MATH 580 LECTURE NOTES 1: EXAMPLES OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In these notes, we learn about several fundamental examples of partial differential equations, and get a glimpse of what will be covered in the course. A tiny bit of historical information is included, and an attempt is made to emphasize the interrelations between the equations.

Contents

1. Transport equations

A linear homogeneous transport equation reads

$$\alpha \cdot \nabla u = 0, \tag{1}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field subject to some smoothness condition (such as continuity), and $\alpha \cdot \nabla u \equiv \sum_{i=1}^n \alpha_i \partial_i u$ is the directional derivative of u along the vector α. Here ∂_i denotes the partial derivative along the i-th coordinate direction. The equation tells us that if we consider a small neighbourhood around any point x, the function u must be constant along the direction $\alpha(x)$. Consequently, if there is a differentiable curve γ going through x, whose tangent at each of its points is aligned with α at that point, then u must be constant along γ. We know from the theory of ordinary differential equations that a unique family of such curves exist, going through every point in space, whenever α is Lipschitz continuous. So in this case, we see that any differentiable function that does not vary along the integral curves of α will be an honest solution of (1). Obviously this is a lot of solutions. To get a precise idea of exactly "how many" solutions are there, we can proceed as follows. Imagine a smooth hypersurface $\Gamma \subset \mathbb{R}^n$ (i.e., a surface of dimension $n-1$) that cuts nontangentially through every integral curve of α exactly once. Whether it is possible to find such a hypersruface, or how to construct them are interesting questions, but for the purposes of this discussion we assume that it is possible (imagine for example, a nonzero constant vector field α). Every solution u of (1) defines a differentiable function f on Γ given by the restriction of u to Γ. Conversely, every function $f : \Gamma \to \mathbb{R}$ can be uniquely extended to $\mathbb{R}^n$ so that the extension is constant along the integral curves of α, and we will prove later in this course that if f is differentiable then the extension is differentiable and satisfies (1). Hence, we conclude that

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the solutions of (1) are in a one-to-one correspondence with the differentiable functions on Γ. Intuitively, the equation (1) restricts the behaviour of u exactly as much as going from functions on $\mathbb{R}^n$ (or on a piece of it) to functions defined on an $n-1$ dimensional surface.

A special case of (1) is

$$\frac{\partial u}{\partial t} + \beta \cdot \nabla u = 0, \tag{2}$$

where one of the independent variables is singled out as "time" t. Here $\beta(x,t) \in \mathbb{R}^n$, and the directional derivative $\beta \cdot \nabla$ is with respect to the variable $x \in \mathbb{R}^n$. In this case, the hyperplane $\{(x,t) : t = 0\}$ can always (i.e., no matter what β is) play the role of the surface Γ as in the previous paragraph. So if we consider the equation (2) together with the condition $u(x,0) = f(x)$, $x \in \mathbb{R}^n$, for some differentiable f, then the resulting problem has a unique solution. This is called the *initial value problem* or *Cauchy problem* for (2) with the *initial data* f. One can give a physical meaning to this problem by saying that the initial mass (or charge) distribution f is being transported under the velocity field β.

Later in this course, we will study the more general inhomogeneous quasilinear transport equation

$$\alpha(x,u) \cdot \nabla u = \phi(x,u). \tag{3}$$

Note that now the velocity field α can depend on u itself, and the growth of u along α is given by a function of x and u.

2. Laplace, Poisson, and Cauchy-Riemann equations

In his investigations on Newtonian gravitation, Pierre-Simon Laplace was led around 1782 to the Laplace equation

$$\Delta u = 0, \tag{4}$$

where $\Delta = \nabla^2 = \partial_1^2 + \ldots + \partial_n^2$ is called the *Laplace operator*, or the *Laplacian*. It should be noted however that the same equation had been written down by Joseph-Louis Lagrange in 1760 in connection with his study of fluid flow problems. By 1687 Isaac Newton had discovered that the force exerted on a point mass Q at $x \in \mathbb{R}^3$ by another point mass q at $y \in \mathbb{R}^3$ is given by

