Exact solutions of the Einstein equations

Christophe Desjardins Morris

December 17, 2012

1 Einstein's equation in General Relativity

The Einstein equations are given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, G is the gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor.

All of these tensors can be written as 4×4 matrices. Since the energy-momentum tensor is symmetric, we have in principle a system of 10 partial differential equations. Solutions to this system of equations are metrics that describe the geometry of space-time, as it is affected by $T_{\mu\nu}$ [1] . To compute $G_{\mu\nu}$ in terms of the metric components, we must first compute the Christoffel symbols [2] :

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} &\equiv \frac{1}{2}g^{\lambda\sigma}[\mu\nu, \sigma] \\ &\equiv \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\end{aligned}$$

Then, we must compute the Riemann tensor elements :

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\lambda}_{\mu\sigma}$$

[1] We will write $T^{\nu 1\ldots\nu n}_{\mu 1\ldots\mu n}$ to denote both a tensor and its components.

[2] We use the Einstein convention, which states that a repeated index, once at the top and once at the bottom, are summed over.

Then, we contract the Riemann tensor to form the Ricci tensor :

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$

After, we compute the Ricci scalar:

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}$$

Finally, we compute $G_{\mu\nu}$ according to equation 1 from $R_{\mu\nu}$. A solution to the Einstein equations must satisfy 1. From the computation of $G_{\mu\nu}$, we see that this system of 10 2^{nd} order PDE is coupled and highly nonlinear. Therefore, exact solutions of 1 are very hard to find and to find those exact solutions, we must restrict ourselves to highly symmetric cases.

Sometimes, it is convenient to work in what is called the trace-reversed Einstein equation. To find it from 1, we first contract it to get

$$\begin{aligned} g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} R g_{\mu\nu} &= 8\pi G g^{\mu\nu} T_{\mu\nu} \\ \leftrightarrow R - \frac{1}{2} \delta^{\mu}_{\mu} R = R - 2R = -R &= 8\pi G T \end{aligned} \tag{2}$$

Then, plugging 2 in 1, we get

$$\begin{aligned} R_{\mu\nu} + \frac{1}{2}(8\pi G T) g_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \leftrightarrow R_{\mu\nu} &= 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) \end{aligned} \tag{3}$$

We will use 3 when $T_{\mu\nu}$ (and hence, T) vanishes.

2 Symmetries and Killing vectors

Definition 1 *Consider a diffeomorphism $\phi : M \to M$. It is a symmetry of a tensor T if*

$$\phi^* T = T$$

where $\phi^* : \bigotimes_{i=1}^{m} V^{(\mu_i)} \times \bigotimes_{j=1}^{n} V^{*(\nu_j)} \to \bigotimes_{i=1}^{m} V^{(\mu_i)} \times \bigotimes_{j=1}^{n} V^{*(\nu_j)}$ is the induced pullback of ϕ.

Definition 2 *A flow is a one-parameter family of diffeomorphisms, represented by a map*

$$\begin{aligned} \Phi : \mathbb{R} \times M &\to M \\ (t, p) &\mapsto \phi_t(p) \end{aligned}$$

where ϕ_t is a diffeomorphism satisfying $\phi_s \circ \phi_t = \phi_{s+t}\ \forall s, t$.

Definition 3 *The generator of a flow is the set of all tangent vectors of the curves*

$$\begin{aligned} \gamma_p : \mathbb{R} &\to M \\ t &\mapsto \phi_t(p) \end{aligned}$$

defined $\forall p \in M$, where the ϕ_t are diffeomorphisms induced by the flow.

Definition 4 *The Lie derivative of a tensor T along a vector field V is given by*

$$\mathcal{L}_V T \equiv \lim_{t \to 0} \left[\phi_t^* T(\phi(p)) - T(p)\right]$$

If there is a one-parameter family of symmetries of a tensor T generated by a vector field V, then the definitions 1 and 4 imply that $\mathcal{L}_V T = 0$.

Definition 5 *A symmetry of the metric tensor is called an isometry.*

Definition 6 *A vector field K is called a Killing vector field if it generates a one-parameter family of isometries, i.e.*

$$\mathcal{L}_K g_{\mu\nu} = 0$$

Theorem 1 *A vector field K is a Killing vector field if it satisfies the Killing equation:*

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{4}$$

Proof 1 *The Lie derivative of a general tensor $T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}$ is*

$$\begin{aligned} (\mathcal{L}_X T)^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m} = {} & \partial_\gamma T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m} X^\gamma \\ & - T^{\gamma \alpha_2 \ldots \alpha_n}_{\beta_1 \ldots \beta_m} \partial_\gamma X^{\alpha_1} - \ldots - T^{\alpha_1 \ldots \alpha_{n-1} \gamma}_{\beta_1 \ldots \beta_m} X^{\alpha_n} \\ & + T^{\alpha_1 \ldots \alpha_n}_{\gamma \beta_2 \ldots \beta_m} \partial_{\beta_1} X^\gamma + \ldots + T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_{m-1} \gamma} \partial_{\beta_m} X^\gamma \end{aligned}$$

By straightforward computation, we can show that we can replace all the partial derivatives by covariant ones : All the connection coefficients cancel to give the above formula anyway. Applying this formula for the metric, we get

$$\mathcal{L}_K g_\mu \nu = \nabla \gamma g_{\mu\nu} K^\gamma + g_{\gamma\nu} \nabla_\mu K^\gamma g_{\mu\gamma} \nabla_\nu K^\gamma = \nabla_\mu K_\nu + \nabla_\nu K_\mu$$

where the 1st term of the right-hand side is zero by metric compatibility. Thus, a non trivial Killing vector field must satisfy $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$ □

Theorem 2 *For a Killing vector field K, we have*

$$\nabla_\sigma \nabla_\mu K_\nu = R^\rho_{\sigma\mu\nu} K_\rho \tag{5}$$

Proof 2 *In a general setting, the Riemann tensor satisfies[1, p.122]*

$$[\nabla_\mu, \nabla_\nu] V_\sigma = R^\rho_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} \nabla_\lambda V^\rho \tag{6}$$

where [,] denotes the commutator. However, in a torsion-free setting, the last term in the right-hand side of equation 6 vanishes. By the antisymmetry of the 2 first indices in $R_{\rho\sigma\mu\nu}$, we can then show that

$$[\nabla_\mu, \nabla_\nu] V^\rho = -R^\rho_{\sigma\mu\nu} V_\rho \tag{7}$$

From equation 4, we then have

$$-R^\rho_{\sigma\mu\nu} K_\rho = [\nabla_\mu, \nabla_\nu] K_\rho = \nabla_\mu \nabla_\nu K_\sigma + \nabla_\nu \nabla_\sigma K_\mu$$

Then, using this equation with the 1st Bianchi identity, we get

$$0 = -(R^\rho_{\sigma\mu\nu} + R^\rho_{\mu\nu\sigma} + R^\rho_{\nu\sigma\mu}) K_\rho = 2(-R^\rho_{\sigma\mu\nu} K_\rho + \nabla_\sigma \nabla_\mu K_\nu)$$

Thus, $R^\rho_{\sigma\mu\nu} K_\rho = \nabla_\sigma \nabla_\mu K_\nu$ □

3 Maximally symmetric spaces

3.1 Consequences of maximal symmetry

Definition 7 *A space is homogeneous if it is invariant under any translation along a coordinate. It is isotropic if it is invariant under any rotation of a coordinate into another coordinate.*

The spaces with highest degree of symmetry are homogeneous and isotropic spaces. Let's count the number of symmetries that such a space has. Suppose first that it is a n-manifold. Thus, each point p of it has a neighbourhood homeomorphic to $\mathbb{R}^n$. Therefore, for that neighbourhood, we pick orthonormal coordinates. Homogeneity implies n one-parameter families of symmetries that are generated by the translation along each coordinate. Then, for each coordinate, there is $(n-1)$ other coordinates in which it can rotated. However, since we do not count a rotation from coordinate x^1 into coordinate x^2 and a rotation from coordinate x^2 into coordinate x^1 as separate one-parameter families of symmetries, we have $\binom{n}{2} = \frac{n(n-1)}{2}$ such families. Thus, the total number of those families are :

$$n + \frac{n(n-1)}{2} = \frac{2n + n^2 - n}{2} = \frac{n(n+1)}{2}$$

Definition 8 *An n-dimensional manifold is maximally symmetric if it has $\frac{n(n+1)}{2}$ linearly independant Killing vectors.*

