}(T) = T$,where the function Alt is defined as

$$\text{Alt}(T)(x_1, \cdots, x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(x_{\sigma(1)}, \cdots, x_{\sigma(k)})$$

where $S_k$ is the group of permutations of numbers 1 to k. From this definition it is clear that space of k differential forms denoted by $\Lambda^k(M)$ is a subspace of $J^k(M)$. In particular Alt has interesting properties as

- If $T \in J^k(M)$ then $\text{Alt}(T) \in \Lambda^k(M)$
- For any k tensor $T$ $\text{Alt}^n(T) = \text{Alt}(T)$ for all $n \geq 2$

Analogous to tensor product we would like to define a product that respects the vector space structure for $\Lambda^k(M)$. In other words we want to define a product such that if we multiply two differential forms we are guaranteed to get another differential form. For this purpose wedge product of differential forms are defined.

**Definition 2.** A wedge product between $\xi \in \Lambda^k(M)$ and $\eta \in \Lambda^r(M)$ is denoted $\xi \wedge \eta \in \Lambda^{k+r}(M)$ satisfying

$$(\xi \wedge \eta)(x_1, ., x_{n+m}) = \frac{1}{m!n!} \sum_{\sigma \in S_{m+n}} \text{sgn}(\sigma) \xi(x_{\sigma(1)}, .., x_{\sigma(m)}) \eta(x_{\sigma(m+1)}, ., x_{\sigma(m+n)})$$

Wedge product is associative and anticommutative: $\xi \wedge \eta = (-1)^{mn} \eta \wedge \xi$. Under local coordinates $x^i$ a general k-form $\eta \in \Lambda^k$ looks like

$$\eta = \sum_{i_1 < \cdots < i_k} T_{i_1, \cdots, i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where $T_{i_1, \cdots, i_k}$ are differentiable real valued functions. Another important map that allows to construct one form to another is a exterior diffentiation we shall see later this allows one to integrate things easier via stokes theorem. Exterior derivatives take k form to k+1 forms. In a local coordinates it will looke like

$$d\eta = \sum_{i_1 < \cdots < i_k} dT_{i_1, \cdots, i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

There are two important special types of forms that one need to know.

**Definition 3.** A differential form $w$ is called *exact form* if $w = d\eta$ for some $\eta$. Also the form $\xi$ is called *closed* if $d\xi = 0$.

By the way we defined, it is clear that exact form is a closed form as well. It is convenient to be able to distinguish them since their integral always turn out to be simple. For now we show how integration of forms work. Recall the change of variables formula in a multiple integral in $\mathbb{R}^n$:

$$\int f(y_1, \cdots, y_n) dy_1 \cdots dy_n = \int f(y_1(x), \cdots, y_n(x)) | \det \frac{\partial y_i}{\partial x_j} | dx_1 \cdots dx_n$$

and compare to the change of coordinates for an n-form on an n-dimensional manifold

$$\begin{aligned} w &= f(y_1, \cdots, y_n) dy^1 \wedge \cdots \wedge dy^n \\ &= f(y_1(x), \cdots, y_n(x)) \sum_i \frac{\partial y^1}{\partial x^i} dx^i \wedge \cdots \wedge \sum_p \frac{\partial y^n}{\partial x^p} dx^p \\ &= f(y_1(x), \cdots, y_n(x)) (\det \frac{\partial y_i}{\partial x_j}) dx^1 \wedge \cdots \wedge dx^n \end{aligned}$$

The only difference is the absolute value, so that if we can sort out a consistent sign, then we should be able to assign a coordinate-independent value to the integral of an n-form over an n-dimensional manifold. By fixing the orientation we get rid of the sign problem

Now we are ready to define an integral of differential forms. For n-dimensional manifold we can define the integral of any n-form that is compactly supported denoted below

$$\int_M w$$

Assume $M$ is orientable, and let $w$ be a non-vanishing n-form. In a coordinate chart $w = f(x^1, \cdots, x^n)dx^1 \wedge \cdots \wedge dx^n$.On each coordinate neighbourhood $U_\alpha$ we have partition of unity $\phi_i$ subordinate to this covering. Then

$$\phi_i w|_{U_\alpha} = g_i(x^1, \cdots, x^n)dx^1 \wedge \cdots \wedge dx^n$$

where $g_i$ is a smooth function of compact support on the whole of $\mathbb{R}^n$ . We then define the integration of forms as

$$\int_M w = \sum_i \int_M \phi_i w = \sum_i \int_{\mathbb{R}^n} g_i(x^1, \cdots, x^n)dx^1 \wedge \cdots \wedge dx^n$$

The integral is well-defined precisely because of the change of variables formula in integration, and the consistent choice of sign from the orientation. Another indispensible tool in integration is stokes theorem.