$$F = \frac{CqQ}{|x-y|^2}\frac{x-y}{|x-y|}, \tag{5}$$

with a universal constant $C < 0$ (like masses attract). Moreover, the same law of interaction between point charges was published in 1785 by Charles Augustin de Coulomb, now with $C > 0$ (like charges repel). Observe that

$$\frac{\partial}{\partial x_1}\left(|x|^2\right)^{-3/2} x_1 = -\frac{3}{2}\left(|x|^2\right)^{-5/2} \cdot 2x_1 \cdot x_1 + \left(|x|^2\right)^{-3/2} = \frac{-2x_1^2 + x_2^2 + x_3^2}{|x|^5}, \tag{6}$$

and so

$$\begin{aligned} \nabla \cdot \frac{x}{|x|^3} &= \frac{\partial}{\partial x_1}\left(|x|^2\right)^{-3/2} x_1 + \frac{\partial}{\partial x_2}\left(|x|^2\right)^{-3/2} x_2 + \frac{\partial}{\partial x_3}\left(|x|^2\right)^{-3/2} x_3 \\ &= \frac{-2x_1^2 + x_2^2 + x_3^2}{|x|^5} + \frac{-2x_2^2 + x_1^2 + x_3^2}{|x|^5} + \frac{-2x_3^2 + x_1^2 + x_2^2}{|x|^5} = 0, \end{aligned} \tag{7}$$

where $\nabla \cdot E = \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3$ is the divergence of the vector field $E = (E_1, E_2, E_3)$. Therefore

$$\nabla \cdot F = 0, \qquad \text{except at} \quad x = y. \tag{8}$$

Observe also that

$$\frac{\partial}{\partial x_1}\frac{1}{|x|} = \frac{\partial}{\partial x_1}\left(|x|^2\right)^{-1/2} = -\frac{1}{2}\left(|x|^2\right)^{-3/2} \cdot 2x_1 = -\frac{1}{|x|^2} \cdot \frac{x_1}{|x|}, \tag{9}$$

and so

$$\nabla\frac{1}{|x|} = -\frac{1}{|x|^2}\cdot\frac{x}{|x|}, \tag{10}$$

where $\nabla v = (\partial_1 v, \partial_2 v, \partial_3 v)$ is the gradient of the scalar field v. This shows that

$$F = -Cq\nabla u, \qquad \text{with} \qquad u(x) = \frac{Q}{|x-y|}, \tag{11}$$

the latter called the *potential*, and therefore

$$\Delta u = \nabla\cdot\nabla u = \nabla\cdot F = 0, \qquad \text{except at} \quad x = y. \tag{12}$$

By the principle of superposition, the field generated by a finite number of point charges also satisfies the same equation in free space. Here free space means the set of points not occupied by the charges. A bit more involved analysis on the extension of the law (5) to continuous distribution of charges will show that the fields generated by continuous distribution of charges in space or on smooth surfaces also satisfies the Laplace equation in free space. Obviously, we do not need the Laplace equation if we know the distribution of charges: We would just use (5) or its counterpart for continuous distribution of charges to calculate the field. However, there are important situations where the density of the distribution must be implied from some indirect information. For example, imagine a closed surface in space, with some charges distributed inside and possibly also outside of it. Then we pose the problem of replacing the charges inside the surface by charges at the surface, so that the potential outside the surface remains the same. This amounts to finding a function u satisfying $\Delta u = 0$ inside the surface, that agrees with the old values of the potential at the surface. The functions satisfying $\Delta u = 0$ are called *harmonic functions*, and the afore-mentioned problem is called the *Dirichlet problem*. Characterizing harmonic functions and solving the Dirichlet problem is much more complicated than characterizing the solutions of (1) and solving the corresponding Cauchy problem. Just to get some feel about harmonic functions, let us consider harmonic functions that are polynomials in $\mathbb{R}^2$. In 1 dimension, all harmonic functions are simply linear functions. Likewise in 2 dimensions, all linear and bilinear polynomials are harmonic. However, there are more harmonic polynomials in $\mathbb{R}^2$, such as $x^2 - y^2$ and $y^3 - 3x^2y$. Playing with some explicit examples will reveal that harmonic polynomials do not have any maximum or minimum points; for example polynomials like $x^4 + y^4$ can never be harmonic. If the gradient of a harmonic polynomial vanishes at some point, then this point is necessarily a saddle point, like the point $(0, 0)$ for $x^2 - y^2$. This is actually the tip of the iceberg known as *mean value property* and *maximum principles*, which hold for general harmonic functions.