Theorem 3 *The Riemann curvature tensor for any maximally symmetric n-manifold M at any point and in any coordinate system is*

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{8}$$

where n is the dimension of the space, R is the Ricci scalar, which is constant over M

Proof 3 *Since the commutator of 2 covariant derivatives is again a covariant derivative, we can apply the product rule in the calculation of $[\nabla_\mu, \nabla_\nu]\nabla_\sigma V_\rho$:*

$$[\nabla_\mu, \nabla_\nu]\nabla_\sigma V_\rho = -(R^\alpha_{\sigma\mu\nu}\nabla_\alpha V_\rho + R^\alpha_{\rho\mu\nu}\nabla_\sigma V_\alpha)$$

However, if $V_\rho \equiv K_\rho$ is a Killing vector field (which we turned into a 1-form by acting with $g_{\mu\nu}$), equations 4 and 5 apply and we can prove that[6, p.47]

$$\left[\nabla_\mu R^\alpha_{\nu\sigma\rho} - \nabla_\nu R^\alpha_{\mu\sigma\rho}\right] K_\alpha + \left[R^\beta_{\nu\sigma\rho}\delta^\alpha_\mu - R^\beta_{\mu\sigma\rho}\delta^\alpha_\nu + R^\alpha_{\sigma\mu\nu}\delta^\beta_\rho - R^\alpha_{\rho\mu\nu}\delta^\beta_\sigma\right] \nabla_\alpha K_\beta = 0 \tag{9}$$

Consider the vector composed of the n components of K_α and the $\frac{n(n-1)}{2}$ antisymmetric components of $\nabla_\alpha K_\beta$ (these are the only linearly independent *components of $\nabla_\alpha K_\beta$ since its symmetric part vanishes by equation 4. Thus, the system of equations 9 is a homogeneous system of equations in $\frac{n(n+1)}{2}$ dimensions. It is a linear system of equations of the form $\mathbf{A}\vec{x} = 0$. However, by the fact that M is a maximally symmetric manifold, it has $\frac{n(n+1)}{2}$ Killing vectors and thus, the rank of that system is zero. Therefore, we must have*

$$\begin{aligned} 0 &= \nabla_\mu R^\alpha_{\nu\sigma\rho} - \nabla_\nu R^\alpha_{\mu\sigma\rho} \\ 0 &= R^\beta_{\nu\sigma\rho}\delta^\alpha_\mu - R^\beta_{\mu\sigma\rho}\delta^\alpha_\nu + R^\alpha_{\sigma\mu\nu}\delta^\beta_\rho - R^\alpha_{\rho\mu\nu}\delta^\beta_\sigma \end{aligned} \tag{10}$$

By taking the 2^{nd} equation of 10 and multiplying it with δ^μ_α and lowering the index β, we get

$$(n-1)R_{\beta\nu\sigma\rho} = R_{\rho\nu}g_{\sigma\beta} - R_{\sigma\nu}g_{\rho\beta} \tag{11}$$

Then, using the antisymmetric proprieties of the Riemann tensor to write $R_{\beta\nu\sigma\rho} = R_{\nu\beta\rho\sigma}$ and contracting with $g^{\nu\rho}$, we get

$$(n-1)R_{\beta\sigma} = Rg_{\sigma\beta} - R_{\sigma\beta} \leftrightarrow nR_{\beta\sigma} - R_{\beta\sigma} = Rg_{\sigma\beta} - R_{\beta\sigma} \leftrightarrow R_{\beta\sigma} = \frac{Rg_{\beta\sigma}}{n} \tag{12}$$

Plugging the expression of $R_{\mu\nu}$ given in equation 12 in equation 11, we show that

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

All that is left to show now is that R is constant all over M. By using equation 11 in the 1^{st} equation of 10 and multiplying and contracting the resulting equation with $g^{\sigma\mu}$ and $g^{\rho\nu}$, we get

$$\begin{aligned} 0 &= \frac{1}{n(n-1)}g^{\sigma\mu}g^{\rho\nu}\left[g_{\sigma\alpha}g_{\rho\nu}\nabla_\mu R - g_{\rho\alpha}g_{\sigma\nu}\nabla_\mu R - g_{\sigma\alpha}g_{\rho\mu}\nabla_\nu R + g_{\rho\alpha}g_{\sigma\mu}\nabla_\nu R\right] \\ &= \frac{1}{n(n-1)}\left[\delta^\mu_\alpha\delta^\nu_\nu\nabla_\mu R - \delta^\nu_\alpha\delta^\mu_\nu\nabla_\mu R - \delta^\mu_\alpha\delta^\nu_\mu\nabla_\nu R + \delta^\nu_\alpha\delta^\mu_\mu\nabla_\nu R\right] \\ &= \frac{1}{n(n-1)}\left[n\delta^\mu_\alpha\nabla_\mu R - \delta^\mu_\alpha\nabla_\mu R - \delta^\nu_\alpha\nabla_\nu R + n\delta^\nu_\alpha\nabla_\nu R\right] \\ &= \frac{n-1}{n(n-1)} * 2\nabla_\alpha R = \frac{2}{n}\nabla_\alpha R \end{aligned}$$

This equations imply that in order for this equation to be true, we must have constant R over the manifold M. □

Maximally symmetric n-manifold can be classified by their signature (ex : Euclidean or Lorentzian signature), their Ricci scalar (positive, null or negative) and discrete information related to the global topology.

Theorem 4 *If we ignore questions about global topology, maximally symmetric n-manifolds with Euclidean signature are classified by their Ricci scalar. Therefore, depending on R, a maximally symmetric n-manifold is one of the followings :*

$$M = \begin{cases} \mathbb{H}^n & , R < 0 \\ \mathbb{R}^n & , R = 0 \\ \mathbb{S}^n & , R > 0 \end{cases}$$

When considering Lorentzian signature, the maximally symmetric space with $R = 0$ is called the Minkowski space, the maximally symmetric space with $R > 0$ is called the de Sitter space and the one with $R < 0$ is called the anti-de Sitter space.

3.2 Maximally symmetric solutions of the Einstein equations

By using equation 12 with $n = 4$, we get :

$$R_{\sigma\nu} = \frac{Rg_{\sigma\nu}}{4} \tag{13}$$

Thus, substitution of 13 in 1 yields

$$\begin{aligned} 8\pi G T_{\mu\nu} = \frac{Rg_{\mu\nu}}{4} - \frac{Rg_{\mu\nu}}{2} &= -\frac{Rg_{\mu\nu}}{4} \\ \leftrightarrow g_{\mu\nu} &= -\frac{32\pi G T_{\mu\nu}}{R} \end{aligned} \tag{14}$$

If $R = 0$, we get a trivial equation, which represent the Minkowski space-time. However, consider a metric $g_{\mu\nu}$ that has Lorentzian signature. Then, equation 14 implies that either $T_{00} < 0$ and $T_{ii} > 0$ for $i \in \{1, 2, 3\}$ or $T_{00} > 0$ and $T_{ii} < 0$ for $i \in \{1, 2, 3\}$, meaning that we can not have both positive energy and positive pressure. Therefore, this solution is regarded as unphysical. However, the Einstein equations are sometimes written with an additional term :

$$R_{\mu\nu} - \frac{Rg_{\mu\nu}}{2} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{15}$$

In that case, the maximally symmetric solutions are regarded as solutions of 15 with $\Lambda = \frac{R}{4}$. This term corresponds to vacuum energy.

3.3 de Sitter space

To obtain a de Sitter space, we have to embed an hyperboloïd given by

$$-u^2 + x^2 + y^2 + z^2 + w^2 = \alpha^2$$

in a 5 dimension Minkowski space with metric $ds_5^2 = -du^2 + dx^2 + dy^2 + dz^2 + dw^2$. Then, we introduction the coordinates $\{t, \xi, \theta, \phi\}$ on the hyperboloïd such that

$$\begin{aligned} u &= \alpha sinh(t/\alpha) \\ w &= \alpha cosh(t/\alpha) cos\xi \\ x &= \alpha cosh(t/\alpha) sin\xi cos\theta \\ y &= \alpha cosh(t/\alpha) sin\xi sin\theta cos\phi \\ z &= \alpha cosh(t/\alpha) sin\xi sin\theta sin\phi \end{aligned} \tag{16}$$

Then, the metric in terms of these coordinates is[3]:

$$ds^2 = -dt^2 + \alpha^2 cosh^2(t/\alpha) \left[d\xi^2 + sin^2\xi (d\theta^2 + sin^2\theta d\phi^2) \right] \tag{17}$$

The expression between the square brackets is the metric on $\mathbb{S}^3$. Thus, we see that the de Sitter space is a 3-sphere that shrinks to a minimum size at $t = 0$ and re-expands.