**Theorem 4.** (Stoke's) Given a differential n-form $w$ whose support is the n-dimensional manifold M then

$$\int_M dw = \int_{\partial M} w$$

The reason that we explained about integration is to persuade the readers that differential forms contain information about the topology of the space since the integral does not depend on the parametrization of the manifold so no matter how crazy parametrization one chooses in the end the integral would be same.

In general n-forms in n-dimensional space represent volume, or other extensive quantities that appears in physics such as mass, pressure and temperature etc. If the reader prefers, value of the n-form at a point is the infinitesimal mass located at the point.

## Homology and Cohomology

Homology is a way of associating free abelian groups or modules with a topological spaces in our case manifolds. In a broad sense nth homology group or class of a manifold measures n dimensional "holes". There are lot of machinery involved in Homology and Cohomology calculations so we will skip the details about how to obtain such groups. Nevertheless I will try to give some intuition about what it is from my own experience.

**Definition 5.** A *chain complex* is a sequence of free abelian groups or modules with

$$\xrightarrow{\partial n+2} C_{n+1}(M) \xrightarrow{\partial n+1} C_n(M) \xrightarrow{\partial n} C_{n-1}(M) \xrightarrow{\partial n-1}$$

Each of the $\partial_i$ is called a *boundary map* which are homomorphism from the chain group (free ablelian group) $C_i$ to $C_{i-1}$ satisfying $\partial_i \partial_{i+1} = 0$ for all $i$. Furthermore *nth homology group* is

$$H_n = \frac{\ker(\partial_n)}{\mathrm{Im}(\partial_{n+1})}$$

The usual way that one calculates homology group is by building a simplicial complexes that is homotopic to the manifold and calculate the homotopy group of the simplicial complex. This works because two topological spaces with same homotopy type has isomorphic Homology groups. Furthermore one uses cellular complexes instead of simplicial complexes which do not change the homology groups. This is advantageous since we know that any manifold is homotopic to some cell complex. Hence we can calculate homology groups of any manifold by using cellular homology.

A most basic example of geometric object is maybe sphere. The homology groups of sphere is $H_m(S^n) = \mathbb{Z}$ if $m = 0, n$ and zero otherwise. For a closed surface of genus 3 the homology groups are $H_2(S_3) = \mathbb{Z}$,$H_1(S_3) = \mathbb{Z}^6$,$H_0(S_3) = \mathbb{Z}$ and zero otherwise. For most of the 2 and 3 dimensional manifold there are not many nonzero homology groups and it is useful to know the interpretations of the lower dimensional ones.

- $H_0(M) \cong \mathbb{Z}^p$ where $p$ is number of path components
- $H_1(M) \cong \mathbb{Z}^g$ where $g$ is the cardinality of the smallest set which generates the fundemental group.
- $H_2(M) \cong \mathbb{Z}^h$ where $h$ is the number of holes needed on the manifold in order to have inside and outside of the manifold to be path connected. If manifold is nonorientable then $H_2(M) = 0$

Few words of caution that our homology groups here are computed using $\mathbb{Z}$ coefficients but we can go back and forward between coefficients using Universal Coefficient Theorem, and coefficients in our case are not very important. Also people may have different views on the interpretations above, it is by no means only way to think about them but they are my way of interpreting it. For $H_2(M)$ a convienant way to imagine it is to fill the manifold with water and count the number of holes to poke in order to completely get rid of all the water inside the manifold.To persuade indeed the interpretation is correct observe that for any closed orientable surface with genus g the homology groups are $H_0(S_g) = \mathbb{Z}, H_1(S_g) = \mathbb{Z}^{2g}$ and $H_2(S_g) = \mathbb{Z}$.