One special case of a Dirichlet problem that can be solved explicitly is when the underlying domain is a ball:

$$u(x) = \int_{S^{n-1}} \frac{1-|x|^2}{\omega_n|x-y|^n} u(y)\,\mathrm{d}S_y, \qquad (|x| < 1), \tag{13}$$

where $S^{n-1} \subset \mathbb{R}^n$ is the $n-1$ dimensional unit sphere in $\mathbb{R}^n$, and ω_n is its surface area. The formula for $n = 2$ and $n = 3$ was discovered by Siméon Denis Poisson in 1820-23. Later in this course, we will study the Dirichlet problem in great detail for more general domains.

The inhomogeneous counterpart of the Laplace equation is the *Poisson equation*

$$\Delta u = f, \tag{14}$$

which is satisfied, e.g., by the electric potential generated by the charge distribution with density proportional to f. This equation was derived by Poisson in 1813.

Finally, Augustin-Louis Cauchy observed in 1814 that with $u(x, y)$ and $v(x, y)$ being differentiable and real valued, complex differentiability of $u+iv$ as a function of $x+yi$ is equivalent to

$$\begin{bmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \equiv \begin{bmatrix} \partial_x u - \partial_y v \\ \partial_y u + \partial_x v \end{bmatrix} = 0. \tag{15}$$

However, Cauchy did not make much use of these differential equations. It was Georg Friedrich Bernhard Riemann who put them at the front stage in his 1851 dissertation to build up his function theory. Hence the equations are nowadays called the *Cauchy-Riemann equations*. Their solutions, i.e., holomorphic functions, share many of the properties of the harmonic functions. In fact, the Cauchy-Riemann system and the Laplace equation are the simplest representatives of the big family of *elliptic equations*, whose theory subsumes to a great extent that of the formers. It should be remarked that the Cauchy-Riemann equations seem to have appeared the first time in 1752 in a work of Jean le Rond d'Alembert on the theory of fluid flow.

Exercise 1. Suppose that u and v satisfy the Cauchy-Riemann equations (15), and that they both have continuous second derivatives. Show that each of them satisfies the Laplace equation.

3. Wave equation

The wave equation in 1 dimension was written down by d'Alembert in 1747 as a model of small vibrations of a stretched elastic string. It was extended to 2 dimensions by Leonhard Euler in 1759 as a model of vibrations of drumheads, and to 3 dimensions by Daniel Bernoulli in 1762 as a model of sound propagation. In general, the homogeneous wave equation reads

$$\frac{\partial^2 u}{\partial t^2} = \Delta u. \tag{16}$$

When the preceding equation is considered in the whole x-space $\mathbb{R}^n$, explicit formulas can be given for the solution $u(x,t)$ in terms of $u(x,0)$ and $u_t(x,0)$, where u_t is of course the t-derivative of u. Hence in the *Cauchy problem for the wave equation*, one needs to give both functions $u(x,0)$ and $u_t(x,0)$ as initial data. The formula for $n=1$ is called *d'Alambert's formula*, which reads

$$u(x,t) = \frac{u(x-t,0)+u(x+t,0)}{2} + \frac{1}{2}\int_{x-t}^{x+t} u_t(y,0)\,\mathrm{d}y, \tag{17}$$

where if $t<0$, the integral over $(x-t,x+t)$ is understood to be the minus of the integral over $(x+t,x-t)$. It is discovered by Euler in 1748 based on the earlier work of d'Alambert. The formula