3.4 Anti-de Sitter space

The anti-de Sitter space is represented by the following hyperboloid

$$-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2$$

embedded in a 5-manifold with metric $ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$. By introducing coordinates $\{t, \rho, \theta, \phi\}$ such that

$$\begin{aligned} u &= \alpha sin(t) cosh(\rho) \\ v &= \alpha cos(t) cosh(\rho) \\ x &= \alpha sinh(\rho) cos(\theta) \\ y &= \alpha sinh(\rho) sin(\theta) cos(\phi) \\ z &= \alpha sinh(\rho) sin(\theta) sin(\phi) \end{aligned} \tag{18}$$

[3] see Appendix 1

. The metric on such hyperboloid is [4]

$$ds^2 = \alpha^2 \left[-cosh^2(\rho)dt^2 + d\rho^2 + sinh^2(\rho)d\Omega_2^2 \right] \tag{19}$$

where $d\Omega_2^2$ represents the metric of $\mathbb{S}^2$.

From the definition of the coordinates, we see that t is periodic, meaning that t and $t + 2\pi$ represents the same point on the hyperboloid. This seems problematic because since ∂_t is a timelike vector [5] and therefore, the curve

$$\gamma : \mathbb{R} \to M \tau \mapsto (\tau, \rho_0, \theta_0, \phi_0)$$

with ρ_0, θ_0 and ϕ_0 constant will be a closed timelike curve, meaning that someone can meet himself in the past just by staying where he is! However, we note that we obtained the metric from a particular embedding and thus, this propriety is not an intrinsic propriety. Therefore, we define the anti-de Sitter space as an infinite covering space of this hyperboloid in which t ranges from $-\infty$ to $+\infty$.

4 Robertson-Walker metrics

A good approximation of our universe is a universe that is *spatially* maximally symmetric, but not maximally symmetric as a whole. Therefore, let's consider such a universe.

Definition 9 *A universe (thus, a Lorentzian manifold) is spatially maximally symmetric if it can be foliated by a one-parameter family of spacelike hypersurfaces Σ_t that are maximally symmetric.*

Thus, for a spatially maximally symmetric universe, we can write $M = \mathbb{R} \times \Sigma_t$. Isotropy is a very restrictive condition. First, we can rule out the $g_{tx^{(i)}}$ [6] components of the metric. Then, we can rule out dependence on space of g_{tt} and we know that any dependence on t of the spacelike components must be the same for each component. We can finally choose coordinate such that the time dependence on the g_{tt} is absorbed in the time dependence

[4] see Appendix 2

[5] see Appendix 2 again

[6] In this discussion, $x^{(i)}$ represents a spacelike coordinate for $i \in \{1, 2, 3\}$.

of the $g_{x^{(i)}x^{(i)}}$ components. Therefore, the metric of such universe can be written as follow :

$$ds^2 = -dt^2 + R^2(t)d\sigma^2 \tag{20}$$

where t is our timelike coordinate, $R(t)$ is called the scale factor and $d\sigma^2$ is the metric on the 3-manifold Σ[7]. Since Σ is maximally symmetric, we have, by theorem 3 that

$${}^{(3)}R_{ijlm} = k(\gamma_{il}\gamma_{jm} - \gamma_{im}\gamma_{jl}) \tag{21}$$

where the superscript $^{(3)}$ denotes a 3 dimensional space and γ_{ij} is the Euclidean metric of Σ. Knowing this and the fact that k is constant, if we can enumerate all spaces (ignoring questions about global topology) with all possible values of k, then we are done. This is precisely what theorem 4 allows us to do : we have $\mathbb{R}^3$ for $k = 0$, $\mathbb{S}^3$ for $k > 0$ and $\mathbb{H}^3$ for $k < 0$. By redefining our coordinates such that $k \in \{-1, 0, 1\}$, the corresponding metrics are given by :

$$d\sigma^2 = d\chi^2 + f(\chi)^2 d\Omega^2 \tag{22}$$

where $d\Omega$ is the metric on $\mathbb{S}^2$ and

$$f(\chi) = \begin{cases} sin(\chi) & k = 1 \\ \chi & k = 0 \\ sinh(\chi) & k = -1 \end{cases} \tag{23}$$

The full spacetime metric is then

$$ds^2 = -dt^2 + R(t)\left[d\chi^2 + f(\chi)^2 d\Omega^2\right] \tag{24}$$

Metrics of the form of equation 24 are called Robertson-Walker metrics. Note that we didn't use the Einstein equations yet : we first have to specify the form of the energy-momentum tensor. On the scales that we are dealing with, we can consider $T_{\mu\nu}$ to be the energy-momentum tensor of a perfect fluid. Then by isotropy (considering the coordinates for which the perfect fluid is at rest), we can write

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & g_{11}p & g_{12}p & g_{13}p \\ 0 & g_{21}p & g_{22}p & g_{23}p \\ 0 & g_{31}p & g_{32}p & g_{33}p \end{pmatrix}$$

[7] We can remove the subscript $_t$ since $R(t)$ controls the time dependance of the spatial components of the metric.

However, from the fact that $g_{\mu\nu}$ is a Robertson-Walker metric, we get from equation 24 that the $T_{ij} = 0$ for $i \neq j$ and that in cartesian coordinates, $T_{11} = T_{22} = T_{33}$. Hence, we are left with 2 independent components of $T_{\mu\nu}$ and hence, 2 Einstein equations : one for T_{00} and one for T_{ii} for a given $i \in \{1, 2, 3\}$. These two equations yield the Friedmann equations that decribe how $R(t)$ is related to the energy and pressure densities.

5 Schwarzschild solution

Spatially homogeneous solutions to the Einstein equations are well suited to describe the entire universe, since the Copernician principle makes us strongly believe that it is on large scales. However, it is not well suited to describe local inhomogeneities such as the exterior proximity of a star.

However, if the star is not moving, we can consider perfect spherical symmetry, which is defined as follow[2, Appendix B, p.369] :

Definition 10 *A Lorentzian manifold (and hence, a spacetime) is spherically symmetric if it admits the group $SO(3)$ as a group of isometries having spacelike 2-surfaces as the group orbits.*

We note that these group orbits are necessarily 2-surfaces of constant positive curvature (and thus, spheres).

Said in another way, it is spherically symmetric if it possess 3 Killing vectors R, S, T satisfying the algebra of angular momentum (these form the Lie algebra of the generators of $SO(3)$) :

$$[R, S] = T$$
$$[S, T] = R$$
$$[T, R] = S$$

5.1 Birkhoff's theorem

We are interested into the Einstein equations in vaccuum. Setting $T_{\mu\nu} = 0$ in 3, we get

$$R_{\mu\nu} = 0 \tag{25}$$

This system of equations is called the vaccuum Einstein equations and its solutions are called vaccuum solutions. It turns out that that the spherically symmetric solution to the vaccuum Einstein equations is *unique*. Even more interesting : it turns out that it is *static*.