Now we shall talk about cohomology since we developed enough knowledge of homology. Basically cohomology is a dual of homology groups.

**Definition 6.** A *cochain complex* is a sequence of free abelian groups or modules $C_i^* = \text{Hom}(C_i, G)$ where G is abelian group and $C_i$ is a chain complex.

$$\xleftarrow{\partial^* n+2} C_{n+1}^*(M) \xleftarrow{\partial^* n+1} C_n^*(M) \xleftarrow{\partial^* n} C_{n-1}^*(M) \xleftarrow{\partial^* n-1}$$

Each of the $\partial_i^*(\alpha) = \alpha(\partial_{i+1})$ is called a *coboundary map* which are homomorphism from the chain group (free ablelian group) $C_i^* - 1$ to $C_i^*$ satisfying $\partial_{i+1}^*\partial_i^* = 0$ for all $i$. Furthermore *nth cohomology group* is

$$H^n = \frac{\ker(\partial_{n+1}^*)}{\mathrm{Im}(\partial_n^*)}$$

There is very useful cohomology that is called De Rham cohomology. It turns out by De Rhams theorem that this cohomology is isomorphic to the singular cohomology groups which is generated from the singular chain complexes. To see what is De Rham cohomology is we will choose the cochain complexes as space of smooth k forms $\Omega_k$ as follows

$$\xrightarrow{d} \Omega_{n-1}(M) \xrightarrow{d} \Omega_n(M) \xrightarrow{d} \Omega_{n+1}(M) \xrightarrow{d}$$

where $d$ is a exterior derivative as we discussed before. This map is natural since exterior derivative of any k-form is k+1-form. Now we can look at the nth De Rham cohomology groups

$$H^n_{DR}(M) := \frac{\ker\left(d : \Omega^n(M) \to \Omega^{n+1}(M)\right)}{\mathrm{Im}\left(d : \Omega^{n-1}(M) \to \Omega^n(M)\right)}$$

We know that $dd = 0$ hence, $\ker\left(d : \Omega^n(M) \to \Omega^{n+1}(M)\right) = \{Closed\, n\; forms\}$ and similarly $\mathrm{Im}\left(d : \Omega^{n-1}(M) \to \Omega^n(M)\right) = \{Exact\ n\ forms\}$. As an example $H^m_{DR}(S^n) = \mathbb{R}$ for $m = 0, n$ and zero otherwise. As we mentioned before that $H^m_{DR}(M) \cong H^m(M, \mathbb{R})$ by De Rhams theorem so it is convenient way to understand the cohomology groups. Also by Poincare duality we have for n-dimensional compact oriented manifold $H^k(M, \mathbb{R}) \cong H_{n-k}(M, \mathbb{R})$. So we can define something called *k-th betti* number which is $\beta_k = \dim(H^m(M, \mathbb{R}))$. Betti numbers are good way to find n dimensional holes. Betti numbers are also used to define Euler characteristics which is another topological invariant

$\chi(M) = \sum_{i=0}^{\dim(M)} (-1)^i \beta_i$. One can classify all closed compact surfaces in terms of their Euler characteristics.

## Hodge Laplace Operator

In order to introduce laplace operator on a manifold we need to introduce a Hodge star operator. Hodge star operator is an isomorphism between smooth p forms to smooth d-p forms on compact d-dimensional Riemannian manifold.

$$* : \Omega_p(M) \to \Omega_{d-p}(M)$$

Throughout this section, we let $M$ be a compact, oriented, Riemannian manifold of dimension $d$. Using this we define an $L^2$ inner product on the space $\Omega^p(M)$ as follows

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta,$$

for any $\alpha, \beta \in \Omega_p(M)$. We can integrate this form since $\alpha \wedge *\beta \in \Omega_d$ which we can see is a top form. From this inner product we define the $L^2$ norms as $||\alpha||_{L^2} = \langle \alpha, \alpha \rangle$.

From the first section recall the exterior derivative $d : \Omega_p(M) \to \Omega_{p+1}(M)$. We define $d^*$ as the adjoint of $d$ with respect to the inner product $\langle ., . \rangle$.

$$(d\alpha, \beta) = (\alpha, d^*\beta)$$

for $\alpha \in \Omega^p(M)$, $\beta \in \Omega^{p+1}(M)$.Hence we see $d^* : \Omega^{p+1}(M) \to \Omega^p(M)$.We use the definitions of $d, d^*$ to introduce the notion of a harmonic $p$-form.