$$u(x,t) = \frac{1}{4\pi t}\int_{|y-x|\leq|t|} \frac{u_t(y,0)\,\mathrm{d}y}{\sqrt{t^2-|y-x|^2}} + \frac{\partial}{\partial t}\left(\int_{|y-x|\leq|t|} \frac{u(y,0)\,\mathrm{d}y}{\sqrt{t^2-|y-x|^2}}\right), \tag{18}$$

for $n=2$ was published by Marc-Antoine Parseval des Chênes in 1800, and the case $n=3$

$$u(x,t) = \frac{1}{4\pi t}\int_{|y-x|=|t|} u_t(y,0)\,\mathrm{d}S_y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|y-x|=|t|} u(y,0)\,\mathrm{d}S_y\right), \tag{19}$$

was published by Poisson in 1819. That said, usually (18) is called *Poisson's formula* and (19) is called *Kirchhoff's formula*. It is notable that the Cauchy problem can be solved *forward and backward in time*, i.e., the formulas are valid for all $t \in \mathbb{R}$. Extensions to inhomogeneous equations, the so-called retarded potentials, and higher dimensions have been achieved by Gustav Robert Kirchhoff, Vito Volterra, and Orazio Tedone during the period 1882-98. A common feature of the preceding formulas is that information has a *finite speed of propagation*, meaning that the solution at (x,t) cannot "feel" the influence of the initial data outside the sphere of radius t centred at x. In other words, if one starts with initial data vanishing outside a small sphere around the origin, then at any instant of time the solution will also be vanishing outside a sphere around the origin, with the radius of the sphere increasing with velocity 1 as time proceeds. One consequence of this is that we can take initial data growing arbitrarily fast as $|x| \to \infty$, and still get well-defined solution of the wave equation. A very notable property of (19) that is not in (17) and (18) is that since the integration is over the *sphere* of radius t, not including the interior of the sphere, the solution at (x, t) depends only on the values of the initial data exactly *at the sphere* of radius t centred at x. This is known as the *strong Huygens' principle*, and is consistent with the familiar experience that when you listen to a sound source that is localized both in time and space, like a thud or a knock, you hear them only for a brief amount of time and then you hear nothing. In contrast, if we lived in two dimensions (or in fact in any even dimensions), every sound would be followed by an infinite echo (because the integral in (18) is over a disk). Language would be nearly impossible since the echo of what you said before would be mingled with what you are saying now. One way to make sense of this is to consider two dimensional waves from point sources as three dimensional waves from line sources that are parallel to each other. Then, thinking of the single source case, one would first hear the wave coming from the nearest point on the line, and then the waves from the other points on the line, with the waves from far away points coming later. Hence, one would hear a sharp beginning of the sound, followed by an infinite echo. This phenomenon is known as *diffusion of waves*. We will study such and related subjects in the next semester.

In order to solve the wave equation in a spatial domain $\Omega \subset \mathbb{R}^n$, we need to specify both a boundary condition, such as $u(x, t) = 0$ for all x at the boundary of Ω and all t, *and* initial data $u(x, 0)$ and $u_t(x, 0)$ for $x \in \Omega$. This problem is called an *initial-boundary value problem*, or a *mixed problem*. One of the first questions one could ask about such a problem is whether there is any *stationary solution*, i.e., whether one can tweak the initial data so that the solution does not depend on t at all. Another way to think of this would be to ask what happens to the system after so long time that all oscillations and motions have been damped out by the frictions and dissipative processes present in the system (assuming that the wave equation describes our system well for short time intervals). So putting the condition $\partial_t u \equiv 0$ into (16), we infer that all stationary solutions must satisfy the Laplace equation. In this light, we can think of the maximum principle for harmonic functions as saying that when an elastic membrane is in equilibrium, it must be in a "maximum stretched" position, in particular without any maximum or minimum in the interior.