Definition 11 *A Lorentzian manifold (a spacetime) is static if it possesses a timelike Killing vector that is orthogonal to a family of hypersurfaces.*

Operationally, this means that we can choose coordinates such that this Killing vector is ∂_t and such that the metric is independent of t. Furthermore, it is orthogonal to a family of hypersurfaces if and only if $g_{tx^{(i)}} = 0$ where $x^{(i)}$ is a spacelike coordinate ($i \in \{1, 2, 3\}$). Thus, we are left with a metric of the following form :

$$ds^2 = g_{tt}(\{x^{(i)}\}_{i=1}^3)dt^2 + g_{ij}(\{x^{(i)}\}_{i=1}^3)dx^i dx^j \tag{26}$$

Theorem 5 (Birkhoff) *Any C^2 solution of the vaccuum Einstein equations which is spherically symmetric in an open set U is locally equivalent to part of the maximally extended Schwarzschild solution in U.[2, p.372]*

We will define the maximally extended Schwarzschild solution in a later subsection. For now, let prove this theorem, following the arguments in [2, Appendix B]

Proof 4 *Consider $I = SO(3)$ to be the group of isometries and from a point $p \in M$, call $\mathscr{S}(p)$ the orbit of p. We have that for each point $q \in \mathscr{S}(q), \exists I_q$, a one-dimensional group of isometries which leaves q invariant. Denote $\mathscr{C}(q)$ the set of all geodesics orthogonal to $\mathscr{S}(q)$. $\mathscr{C}(q)$ locally form a 2-surface left invariant by I_q since I_q changes only the directions in $\mathscr{S}(q)$. Then, we can consider U_p to be an neighbourhood of p that is left invariant by I_q.*

Then, consider $r \in U_p$ and the set $V_r \subset T_rM$ of tangent vectors orthogonal to U_p. Since U_p is left invariant by R_q, R_q is a map from V_r to V_r. However, R_q acts in the group orbit $\mathscr{S}(r)$. Thus, $\mathscr{S}(r)$ is orthogonal to U_p. Thus, we have shown that the group orbits $\mathscr{S}$ are everywhere orthogonal to $\mathscr{C}$, the surface formed from the union of neighbourhoods similar to U_p.

$\forall r \in U_p, \exists! C_{q-r} \in \mathscr{C}(q)$, a geodesic from q to r that is left invariant by I_q. Thus, we have a bijective map $f : \mathscr{S}(q) \to \mathscr{S}(r)$ where $f(q)$ is the intersection of $\mathscr{C}(q)$ with $\mathscr{S}(r)$. This map is invariant under I_q. Thus, vectors of equal magnitudes in $\mathscr{S}(q)$ are mapped into vectors of equal magnitudes in $\mathscr{S}(r)$. Furthermore, since M is invariant under $SO(3)$, this map is independent of the point in $\mathscr{S}(q)$ and thus, the same magnitude multiplication factor

occurs for $f_* : T_pM \to T_rM$ regardless of $p \in \mathscr{S}(p)$ and its image in $\mathscr{S}(r)$. Thus the surfaces $\mathscr{C}$ map trajectories conformally onto each other and thus, M is foliated by 2-spheres.

By choosing coordinates $\{t, r, \theta, \phi\}$ such that $\mathscr{S}$ are the surfaces $\{t,\ r =$ constant$\}$ and such that $\mathscr{C}$ are the surface $\{\theta, \phi =$ constant$\}$, we have that

$$ds^2 = d\tau^2(t,r) + Y^2(t,r)d\Omega^2$$

Furthermore, if we choose r and t that are orthogonal to each other, we have

$$ds^2 = \frac{-dt^2}{F^2(t,r)} + X^2(t,r)dr^2 + Y^2(t,r)d\Omega^2$$

Using the Einstein equations, we get the following 4 equations :

$$0 = \frac{2X}{R}\left(\frac{\dot{Y}'}{Y} - \frac{\dot{X}Y'}{XY} + \frac{\dot{Y}F'}{YF}\right) \tag{27}$$

$$0 = \frac{1}{Y^2} + \frac{2}{X}\left(-\frac{Y'}{XY}\right)' - 3\left(\frac{Y'}{XY}\right)^2 + 2F^2\frac{\dot{X}\dot{Y}}{XY} + F^2\left(\frac{\dot{Y}}{Y}\right)^2 \tag{28}$$

$$0 = \frac{1}{Y^2} + 2F\left(F\frac{\dot{Y}}{Y}\right)^{\cdot} + 3\left(\frac{\dot{Y}}{Y}\right)^2 F^2 + \frac{2}{X^2}\frac{Y'F'}{YF} - \left(\frac{Y'}{XY}\right)^2 \tag{29}$$

$$0 = \frac{1}{X}\left(-\frac{F'}{FX}\right)' - F\left(F\frac{\dot{X}}{X}\right)^{\cdot} - 2F\left(F\frac{\dot{Y}}{Y}\right)^{\cdot} - F^2\left(\frac{\dot{X}}{X}\right)^2 - 2F^2\left(\frac{\dot{Y}}{Y}\right)^2 + \frac{1}{X^2}\left(\frac{F'}{F}\right)^2 - \frac{2}{X^2}\frac{Y'F'}{YF} \tag{30}$$

where $'$ denotes ∂_r and $\dot{}$ denotes ∂_t.

The local solution depends on the nature of the hypersurfaces $\{Y =$ constant$\}$ (timelike, spacelike, null or undefined). However, if these hypersurfaces are null or undefined on some open set U, then we have

$$\frac{Y'}{X} = F\dot{Y}$$

in U. However, when this equation holds, equation 28 is inconsistent with equation 27. Thus, we have proven that $\forall p \in M | Y^{;\mu}_{;\mu} > 0 or Y^{;\mu}_{;\mu} < 0, \exists U$, neighbourhood of p such that $Y^{;\mu}_{;\mu} > 0$ (resp.$Y^{;\mu}_{;\mu} < 0$) in U [8].

We consider first the case $Y^{;\mu}_{;\mu} < 0$. Then the hypersurfaces $\{Y =$ constant$\}$ are timelike in U and we can choose $Y = r$. Thus, $\dot{Y} = 0$ and $Y' = 1$.

[8]Here, ; represents the covariant derivative.

Replacing these expression in 27 reveals $\dot{X} = 0$. *Equation 29 then shows* $\left(\frac{F'}{F}\right)^{\cdot} = 0$. *Thus, we can set* $F = F(r)$. *Thus, the metric can now be written as*

$$ds^2 = \frac{-dt^2}{F^2(r)} + X^2(r)dr^2 + r^2 d\Omega^2 \tag{31}$$

. *Hence, we have proven one of the most interesting parts of the theorem : M is* static. *Then, equation 28 shows* $\frac{d(rX^2)}{dr} = 1$. *Solving this for* X^2 *is straightforward and one gets*

$$X^2 \left(1 - \frac{2m}{r}\right)^{-1}$$

where $2m$ *is a constant of integration. Equation 29 can then be integrated to give* $F^2 = X^2$, *with equation 30 automatically satisfied. Thus, our metric takes the form*

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2$$

which is the Schwarzschild solution for $r > 2m$.

Now consider the case $Y_{;\mu}^{;\mu} > 0$. *Then, the surfaces* $\{Y = constant\}$ *are spacelike in* U *and we can choose* $Y = t$. *We then have* $\dot{Y} = 1$ *and* $Y' = 0$. *Equation 27 show that* $F' = 0$. *By choosing* $X = X(t)$, *we have* $F = F(t), X = X(t)$ *and* $Y = t$. *By integrating 29 and 30, we get that the metric is*

$$ds^2 = -\frac{dt^2}{\frac{2m}{t} - 1} + \left(\frac{2m}{t} - 1\right) dr^2 + t^2 d\Omega^2$$

By making the transformations $t \to r'$ *and* $r \to t'$, *we readily see that we recover the Schwarzschild solution for* $r < 2m$.