**Lemma 7.** $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ satisifies

$$d^* = (-1)^{d(p+1)+1} * d*$$

*Proof.* Let $\mu \in \Omega^{p-1}(M), \nu \in \Omega^p(M)$

$$\begin{aligned} d(\mu \wedge *\nu) &= d\mu \wedge *\nu + (-1)^{(p-1)}\mu \wedge d * \nu \\ &= d\mu \wedge *\nu + (-1)^{(p-1)}(-1)^{d-p+1}\mu \wedge * * (d * \nu) \\ &= d\mu \wedge *\nu - (-1)^{d(p+1)+1}\mu \wedge * * d * \nu \\ &= \pm * (\langle d\mu, \nu\rangle - (-1)^{d(p+1)+1}\langle \mu, *d * \nu\rangle \end{aligned}$$

Use Stokes's theorem to integrate and integral on the left hand side vanishes giving the desired equality. □

**Definition 8.** The Laplace-Beltrami or Hodge-Laplace operator $\Delta : \Omega^p(M) \to \Omega^p(M)$ is defined as

$$\Delta\alpha = dd^*\alpha + d^*d\alpha.$$

A form $w \in \Omega^p(M)$ is called harmonic if $\Delta w = 0$

**Lemma 9.** Let $w \in \Omega^p(M)$ then

$$(\Delta w, w) = (dd^*w, w) + (d^*dw, w) = (d^*w, d^*w) + (dw, dw) \geq 0$$

In particular, $\Delta w = 0$ if and only if $dw = 0$ and $d^*w = 0$

## Hodge Theorem

We are finally ready to see the statement of Hodge theorem and its consequences. However in order to prove Hodge theorem we need tools from differential equations and functional analysis. These results will be directly applied in the proof of Hodge later on.

## Analytic Preliminaries

We shall state few theorems that are going to be used in the proof of Hodge.

**Theorem 10.** (Sobolev Embedding) Let $\Omega \subset \mathbb{R}^n$ open and bounded, then

$$H_0^{k,p} \subset C^m(\bar{\Omega})\ for\ 0 \leq m < k - \frac{n}{p}$$

in particular, if $f \in H_0^{k,p}(\Omega)$ for all $k \in \mathbb{N}$ and some fixed p, then $f \in C^\infty(\bar{\Omega})$

**Theorem 11.** (Rellich-Kondrachov Compactness Theorem) Let $\Omega \subset \mathbb{R}^n$ open and bounded. Suppose $1 \leq q < \frac{np}{n-p}$ if $p < d$ and $1 \leq q < \infty$ if $p \geq d$. Then $H_0^{1,p}$ is compactly embedded in $L^q(\Omega)$. In other words $(f_n)_{n\in\mathbb{N}} \subset H_0^{1,p}$ satisfies

$$\|f_n\|^2_{W^{1,2}(\Omega)} \leq K\ for\ some\ K \in \mathbb{R}$$

then the subsequence converges in $L^q(\Omega)$

## Hodge Theorem and Applications

The basic idea of Hodge theory is that differentiability gives an information about continuouty which encodes informations about the underlying topology.

**Theorem 12.** (Hodge) Let M be a compact Riemannian manifold.Then every cohomology class $H^p(M)$ contains unique harmonic representative.

*Proof.* We will divide the proof into two parts that of existence and uniqueness. Between the two, uniqueness is much easier than existence hence we will show uniqueness proof first.

For the uniqueness, let $\xi_1$ $\xi_2$ be both harmonic p-form such that $\xi_1 - \xi_2$ is an exact form (this condition is called cohomologous ). We have two possible scenarios wether $p = 0$ or $p \geq 1$. If $p = 0$ then $\xi_1 = \xi_2$ trivially. So assume

$p \geq 1$ then

$$\begin{aligned}(\xi_1 - \xi_2, \xi_1 - \xi_2) &= (\xi_1 - \xi_2, d\eta) \\ &= (d^*(\xi_1 - \xi_2), \eta) \\ &= 0\end{aligned}$$

For some $\eta \in \Omega^{p-1}(M)$ and the last equality follows from the fact that $\xi_1, \xi_2$ are assumed to be harmonic which means $d^*\xi_i = 0$. Recall that the norm is positive definite hence we conclude that $\xi_1 = \xi_2$.