More generally, we can look for solutions of the form $u(x, t) = v(x)e^{i\omega t}$, i.e., solutions that oscillate harmonically in time, and arrive at the equation

$$\Delta v + \omega^2 v = 0, \tag{20}$$

called the *Helmholtz equation*. In certain situations, e.g., when the domain Ω is bounded and the condition $u \equiv 0$ is imposed at the boundary of Ω, the equation (20) has no nonzero solutions except for countably many values of ω. In other words, for time-harmonic solutions most of the frequencies are forbidden, except countably many values, that are called the *natural* or *resonant frequencies* of Ω. These frequencies are determined by the shape and size of the domain Ω. For example, in a typical 1 dimensional situation the natural frequencies are integer multiples of some fundamental frequency. Most musical instruments that use vibrations of stretched string or air columns produce such (harmonic) frequencies. For higher dimensions, like in the case of drumheads and bells, the natural frequencies are far from being integer multiples of a fundamental tone. Posed as a problem of finding the resonant frequencies, (20) is known as the *eigenvalue problem* for the Laplace operator. Then the *eigenvalues* of Δ would be the numbers $\lambda_i = -\omega_i^2$, where ω_i are the resonant frequencies. Note that in the literature $\lambda_i = \omega_i^2$ are sometimes defined as the eigenvalues, in order to have them nonnegative. To avoid any confusion, let me remark that we do not know *a priori* if the eigenvalues are even real, let alone positive, but rather one must study the problem (20) to address such questions. Now suppose that v_i is a nontrivial solution of (20) corresponding to the eigenvalue λ_i. Then the corresponding time-harmonic solutions $v_i(x)e^{\pm it\sqrt{-\lambda_i}}$, or equivalently, $v_i(x)\cos(t\sqrt{-\lambda_i})$ and $v_i(x)\sin(t\sqrt{-\lambda_i})$, of the wave equation are called the *i-th normal modes* of Ω. In 1750, Daniel Bernoulli essentially asked if we can write an arbitrary solution of the wave equation as an infinite linear combination of the normal modes, as in

$$u(x,t) = \sum_i A_i v_i(x)\cos(t\sqrt{-\lambda_i}) + B_i v_i(x)\sin(t\sqrt{-\lambda_i}), \tag{21}$$

where the (infinite collection of) constants A_i and B_i are to be determined from the initial data. The question was quickly dismissed by Euler, as solutions of the wave equation can be nonsmooth while every term in the above series is smooth in time, but later picked up by Jean Baptiste Joseph Fourier, who at least managed to give some evidence for an affirmative answer.

4. Heat equation

In his study of heat conduction, presented in 1807 to the French Academy of Sciences and published in 1822, Jean Baptiste Joseph Fourier derived the heat equation

$$\frac{\partial u}{\partial t} = \Delta u. \tag{22}$$

Here $u(x,t)$ is interpreted as the temperature at the point $x \in \mathbb{R}^n$ and time moment t. Just after Fourier's results were announced, in 1809, Laplace derived the $n = 1$ case of the solution formula

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) u(y,0)\,\mathrm{d}y, \tag{23}$$

for the initial value problem. The extensions to higher dimensions were published by Poisson in 1819 and by Fourier in 1822. It is not clear they knew each other's results, but both acknowledge that their result is an easy extension of Laplace's formula. Looking at this formula, we note some striking differences between the wave and heat equations. First of all, information has *infinite speed of propagation* for the heat equation, because however large the distance between x and y is, the weight $\exp(-|x-y|^2/4t)$ in the integral (23) is nonzero. So one needs to restrict the growth of the initial datum $u(y,0)$ as $|y| \to \infty$, since otherwise contributions to $u(x,t)$ from far away points y may build up to the point the integral in (23) does not make sense. In particular, it is possible for a solution ceases to exist after a finite time t, as the weight $\exp(-|x-y|^2/4t)$ actually grows with time if $|x-y|$ is large. A related issue is that there can be more than one solution to the heat equation. All these issues can be avoided by restricting ourselves to the situation where the solution cannot grow too fast at spatial infinity. The second difference to the wave equation is that the heat equation has a *direction of time*, i.e., the integral in (23) is well-behaved only for $t > 0$. Moreover, even if the initial datum decays sufficiently fast so that the integral is convergent for $t < 0$, the limit $u(x,t)$ as $t \to 0$ does not exist. This sense of time direction is actually a feature that makes the heat equation a good candidate for describing irreversible processes, such as heat propagation and the diffusion of an ink drop in a glass of water. Finally, the third point we wanted to discuss, that is also related to irreversibility, is that the heat equation is *smoothing*, in the sense that the solution is infinitely often differentiable as long as $t > 0$ even if one starts with, say, only continuous initial datum $u(y,0)$. For the wave equation, the solution is not smooth unless the initial data are smooth.