Finally, when the hypersurfaces $\{Y = constant\}$ *are spacelike in some part of* U *and timelike in another part of* U, *we simply obtain the solutions for each part separately and glue them along* $Y_{;\mu}^{;\mu} = 0$. *Thus,* $\forall U \in M$ *such that* M *is a solution of* $R_{\mu\nu} = 0$ *that is spherically symmetric in* U, M *is equivalent to a part of the maximally extended Schwarzschild solution in* U. *In the case where* $U = M$, *then* M *is the maximally extended Schwarzschild solution.* $\square$

5.2 Analysis of the Schwarzschild solution

As derived in the previous section, the Schwarzschild solution is given by the following metric :

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)}dr^2 + r^2 d\Omega^2 \tag{32}$$

To understand the geometry of this solution, we would like to study the causal structure caused by null geodesics. Since the Schwarzschild solution has spherical symmetry, we are led to consider radial null geodesics : those for which $ds^2 = 0$, $d\theta = 0$ and $d\phi = 0$. Thus, equation 32 becomes :

$$\begin{aligned} 0 &= -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)}dr^2 \\ \leftrightarrow \frac{dt}{dr} &= \pm\left(1 - \frac{2GM}{r}\right)^{-1} \end{aligned} \tag{33}$$

where the $+$ sign refers to outgoing radial null geodesics and the $-$ sign refers to infalling radial null geodesics. From equation 33, we see that as $r \to 2GM^+$, $\frac{dt}{dr} \to \infty$. Thus, a light ray seems to never reach $r = 2GM$. However, this is only what seems to happen for an outside observer. We will show in this section that a radial null geodesic actually reaches $r = 2GM$, where strange things happen.

we begin by integrating 33 to get

$$t = \pm\left(r + 2GMln\left(\frac{r}{2GM} - 1\right)\right) + C \equiv \pm r^* + C \tag{34}$$

Where C is a constant. We then defined a new coordinate in terms of which the Schwarzschild metric becomes :

$$ds^2 = \left(1 - \frac{2GM}{r}\right)(-dt^2 + dr^{*2}) + r(r^*)^2 d\Omega^2$$

Since $-dt^2$ and dr^{*2} are multiplied by the same factor, the light cones do not close up as $r \to 2GM$. However, we must still choose other coordinate system since from equation 34, we see that the hypersurface $r = 2GM$ goes to $-\infty$.

To characterise the radial null geodesics, we define the light cone coordinates

$$u = t - r^*$$
$$v = t + r^*$$

such that $u = C$ for outgoing radial null geodesics and $v = C$ for infalling radial null geodesics (see equation 34). We then express r^* in terms of r to get the metric in what is called the Eddington-Finkelstein coordinates $\{v, r, \theta, \phi\}$:

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dv^2 + (dvdr + drdv) + r^2 d\Omega^2 \tag{35}$$

From equation 35, we have $Det(g_{\mu\nu}) = -r^4 sin^2\theta$ and thus, as long as $\theta \neq 0$ or $\theta \neq \pi$, the metric is invertible and $r = 2GM$ doesn't represent a geometrical singularity. Equation 33 can be solved in those coordinates to give :

$$\frac{dv}{dr} = \begin{cases} 0 & \text{infalling null geodesics} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{outgoing null geodesics} \end{cases} \tag{36}$$

From equation 36, we have that the curves of constant v on a graph of (v, r) are horizontal lines. We can also find the curves of constant u by recalling that $u = t - r^* = v - 2r^*$. The curves of constant u are then given by

$$v(r) = C + 2\left(r + 2GMln\left(\left|\frac{r}{2GM} - 1\right|\right)\right) \tag{37}$$

A graph of (v, r) is represented in Appendix 3 for $C = 0$ (both for the curves of constant u and those of constant v) and $2GM = 1$. Timelike vectors are therefore inside the angles made by the tangent vectors of both curves. The result is the following :

1. In the region $r > 2GM$, the curves of constant u have positive slopes. Thus, future directed light rays (that must go up in the (v, r) graph) can go away from $r = 0$.

2. As $r \to 2GM$, the slopes of the curves of constant u have $\frac{dv}{dr} \to \infty$. Thus, future directed light rays at $r = 2GM$ cannot go away from $r = 0$. At most, they can stay at $r = 2GM$.

3. In the region $r < 2GM$, the curves of constant u have negative slopes and thus, future directed light rays have choice, but to go towards $r = 0$.

From this discussion, we see that $r = 2GM$ is a point of no return, in the sense that no future directed timelike path starting from the region $r < 2GM$ can go outside of that region. Thus, $r = 2GM$ defines what is called an event horizon and the region $r < 2GM$ is called a black hole.

5.3 Maximal extension of the Schwarzschild solution

Now, we start by using both u and v coordinates. However, we have to do more, since in those coordinates, the hypersurface $r = 2GM$ is at infinity. To bring it back to a finite value of r, we define the following coordinates :

$$v' = e^{v/4GM}$$
$$u' = e^{-u/4GM}$$

Then, to analyse the Schwarzschild solution, we are more comfortable using a coordinate system for which one coordinate is timelike and the others are spacelike. Therefore, we define the Kruskal coordinates

$$\begin{aligned} T &= \frac{v' + u'}{2} = \left(\frac{R}{2GM} - 1\right)^{1/2} e^{r/4GM} sinh\left(\frac{t}{4GM}\right) \\ R &= \frac{v' - u'}{2} = \left(\frac{R}{2GM} - 1\right)^{1/2} e^{r/4GM} cosh\left(\frac{t}{4GM}\right) \end{aligned} \tag{38}$$

in terms of which the metric becomes

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dR^2) + r^2 d\Omega^2$$

r is implicitly defined from

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} \tag{39}$$

The curves of constant r satisfy $T^2 - R^2 = C, C \in \mathbb{R}$. They are therefore hyperbolae in the (R, T) plane. Then, from equation 38, the curves of constant t satisfy $\frac{T}{R} = tanh\left(\frac{t}{4GM}\right)$. Thus, they are straight lines with slope $tanh\,(t/4GM)$.

We note that $\lim_{t\to\pm\infty} tanh\,(t/4GM) = \pm 1$ Thus, the line for which $t \to \pm\infty$ satisfy $T = \pm R$. From equation 39, we note that the event horizon $r = 2GM$ also satisfy that equation.

Since the only geometrical singularity of the Schwarzschild solution is represented by the curve of constant $r = 0$, we can allow R to range from $-\infty$ to ∞. Then, T can be calculated from 39 and must satisfy

$$\begin{aligned}
T^2 - R^2 &= \left(1 - \frac{r}{2GM}\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{r}{2GM}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{r}{2GM}\right)^n - \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{r}{2GM}\right)^{n+1} \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{r}{2GM}\right)^n - \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\frac{r}{2GM}\right)^n \\
&= 1 + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \left(\frac{1}{n!} - 1\right) \left(\frac{r}{2GM}\right)^n < 1
\end{aligned}$$

Thus, we have extended the Schwarzschild solution to the space inside the hyperbola $r = 0$ (see figures 1 and 2.

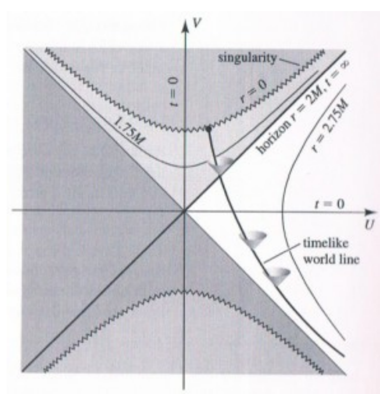

Figure 1: Kruskal diagram of the Schwarzschild solution [http://www.askamathematician.com/2009/11/q-if-black-holes-are-rips-in-the-fabric-of-our-universe-does-it-mean-they-lead-to-other-universes-if-so-then-did-time-begin-in-that-universe-at-the-inception-of-the-black-hole-could-we-be-in/]

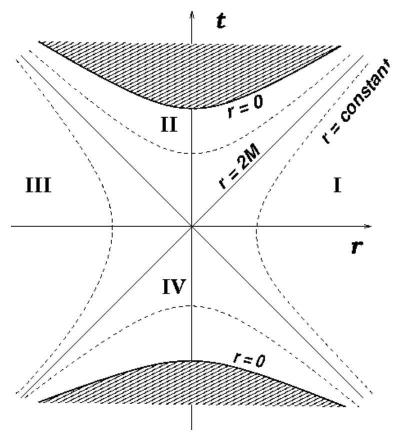

Figure 2: Kruskal diagram of the maximally extended Schwarzschild solution [http://faculty.iisertvm.ac.in/shanki/thesis.html]

6 Reissner-Nordström solution

We now turn to a spherically symmetric solution which is not a vaccuum solution. In that case, spherical symmetry still allows us to write the metric in the following form

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\Omega^2$$

For the energy-momentum tensor, we consider an electromagnetic field. In that case, we have

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

where is the eletromagnetic tensor [5]. We don't know its form yet (see [5] for its expression in the Minkowski space) since it is determined by the solution, but we can deduce some of its proprieties by spherical symmetry :

$$\begin{aligned} F_{tr} \equiv f(r,t) = -F_{rt} \\ F_{\theta\phi} \equiv g(r,t) sin\theta = -F_{\phi\theta} \\ F_{tt} = F_{t\theta} = F_{t\phi} = F_{rr} = F_{r\theta} = F_{r\phi} = F_{\theta\theta} = F_{\phi\phi} = 0 \end{aligned} \tag{40}$$

To find our solution, we must then solve the set of Einstein equations and the following Maxwell equations :

$$\begin{aligned} g^{\mu\nu}\nabla_{\mu}F_{\nu\rho} = 0 \\ \nabla_{[\mu}F_{\nu\rho]} = 0 \end{aligned} \tag{41}$$

The solution can be found by a similar procedure that the one followed to find the Schwarzschild solution. That solution is called the Reissner-Nordström solution and it is given by

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega \tag{42}$$

where

$$\Delta = 1 - \frac{2GM}{r} + \frac{G(Q^2 + P^2)}{r^2} \tag{43}$$

In equation 43, Q is the total electric charge and P is the total magnetic charge.