We will prove now the existence, let $\xi_0$ be closed form representing given cohomology class $H^p(M)$. Now consider set of all forms that are cohomologous to $\xi_0$ which can be written as $\xi = \xi_0 + d\eta$ for some $\eta \in \Omega^{p-1}(M)$. We will minimize $L^2$ norm in the class of all such forms.

In order to minimize we shall make use of Sobelev spaces, hence we define a Sobelev norm $\|.\|_{H^{1,2}}$:

$$\|\xi\|^2_{H^{1,2}} = (d\xi, d\xi) + (d^*\xi, d^*\xi) + (\xi, \xi)$$

We complete $\Omega^p(M)$ with respect to the norm we just defined, which gives us a Hilbert space $H^{1,2}(M)$ of p-forms. In order to make use of the tools we developed previously we need to be able to compare the norms. To recall the Sobolev norm in Euclidean space, let $V \in \mathbb{R}^d$ be open and $f : V \to \mathbb{R}^n$ then

$$\|f\|^2_{H^{1,2}_{\mathbb{R}}(V)} = \int_V f \cdot f + \int_V \partial_i f \cdot \partial_i f$$

notice we are using Einstein's summation convention and dot product as Euclidean scalar product. Since M is smooth manifold there exist charts and bundle charts for every $p \in M$, hence we can find an open neighborhood $U$

and corresponding diffeomorphism such that

$$\phi : \Lambda^p(M)|_U \to V \times \mathbb{R}^n$$

notice the dimension of fibers of $\Lambda^p$ is $n = \binom{n}{m}$. The fiber $x \in U$ is mappet to a fiber $\{\pi(\phi(x))\} \times \mathbb{R}^n$ where $\pi : V \times \mathbb{R}^n \to V$ is the projection map to the first component. Using this information we have the following useful lemma to bridge between sobolev space on manifolds and euclidnian spaces.

**Lemma 13.** On any $U' \Subset U$ the norms

$$\|\xi\|_{H^{1,2}_{\mathbb{R}}(U')} \ and \ \|\phi(\xi)\|_{H^{1,2}_{\mathbb{R}}(V')}$$

where $V' := \pi(\phi(U'))$ are equivalent

*Proof.* The theorem says that as long as we restrict ourselves on compact subsets of $U'$ then we will get an equivalence of norms. This is equivalent to showing that for any $x \in \bar{U}'$ there is a neighborhood $W$ such that above equivalence holds. Hence without loss of generality assume that $\pi \circ \phi$ is a map onto normal coordinates with center $x_0$ secondly metric in our neighborhood of $x_0$ we have the following

$$|g_{ij}(x) - \delta_{ij}| < \epsilon \ and \ |\Gamma^i_{j,k}| < \epsilon \ for \ i, j, k = 1, \cdots, d$$

Given this we see that if $(\alpha, \beta) = \alpha_{i_1,\cdots,i_p}\beta^{i_1,\cdots,i_p}$ then

$$\begin{aligned}
(d\alpha, d\beta) &= \frac{\partial \alpha_{ki_1,\cdots,i_p}}{\partial x^k}\frac{\partial \beta_{j_1,\cdots,j_p}}{\partial x^l} g^{kl} g^{i_1 j_1} \cdots g^{i_p j_p} \\
(d^*\alpha, d^*\beta) &= \Big( g^{kl}(\frac{\partial \alpha_{ki_1,\cdots,i_p}}{\partial x^l} - \Gamma^j_{kl}\alpha_{ji_1,\cdots,i_{p-1}})e_{i_1} \wedge \cdots \wedge e_{i_{p-1}}, \\
& \quad g^{mn}\frac{\partial \beta_{j_1,\cdots,j_p}}{\partial x^n} - \Gamma^r_{mn}\beta_{mj_1,\cdots,j_{p-1}})e_{j_1} \wedge \cdots \wedge e_{j_{p-1}} \Big)
\end{aligned}$$

as we can see under our assumtion for sufficiently small $\epsilon > 0$ above implies two neighborhood agrees. Since $\bar{U}' \subset U$ by initial assumption, claim for $U'$ follows by a covering argument. □

Above lemma implies that all results we introduced for sobolev spaces in the Euclidean setting is applicable to Riemannian situation . Specifically, Rellich-Kondrachov compactness theorem and Sobolev embedding theorem can be used in our proof.