Similarly to the wave equation, one can study the initial-boundary value problem for the heat equation on a spatial domain $\Omega \subset \mathbb{R}^n$. Then the stationary solutions are given by the solutions of the Laplace equation, with the boundary condition inherited from the heat equation. In this setting, the maximum principle for harmonic functions has the interpretation that in equilibrium, the temperature field cannot have local maximum or minimum, which is consistent with physical intuition that such local maximum or minimum cannot be stationary. The substitute of the time-harmonic solutions for the heat equation would be the exponentially decaying solutions $u(x,t) = v(x)e^{-\omega t}$, which leads again to the Helmholtz equation and the eigenvalue problem for the Laplace operator. Then the analogue of the normal mode expansion (21) is

$$u(x,t) = \sum_i A_i v_i(x) e^{\lambda_i t}, \tag{24}$$

where v_i and λ_i are the eigenfunctions and eigenvalues of the Laplacian as before, and the constants A_i are to be determined from the initial datum

$$u(x,0) = \sum_i A_i v_i(x). \tag{25}$$

In his 1822 manuscript Fourier gave formulas to calculate A_i in terms of $u(x,0)$ in several special settings, and by way of examples showed that the expansion (25) converges even when $u(x,0)$ is discontinuous. This initiated the study of trigonometric series, as in a typical one dimensional setting (25) is such a series. We will start tackling this question towards the end of this semester.

5. Schrödinger equation

The basic equation of the non-relativistic quantum mechanics

$$i\frac{\partial u}{\partial t} = -\Delta u + Vu, \tag{26}$$

was discovered by Erwin Rudolf Josef Alexander Schrödinger in 1926. Here $i = \sqrt{-1}$, V is a given (scalar) function depending on $x \in \mathbb{R}^n$ and t. The solution u has the interpretation that $|u(x,t)|^2$ is the probability density of the particle being found at (x,t), and V represents the potential of the external field. The equation can be generalized to many particle systems, where now x varies in $\mathbb{R}^{n\times m}$ for m-particle systems, and certain symmetry conditions must be imposed on u depending on the spins of the particles. For some special cases of the potential V, the solution can be expressed in terms of explicit formulas similar to (23). The free Schrödinger equation (i.e., with $V \equiv 0$) has a vague resemblance to both the heat and the wave equations. It has *infinite speed of propagation*, as well as a *smoothing* property. Connected to this, one has issues with solutions that grow too fast at infinity. On the other hand, the Schrödinger equation is *reversible*. A finer inspection will show that high frequency components of the initial datum travels faster than the low frequency components, so the equation is *dispersive*. We will look closely at these properties in the next semester.

In this context, the time-harmonic solutions $u(x,t) = v(x)e^{-i\omega t}$ are called the *stationary solutions*, because for these solutions the quantity $|u|^2$ is constant in time. The stationary solutions satisfy the *stationary Schrödinger equation*

$$-\Delta u + Vu = \omega u, \tag{27}$$

which is the eigenvalue problem for the Schrödinger operator $-\Delta + V$. This can explicitly be solved for a few simple cases, including the harmonic oscillator, $V(x) = |x|^2$, and the hydrogen atom, $V(x) = -1/|x|$. We may well be able to say a few things about some more general cases later.