6.1 Location of an event horizon

Definition 12 *A Lorentzian manifold (spacetime) is said to be stationary if it possess a Killing vector that is timelike near infinity.*

In such metrics, we can choose coordinates (t, x^1, x^2, x^3) in which that Killing vector is ∂_t and the metric components are independent of t.

Definition 13 *A Lorentzian manifold (spacetime) is said to be asymptotically flat if*

$$\lim_{r \to \infty} g_{\mu\nu} = \eta_{\mu\nu}$$

where r is understood to be a radial coordinate and $\eta_{\mu\nu}$ is the Minkowski metric.

The two properties we have just defined allow us to quickly find event horizons in some cleverly chosen coordinates. First, we express the metric in a way such that its components are time-independent and that ∂_t is a Killing vector. Then, we choose coordinates (r, θ, ϕ) such that the metric tends to the Minkowski metric in spherical polar coordinates. Finally, we can choose coordinates such that as r decreases, the $r = C, C \in \mathbb{R}$ hypersurfaces remain timelike $\forall(\theta, \phi)$ until some radius $r = r_H$ is reached. Then, for $r < r_H$, ∂_t will become a spacelike vector and thus, future-directed timelike paths will never escape back to infinity. Therefore, r_H is an event horizon.

In particular, $r = r_H$ is a null hypersurface. Therefore, its tangent vectors satisfy $\xi_\mu \xi^\mu = 0$, meaning that they also serve as normal vector to that hypersurface. However, since we are considering $r = C, C \in \mathbb{R}$ hypersurfaces, $\partial_\mu r$ is a one-form normal to these hypersurface and we have :

$$g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{\mu\nu}(\delta^r_\mu)(\delta^r_\nu) = g^{rr}$$

Thus, the condition for r_H to be an event horizon is

$$g^{rr}(r_H) = 0 \tag{44}$$

6.2 Penrose diagrams

The Reissner-Nordström solution has some complication that call for some extra tools for their analysis. In this section, we consider again the light-cone coordinates (u', v') introducted in subsection 5.3 for which the hypersurface $r = 2GM$ was situated at finite coordinate values, but we apply an additional change of coordinates before introducing the Kruskal coordinates. This additional change is designed to bring the range of our coordinates to finite values. This allow us to better explore the causal structure of our solution and to extend it more that we would have done if we had coordinate with infinite ranges.

The way to do it is first to have good null coordinates such as (u', v'), for which the event horizons are not situated at infinity. Then, we use the arctangent to bring infinities into finite coordinate values. The exact form of the arctangent will depend of the form of the metric in the null coordinates, but the idea is always the same. Let's give two examples.

6.2.1 Minkowski spacetime

In the Minkowski spacetime, the metric in null coordinates is given by[1, p.473]:

$$ds^2 = -\frac{1}{2}(dudv + dvdu) + \frac{1}{4}(v-u)^2 d\Omega^2$$

Then, the appropriate coordinate change to bring infinity into finite coordinate values is:

$$U = arctan(u)$$
$$V = arctan(v)$$

To revert to a coordinate system in which one coordinate is timelike and one coordinate is spacelike, we simply define :

$$T = V + U$$
$$R = V - U$$

The Minkowski metric in those coordinates is $ds^2 = -dT^2 + dR^2 + sin^2 R d\Omega^2$. The Penrose diagram that is represented at figure 3 is an extended Minkowski space (allowing $R < 0$).

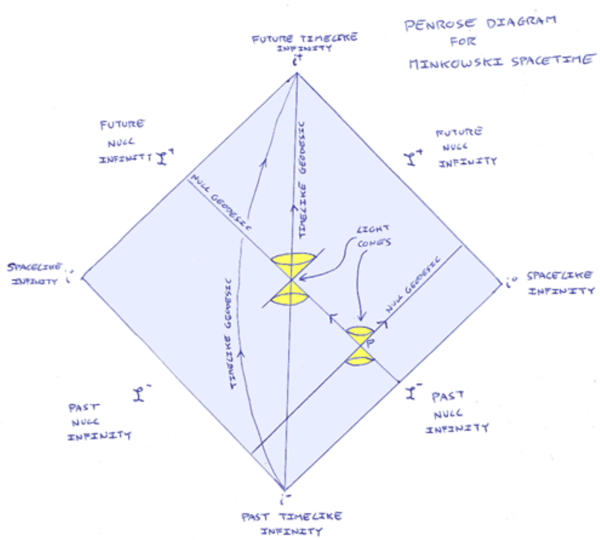

Figure 3: Penrose diagram for the Minkowski spacetime [http://www.theculture.org/rich/sharpblue/archives/000189.html]

6.2.2 Schwarzschild solution

In terms of the null coordinates (u', v') given in subsection 5.3 for the Schwarzschild solution, the metric could be written as follow :

$$ds^2 = -\frac{16G^3M^3}{r} e^{-r/2GM} (dv'du' + du'dv') + r^2 d\Omega^2$$

Then, the appropriate coordinate change to bring infinities into finite coordinate values is:

$$v'' = arctan\left(\frac{v'}{\sqrt{2GM}}\right)$$
$$u'' = arctan\left(\frac{u'}{\sqrt{2GM}}\right)$$

After the following change of coordinates :

$$T = \frac{1}{2}(v'' + u'')$$
$$R = \frac{1}{2}(v'' - u'')$$

We get the Penrose coordinates for the Schwarzschild solution (see figure 4).

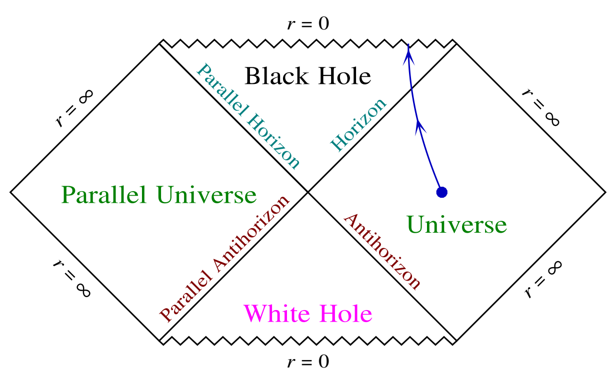

Figure 4: Penrose diagram for the Schwarzschild solution [http://jila.colorado.edu/ ajsh/insidebh/penrose.html]

6.3 Finding of an event horizon of the Reissner-Nordström solution

From equation 42, we first see that the metric components are independant of t and that ∂_t is a Killing vector. By taking the limit of that metric as $r \to \infty$, we also see that ∂_t is timelike near ∞ and that the metric tends to the Minkowski metric in spherical polar coordinates as $r \to \infty$. Finally, as we decrease r by leaving the other coordinates fixed, we see that the $r = C, C \in \mathbb{R}$ hypersurfaces remain timelike until a certain r_H. Therefore, we can apply the condition found in subsection 6.1 to find an event horizon of the Reissner-Nordström solution. Thus, we want to find r_H such that

$$\begin{aligned} g^{rr}(r_H) = \Delta(r_H) = 1 - \frac{2GM}{r_H} + \frac{G(Q^2 + P^2)}{r^2} &= 0 \\ \leftrightarrow r^2 - 2GMr + G(Q^2 + P^2) &= 0 \end{aligned} \tag{45}$$

This equation can easily be solved by using the discriminant method :

$$r_\pm = GM \pm \sqrt{G^2M^2 - G(Q^2 + P^2)} \tag{46}$$

Therefore, the Reissner-Nordstrom solution has 0, 1 or 2 event horizons and we have to consider each case separately.