**Lemma 14.** (Application of Rellich-Kondrachov) Let $(\xi_n)_{n\in\mathbb{N}} \subset H^{1,2}(M)$ be a bounded sequence, then subsequence of $(\xi_n)$ converges with respect to the $L^2$ norm

$$\|\xi\|_{L^2} := (\xi, \xi)$$

for some $\xi \in H^{1,2}(M)$

**Corollary 15.** There exist a constant $K$ such that for all closed forms $\eta$ that are orthogonal to the kernel of $d^*$ following holds

$$(\eta, \eta) \leq K(d^*\eta, d^*\eta)$$

where $K$ only depends on the Riemannian metric

*Proof.* Suppose on the contrary there above is not true then there would exist a sequence of closed forms $\eta_n$ orthogonal to the kernel of $d^*$ such that following is true

$$(\eta_n, \eta_n) \geq n(d^*\eta_n, d^*\eta_n)$$

Denote $\lambda_n := (\eta_n, \eta_n)^{-\frac{1}{2}}$ then we see that

$$(\lambda_n\eta_n, \lambda_n\eta_n) \geq n(d^*(\lambda_n\eta_n), d^*(\lambda_n\eta_n))$$

Since $\eta_n$ is closed by assumption we have

$$\|\lambda_n \eta_n\|_{H^{1,2}} \geq 1 + \frac{1}{n}$$

by the prevous lemma $\lambda_n \eta_n$ converges in $L^2$ to some form $\psi$. From above inequalities we see that $d^*(\lambda_n \eta_n)$ converges to 0 in $L^2$. Furthermore for all $\phi$

$$\lim_{n\to\infty}(d^*\lambda_n\beta_n, \phi) = \lim_{n\to\infty}(\lambda_n\beta_n, d\phi) \quad (1)$$

$$= (\psi, d\phi) = (d^*\psi, \phi) = 0 \quad (2)$$

Hence we see that $d^*\psi = 0$. Now since $d^*\psi = 0$ and $\eta_n$ are orthogonal to the kernel of $d^*$ we have

$$(\psi, \lambda_n\eta_n) = 0 \ for \ all \ n$$

On the other hand, $(\lambda_n\eta_n, \lambda_n\eta_n) = 1$ and $L^2$ convergence of $\lambda_n\eta_n$ to $\psi$ imply that

$$\lim_{n\to\infty}(\psi, \lambda_n\eta_n) = 1$$

but this is impossible. Hence by contradiction the inequality in the corollary holds. □

We are now finally have enought tools to complete Hodge. Recall we were trying to minimize over $L^2$ norm of forms $\xi = \xi_0 + d\eta$ for some $\eta \in \Omega^{p-1}(M)$. Let $(\xi_n)_{n\in\mathbb{N}}$ be minimizing sequence for $L^2$ norms $D := (\xi_n, \xi_n)$ in a fixed cohomology class. Using Dirichlet's principle in $\mathbb{R}^d$, the sequence $\xi_n$ converges weakly to some $\xi$. After selection of a subsequence we have

$$(\xi - \xi_0, \phi) = 0 \ for \ all \ \phi \in \Omega^p(M) \ with \ d * \phi = 0$$

above is because w $(\xi_n - \xi_0, \phi) = (d\eta_n, \phi) = 0$ hence $\xi - \xi_0$ is weakly exact.