6. Maxwell equations

The basic equation of the classical electrodynamics

$$\partial_t E = \nabla \times B - J, \qquad \partial_t B = -\nabla \times E, \qquad \nabla \cdot E = \rho, \qquad \nabla \cdot B = 0, \tag{28}$$

was published by James Clerk Maxwell in 1861. Here J is the electric current density, and ρ is the charge density, which are the sources of the electromagnetic field. The field itself is described by the electric field $E : \mathbb{R}^3 \to \mathbb{R}^3$ and the magnetic field $B : \mathbb{R}^3 \to \mathbb{R}^3$, whose physical meaning is given by

$$F = qE + qv \times B, \tag{29}$$

where F is the electromagnetic force acted on the test charge q with velocity v. In particular, the electric field is the force per unit charge when the test charge is at rest. By differentiating the first equation of (28) in time, and with the help of the second and third equations, we can derive

$$\partial_t^2 E = -\nabla \times \nabla \times E - \partial_t J = \Delta E - \nabla \rho - \partial_t J, \tag{30}$$

where we have used the identity

$$-\nabla \times \nabla \times E = \Delta E - \nabla(\nabla \cdot E). \tag{31}$$

In particular, the components of the electric field satisfies the homogeneous wave equation in the region of space-time where $\nabla \rho = 0$ and $\partial_t J = 0$.

Exercise 2. Derive the analogous wave equation for the magnetic field B.

Now we relate Maxwell's equations to our earlier discussion about the electric field potential. By the last equation of (28), B is divergence free, so there is a vector field A such that $B = \nabla \times A$. Plugging this into the second equation, we get $\nabla \times (E + \partial_t A) = 0$. Hence there is a scalar field u such that $E + \partial_t A = \nabla u$. To summarize, there is a vector field A and a scalar field u, called respectively *vector* and *scalar potentials*, such that

$$B = \nabla \times A, \qquad E = \nabla u - \partial_t A. \tag{32}$$

Then the first and third equations of (28) become

$$\partial_t^2 A = -\nabla \times \nabla \times A + \nabla \partial_t u + J, \qquad \Delta u = \partial_t \nabla \cdot A + \rho. \tag{33}$$

Exercise 3. Show that with the definitions (32), the equations (33) imply the Maxwell equations (28), hence demonstrating the equivalence of the two formulations.

It is possible that two different pairs (A, u) and (A', u') give rise to the same physical fields (E, B). In this situation, we need to consider the two different mathematical configurations as describing the same physical reality. If there is enough flexibility in choosing a pair (A, u) among all pairs that describe the same physics, we can use this freedom to simplify the mathematical formulation (33). Such a freedom in general is called *gauge freedom*. Let us identify the gauge freedom of the formulation (33). First of all, we can add the gradient of any scalar field to A, since it would not alter $\nabla \times A$. Conversely, if $\nabla \times A = \nabla \times A'$, then $A' = A + \nabla \chi$ for some scalar field χ. Secondly, such a transformation would affect E as $E' = \nabla u - \partial_t A - \partial_t \nabla \chi$. So in order to have $E = E'$, we need to transform u as $u' = u - \partial_t \chi$. To conclude, any transformation $(A, u) \mapsto (A', u')$ that leaves E and B invariant in (32) can be written as

$$A' = A + \nabla \chi \qquad u' = u - \partial_t \chi, \tag{34}$$

with some scalar function χ.

A convenient gauge choice is the *Coulomb gauge* given by the condition $\nabla \cdot A = 0$. In this gauge, the equations (33) read

$$\partial_t^2 A = \Delta A + \nabla \partial_t u + J, \qquad \Delta u = \rho. \tag{35}$$

We see that the scalar potential satisfies the Poisson equation. Note that here the gauge is set up so that even in the case of dynamic fields, the Poisson equation is satisfied by the electric potential. If we require $\partial_t A \equiv 0$ and $\partial_t u \equiv 0$, the Poisson equation would be satisfied by the components of the vector potential as well. The Coulomb gauge is realizable, since from any given gauge we can shift to the Coulomb gauge by solving the Poisson equation $\Delta\chi + \nabla \cdot A = 0$. Another popular gauge is the so-called *Lorenz gauge*, given by $\partial_t u - \nabla \cdot A = 0$, which gives rise to

$$\partial_t^2 A = \Delta A + J, \qquad \Delta u = \partial_t^2 u + \rho. \tag{36}$$

Hence both the scalar potential and the vector potential satisfy (inhomogeneous) wave equations.