6.3.1 $GM^2 < Q^2 + P^2$

In this case, Δ is always positive and thus, never 0. Therefore, this Reissner-Nordström is perfectly regular until $r = 0$. The causal structure of this spacetime is the same as the one of Minkowski spacetime except for that singularity. Thus, this solution is not very interesting.

6.3.2 $GM^2 > Q^2 + P^2$

In this case, we have 2 event horizons given by equation 46. Thus we have that the Killing vector ∂_t is

- Timelike for $r > r_+$ and $r < r_-$.
- Spacelike for $r_- < r < r_+$.

The maximal extension of this solution is quite interesting. We first start with 3 regions :

1. from infinity to r_+ : away from the black hole
2. from r_+ to r_- : black hole
3. from r_- to 0 : wormhole

Then, we extend exactly as in the Schwarzschild case to get 3 other regions, which are the mirror regions of the 3 regions we started with. Thereafter, we realize that we can extend more : what was future null infinity in the very first region is also actually the r_- event horizon of a white hole in another universe. Therefore, we can extend the Reissner-Nordström solution to include that white hole and a new region away from holes. Likewise, in the other way, what was past null infinity in the first mirror region is also the r_- event horizon of a mirror black hole in a third universe. Thus, we can extend further the Reissner-Nordström solution to include this mirror black hole as well as a mirror region away from that black hole.

By continuing the extension, we get a manifold representing an infinite amount of universes which are connected by what is called wormholes. This name comes from the fact that even if by crossing $r = r_+$, a future directed timelike path must eventually cross $r = r_-$, once it has crossed it, it doesn't have to go to $r = 0$. It can turn in order to cross the $r = r_-$ event horizon of a white hole in another universe, after which it must eventually cross $r = r_+$ of that white hole and emerge into this other universe (see figure **??**).

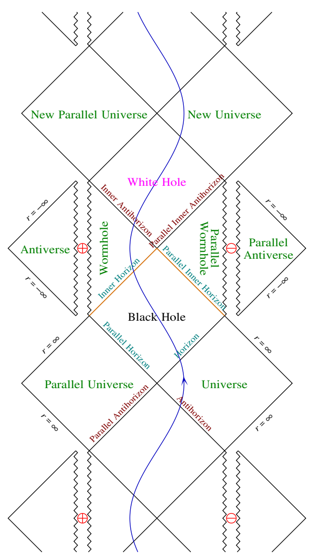

Figure 5: Penrose diagram for the Reissner-Nordström solution [http://jila.colorado.edu/ ajsh/insidebh/penrose.html]

6.3.3 $GM^2 = Q^2 + P^2$

This last case is called the extreme Reissner-Nordström solution. One of its main characteristics is that the radial coordinate r never becomes timelike. It does become null as $r = GM$, but it is spacelike on both sides. It is extended in the same way as the last case, but the $r = 0$ always stays at the same side of the extension.

7 Kerr solution

Another interesting case to consider is the proximity of a rotating star. It is physically motivated by the fact that stars have in general non-zero angular momentum and thus, they are spinning along some axis. However, in this case, the solution is not be spherically symmetric. Therefore, it doesn't possess $SO(3)$ as a group of isometries and we can not, as in the spherically symmetric case, find 3 Killing vector field satisfying the albegra of angular momentum. However, the solution is axisymmetric, with regard to the axis of rotation. Therefore, we can keep one of the 3 generators of $SO(3)$ as a Killing vector field of the solution.

Since we are considering asymptotically flat solutions, we consider coordinates (t, r, θ, ϕ) that tend to the spherical polar coordinates in the Minkowski spacetime. In that case, we can choose ∂_ϕ to be the generator of $SO(3)$ that we keep as a Killing vector of our solution and expect a metric having components independent of ϕ.

Another symmetry is lost : the solution will not be static, as the Schwarzschild and the Reissner-Nordström solutions, since a static solution must be symmetric under time reversal $t \to -t$, which is clearly not the case for a space with a rotating star, since that time reversal would reverse the angular momentum of that star. However, it remains stationary and thus, $\exists$ a Killing vector, which we take to be ∂_t, that is timelike near infinity. Therefore, we expect a solutions with components that are independent of t.

The general form of a stationary metric in coordinates (t, x^1, x^2, x^3) for which ∂_t is a Killing vector is the following :

$$ds^2 = g_{tt}(\{x^i\}_{i=1}^3)dt^2 + g_{ti}(\{x^i\}_{i=1}^3)(dtdx^i + dx^i dt) + g_{ij}(\{x^i\}_{i=1}^3)dx^i dx^j \quad (47)$$

Therefore, we expect cross terms of the form $g_{ti}(\{x^i\}_{i=1}^3)(dtdx^i + dx^i dt)$ that were absent both in the Schwarzschild and Reissner-Nordström solutions. We actually expect a term of the form $g_{t\phi}(r,\theta)(dtd\phi + d\phi dt)$ in the solutions.

The Kerr solution is a lot more complicated than the other solutions found so far. It is the following :

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{2GMarsin^2\theta}{\rho^2}(dtd\phi + d\phi dt) \\ &+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta sin^2\theta\right] d\phi^2 \end{aligned} \tag{48}$$

where

$$\Delta(r) = r^2 - 2GMr + a^2 \tag{49}$$

and

$$\rho^2(r,\theta) = r^2 + a^2cos^2\theta \tag{50}$$

Let us note that there is also a charged version of this solution that is called the Kerr-Newman solution. However, all of the essential phenomena persist when we remove the charges and thus, we will consider only the Kerr solution.

a represents the angular momentum per unit mass of the rotating star. We can check easily that as $a \to 0$, we recover the Schwarzschild solution. What is more interesting is what happens when we let $M \to 0$. In that case, we should recover the Minkowski spacetime. However, by doing so, we realize that (t, r, θ, ϕ) are not spherical polar coordinates, but rather ellipsoidal coordinates. These are related to Cartesian coordinates in $\mathbb{R}^3$ by the following coordinate transformations :

$$\begin{aligned} x &= (r^2 + a^2)^{1/2} sin\theta cos\phi \\ y &= (r^2 + a^2)^{1/2} sin\theta sin\phi \\ z &= rcos\theta \end{aligned} \tag{51}$$

In term of these coordinates, the metric of the Minkowski space is

$$ds^2 = -dt^2 + \frac{r^2 + a^2cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2cos^2\theta)^2 d\theta^2 + (r^2 + a^2)sin^2\theta d\phi^2$$

7.1 Event horizons and ergosphere

By straightforward computations, we can show that

$$\lim_{r\to\infty} g_{tt} = -1$$
$$\lim_{r\to\infty} g_{t\phi} = \lim_{r\to\infty} g_{\phi t} = 0$$
$$\lim_{r\to\infty} g_{rr} = 1$$
$$\lim_{r\to\infty} g_{\theta\theta} = \lim_{r\to\infty} r^2$$
$$\lim_{r\to\infty} g_{\phi\phi} = \lim_{r\to\infty} r^2 sin^2\theta$$

Thus, as $r \to \infty$, the metric tends to the Minkowski metric in spherical polar coordinates. We have tried to show that as r decreases, the hypersurfaces $r = C, C \in \mathbb{R}$ remain everywhere timelike until some fixed $r = r_H$ where it becomes everywhere null in appendix 4. Do to so, we tried to show that the normal vector fields of those hypersurfaces, given by [1, p.443]

$$\xi^\mu = g^{\mu\nu}\nabla_\nu r$$

remains everywhere timelike until r_H where it becomes everywhere null. However, since the ξ^μ vector that we have found depends on θ, we couldn't end the proof. However, it seems to be true according to [1, p.263]. Thus, the event horizons occur at values of r_H such that $g^{rr}(r_H) = 0$. However, $g^{rr} = \frac{\Delta}{\rho^2}$ (see appendix 4) and $rho^2 > 0$ for $r > 0$. Thus,

$$g^{rr}(r_H) = 0 \leftrightarrow \Delta(r_H) = r_H^2 - 2GMr_H + a^2 = 0 \tag{52}$$

This equation has at most 2 solutions that are found by the discriminant method. We will consider the case where there is 2 solutions. They are given by