We define a linear functional on $d^*(\Omega^p(M))$ by

$$L(\delta\phi) := (\eta, \phi)$$

One can check that above linear functional is well defined. For $\phi \in \Omega^p(M)$ let $\pi(\phi)$ be the orthogonal projection onto the kernel of $d^*$, and $\psi := \phi - \pi(\phi)$, which imply $d^*\psi = d^*\phi$. Then we have $L(d^*\phi) = L(d^*\psi) = (\eta, \psi)$ since $\psi$ is orthogonal to the kernel of $\delta$. By corollary 12 there is a constant $K$ such that

$$\|\psi\|_{L^2} \leq K \|d^*\psi\|_{L^2} = K \|d^*\phi\|_{L^2}$$

which imply that

$$|L(d^*\phi)| \leq K \|\eta\|_{L^2} \|d^*\phi\|_{L^2}$$

Hence the functional $L$ on $d^*(\Omega^p(M))$ is bounded which means that it can be extended to the $L^2$ closure of $d^*(\Omega^p(M))$. We will use Riesz Representation theorem which says that a bounded linear functional on a Hilber space is representable as the scalar product with an element of the space itself. By this theorem there exist $w$ such that

$$(w, d^*\phi) = (\eta, \phi)$$

for all $\phi \in \Omega^p(M)$. Thus we have $dw = \eta$ weakly. Thus $\xi = \xi_0 + \eta$ is contained in the closure of the considered class. Hence instead of minimizing over $\xi$ that is cohomologous to $\xi_0$ we can just as well minimize over space of all $\xi$ such that there exist some $w$ with

$$(w, d^*\phi) = (\eta, \xi - \xi_0) \; for \; all \; \phi \in \Omega^p(M)$$

Then $xi$ the weak limit of a minimizing sequence is contained in the class. In other words that suppose $\xi_n = \xi_0 + dw_n$ weakly meaning

$$L_n(d^*\phi) := (w_n, d^*\phi) = (w^n - w, \phi) \quad \forall \phi \in \Omega^p(M)$$

By the same estimate above linear functionals $L_n$ converge to some functional $L$ which again represented by some $w$. Since D is also weakly lower semi-continuous with respect to weak convergence, it follows that $c \geq D(\xi) \geq \lim_{n\to\infty} \inf D(\xi_n) = c$ which imply that $D(\xi)$. Furthermore by the Euler-Lagrange equations for D we see that if smooth form $\xi$ is the infimum achieved then

$$\begin{aligned} 0 &= \frac{d}{dt}(\xi + td\tau, \xi + td\tau)|_{t=0} && (3) \\ &= 2(\xi, d\tau) && (4) \\ &= 2(d^*\xi, \tau) \ \forall \tau \in \Omega^{p-1}(M) && (5) \end{aligned}$$

To show indeed that $\xi$ is smooth we use Sobolev embedding theorem. Hence we see that the minimizer of $D(\xi)$ is a harmonic form that always exist which completes the proof. □

The similar proof can be found using heat equations. This theorem is important since it gives rise to a nice theorem that allows one to classify all square integrable forms on the manifold.

**Theorem 16.** (Hodge Decomposition) Let $B_p$ be a the $L^2$ closure of $\{d\alpha : \alpha \in \Omega^{p-1}\}$ and similarly $B_p^*$ closure of $\{d^*\beta \in \Omega^{p+1}\}$ then space of square integrable p-form admits orthogonal decomposition

$$L_p^2(M) = B_p \oplus B_p^* \oplus \ker(\Delta_p)$$

Above classification of squere integrable forms have some interesting corollaries. Among these includes rather surprising facts about topology emerges, for instance if $M$ is a odd dimensional compact manifold then $\chi(M) = 0$. Another eye opening fact that follows from Hodge is that $\beta_n(M \times N) = \sum_{i=1}^{n} \beta_i(M)\beta_{n-i}(N)$ where $M, N$ are compact manifolds. This implies that $\chi(M \times N) = \chi(M)\chi(N)$ which simplifies life of topologists much easier.

The Hodge decomposion has also interesting applications in partial differntial equations. An obvious corrolary of hodge to differential equations is that the poisson equation $\Delta w = \alpha$ has a solution if and only if $\alpha$ is orthogonal to $\ker(\Delta)$ and a solution has a degree of smoothness two more than $\alpha$. These results are generalized using theory of pseudo differential operators, and not surprisingly one can prove Hodge using the language of psuedo differential operators.

As seen from these applications Hodge theorem is an important bridge that gaps the field of Partial Differential Equations and Algebraic Topology. Some problems that are hard in the nature of PDE's can easily approached via Algebraic topology and vice versa, hence a clever use of Hodge can open up many new possibilities in both fields.

## References


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