Exercise 4. Show that the Lorenz gauge is realizable.

7. Incompressible fluids

The fundamental equation for incompressible Newtonian fluids

$$\partial_t u + (u \cdot \nabla)u = \nu\Delta u - \nabla p + f, \qquad \nabla \cdot u = 0, \tag{37}$$

was derived by Claude-Louis Navier in 1822. Here the unknowns are $u(x, t)$ the velocity of the fluid element at (x, t), and $p(x, t)$ the pressure at (x, t). The viscosity coefficient $\nu \geq 0$ and the external force field f are considered to be given. The equation is called the *Navier-Stokes equations*, honouring Navier and Sir George Gabriel Stokes, whose 1845 work was important in understanding the nature of the equations. For inviscid fluids, i.e., for $\nu = 0$, the equations reduce to the *Euler equations*

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + f, \qquad \nabla \cdot u = 0, \tag{38}$$

which were derived by Euler in 1757. Note that there are also compressible versions of the above equations, where the condition $\nabla \cdot u = 0$ is suitably relaxed. Then for example, under a certain approximate regime, the compressible Euler equations give rise to the wave equation, providing a basis for the study of sound in fluids as pressure waves. However, we will not have much to say about the compressible equations themselves in this sequence. Many people contributed to the realization that adding the viscosity term $\nu\Delta u$ to the Euler equation was important, most notably including d'Alambert and Jean Claude Barré de Saint-Venant. Some basic properties of fluids can be understood by studying *irrotational* flows, meaning that the flow is such that $\nabla \times u = 0$. In this case, we can write $u = \nabla\varphi$ for some scalar function φ, called the *flow potential*, hence the incompressibility condition $\nabla \cdot u = 0$ becomes the Laplace equation $\Delta\varphi = 0$ for the potential. This is the reason why d'Alambert and Lagrange encountered Cauchy-Riemann and Laplace equations in their study of fluid flows. For completeness, we can simplify the other equation as

$$\partial_t \nabla\varphi + (u \cdot \nabla)u + \nabla p + f = \nabla(\partial_t\varphi + \tfrac{1}{2}u \cdot u + p + \Phi) = 0, \tag{39}$$

where we assumed that $f = \nabla\Phi$, and have used the identity

$$\tfrac{1}{2}\nabla(u \cdot u) = u \times (\nabla \times u) + (u \cdot \nabla)u. \tag{40}$$

The equation (39) requires that the quantity $\partial_t\varphi + \frac{1}{2}u \cdot u + p + \Phi$ be constant throughout space, with a possible dependence only on time. Now similarly to the Maxwell equations, we have a freedom to add any function of time to φ (as well as Φ), so that under a certain gauge we get the *Bernoulli equation*

$$\partial_t\varphi + \tfrac{1}{2}|\nabla\varphi|^2 + p + \Phi = 0, \tag{41}$$

for inviscid, incompressible, irrotational fluid flows. When nonlinear effects are negligible, i.e., the term $(u \cdot \nabla)u$ is small, the Navier-Stokes equations simplify to give the *Stokes equations*

$$\partial_t u = \nu \Delta u - \nabla p + f, \qquad \nabla \cdot u = 0. \tag{42}$$

Before studying the Navier-Stokes equations, it is important to understand the Stokes equations. As with the story of the heat, Laplace and Helmholtz equations, the first steps to getting a grip of the Stokes equations are the stationary Stokes problem, and the corresponding Stokes eigenvalue problem. In a fluid mechanics course you would study special solutions to the afore-mentioned equations, and learn heuristic reasonings based mainly on physical intuition. In this course, we will study the equations from a purely mathematical point of view, concentrating on proving the existence and uniqueness of solutions for a wide range of cases, as a necessary ingredient in a full mathematical understanding of the equations. In fact, there are many open mathematical problems related to the existence and uniqueness of solutions for fluid equations, that resisted the attacks of the best mathematicians for generations. Note that this is not merely a hobby of mathematicians, as the existence theory would shed light on the correctness of the model, and more importantly, the knowledge and experience gained from trying to establish such a theory would be priceless.