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2} \tag{53}$$

Definition 14 *A null hypersurface Σ is said to be a Killing horizon of a Killing vector χ^μ if χ^μ is null everywhere on Σ.*

In the Schwarzschild and in the Reissner-Nordström solutions, all of the event horizons were Killing horizons of ∂_t. However, this is not the case in the Kerr solution because of its stationary nature. By computing the norm of ∂_t in the Kerr solution, we get :

$$\partial_t.\partial_t = g_{tt} = -\left(1 - \frac{2GMr}{r^2 + a^2cos^2\theta}\right) \tag{54}$$

Setting 54 equal to 0, we get that

$$\partial_t \text{is null} \leftrightarrow r^2 + a^2cos^2\theta - 2GMr = 0 \tag{55}$$

By rearranging this equation, we see that the Killing horizon of ∂_t is the locus of points satisfying

$$(r^2 - GM)^2 = G^2M^2 - a^2cos^2\theta \tag{56}$$

By rearranging equation 53, we get that the event horizons are rather locus of points satisfying

$$\begin{aligned}(r_\pm - GM)^2 &= G^2M^2 - a^2 \\ r_\pm &= GM \pm \sqrt{G^2M^2 - a^2}\end{aligned} \tag{57}$$

Thus, ∂_t becomes spacelike before the event horizon. The Killing horizon of ∂_t is called the ergosphere and the space between the r_+ event horizon and the ergosphere is called the ergoregion.

An interesting thing about the ergoregion is that no geodesic contained inside of it can move against the rotation of the star that generates the Kerr metric. To see this, we consider a photon emitted in the ϕ direction in the $\theta = \frac{\pi}{2}$ plane. It follows a null geodesic and hence,

$$ds^2 = 0 = g_{tt}dt^2 + g_{t\phi}(dtd\phi + d\phi dt) + g_{\phi\phi}d\phi^2$$

This equation is solved to obtain

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \tag{58}$$

On the surface defining the Killing horizon of ∂_t, we have $g_{tt} = 0$ and the solutions are

$$\frac{d\phi}{dt} = 0$$

$$\frac{d\phi}{dt} = \frac{a}{2G^2M^2 + a^2}$$

The 2^{nd} solution is interpreted as a photon moving in the same direction as the star's rotation, since it has the same sign as a. The 1^{st} solution means that the photon directed against the hole's rotation doesn't move at all. It is straightforward to show that for $r > r_+$, $g_{\phi\phi} > 0$ therefore, inside the ergoregion, equation 58 tells us that even for a photon directed against the star's rotation, $\frac{d\phi}{dt} < 0$, meaning that nothing can move against the star's rotation inside of the ergoregion.

7.2 Maximal extension of the Kerr solution

First, we can realize, by calculating $R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu}$ that $\rho = 0$ represents a geometrical singularity. Since $\rho^2(r,\theta) = r^2 + a^2cos^2\theta$, the only way that ρ can be zero is if $r = 0$ and $\theta = \frac{\pi}{2}$. By looking at equation 51, we realize that $(r,\theta) = \left(0, \frac{\pi}{2}\right)$ actually represents a ring. Therefore, we can extend the Kerr solution to the interior of that ring (i.e. in the region $r < 0$).

To do so, we first transform to Kerr-Schild coordinates $(x, y, z, \bar{t})$ defined as follow :

$$\begin{aligned} x + iy &= (r + ia)sin\theta e^{i\int(d\phi + a\Delta^{-1}dr)} \\ z &= rcos\theta \\ \bar{t} &= \int(dt + (r^2 + a^2)\Delta^{-1}dr) - r \end{aligned} \tag{59}$$

For which r is implicitly determined up to a sign in term of x, y, z by

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2z^2 = 0$$

In those coordinates, the surface $r = C, C \in \mathbb{R}$ are confocal ellipsoids in the (x, y, z) space that degenerate to the disk $x^2 + y^2 \leq a^2, z = 0$ when $r = 0$. Then, we consider two such spaces : (x, y, z) for which $r \geq 0$ and (x', y', z') for which $r \leq 0$. We identify a point on the top side of the disc $x^2 + y^2 < a^2$ with the point with the same coordinates in (x', y', z') at the bottom side of the disc $x'^2 + y'^2 < a^2$. Likewise, we identify a point on the bottom side of the disk $x^2 + y^2 < a^2$ with the point with the same coordinates in (x', y', z') at the top side of the disk $x'^2 + y'^2 < a^2$.

Then, the metric in Kerr-Schild coordinates extend to this new manifold by letting r range from $-\infty$ to $+\infty$ and the metric on the region (x', y', z') is a Kerr metric, but with $r < 0$. Thus, there is no event horizon in the region (x', y', z'). This region also contains closed timelike curves. To see this, consider the curve that winds around in ϕ, keeping $\theta = \frac{\pi}{2}$ and t constant with a very small negative value of r (relative to a). Then, the line element of such a path can be approximated as follow :

$$
\begin{aligned}
ds^2 = g_{\phi\phi}\left(r, \frac{\pi}{2}\right) d\phi^2 &= \frac{1}{r}(r^3 + a^2(r + 2GM))d\phi^2 \\
&=\approx a^2 \left(1 + \frac{2GM}{r}\right) d\phi
\end{aligned}
$$

This path is obviously timelike for $0 < r < 2GM$ and it is closed since ϕ is a periodic coordinate.

we can also extend the Kerr solution by choosing coordinates that are regular at the event horizons. The result is similar to the Reissner-Nordström case in the sense that the Penrose diagram of the Kerr solution can be extended to include an infinite amount of regions of spacetime. The only significant difference is that each universe (represented by a single Kerr metric that has been extended beyond the ring singularity) has a region beyond its ring singularity that contains closed timelike curves.

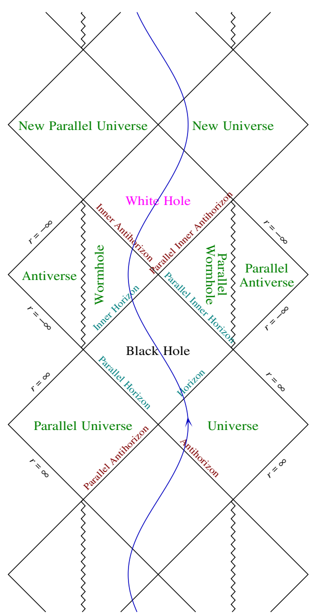

Figure 6: Penrose diagram for the Kerr solution [http://jila.colorado.edu/ ajsh/insidebh/penrose.html]

8 Conclusion

In this essay, we have spoken about the most important exact solutions to the Einstein equations (with the exception of the Minkowski spacetime, which is the most important) and have introduced a lot of geometrical formalism along the way. We enumerate them here in order of complexity and of treatment. Note that they are also in order from the most symmetrical to the least symmetrical.

1. The Minkowski solution
2. The Sitter and anti-de Sitter solution
3. The Robertson-Walker solution
4. The Schwarzschild solution
5. The Reissner-Nordström solution
6. The Kerr solution

A more complete treatment of these solutions and of many others can be found in [2, Chapter 5].

We strongly encourage those interested to read more about them to have a better understanding of their geometrical structure. There is something inspiring about them.

References

[1] SEAN M. CARROLL, *Spacetime and Geometry, An Introduction to General Relativity*, Addison-Wesley, 2004, 513 pages

[2] HAWKING, ELLIS, *The Large Scale Structure of Space-Time*, Cambridge University Press, 1973, 391 pages

[3] ROBERT M. WALD, *General Relativity*, The University of Chicago Press, 1984, 495 pages

[4] PRUDENCE CATHAWAY-KJONTVEDT, *Schwarzschild Geometry in association with Black Holes*, California Lutheran University, 2007/05/09,

15 pages, http://www.callutheran.edu/schools/cas/programs /mathematics/people/documents/finalpresCathaway.pdf, page consulted on 2012-12-11

[5] ROBERT G. BROWN, *Covariant Formulation of Electromagnetism*, Duke University Physics Departement, http://www.phy.duke.edu/ rgb/Class/phy319/phy319/node135.html, page consulted on 2012-12-10

[6] MARKUS DESERNO, *Notes on differential geometry with special emphasis on surfaces in* $\mathbb{R}^3$, Department of chemistry and biochemistry, UCLA, Max-Planck Institut für Polymerforschung, 2004, 64 pages