MATH 580 ASSIGNMENT 4

DUE THURSDAY OCTOBER 27

1. With $\Omega \subset \mathbb{R}^n$ a bounded domain, the *Robin problem* for the Poisson equation is
$$\Delta u = f, \quad \text{in } \Omega, \qquad \partial_\nu u + ku = g, \quad \text{on } \partial\Omega,$$
where f and g are functions defined on Ω and $\partial\Omega$, respectively, and $k > 0$ is a constant. Devise an approach analogous to Green's functions for the Robin problem. The resulting functions are called *Robin's functions*, or *Green's functions of the third kind.* Make some preliminary observations on the behaviour of these functions. Can you produce more than one approach? Give explicit formulas for the Robin function(s) and the solution of the Robin problem in the case $n = 1$.
2. The same as above, but for the Neumann problem, i.e., the case $k = 0$. The resulting functions are called *Neumann's functions*, or *Green's functions of the second kind.*
3. We say $u \in C(\Omega)$ is *subharmonic* in Ω if for each $y \in \Omega$ there exists $r^* > 0$ such that
$$u(y) \leq \frac{1}{|B_r|} \int_{B_r(y)} u, \qquad \forall r \in (0, r^*).$$
Prove the following statements.
   (a) A function $u \in C^2(\Omega)$ is subharmonic in Ω iff $\Delta u \geq 0$ in Ω.
   (b) If u is harmonic in Ω, then $|\nabla u|^2$ is subharmonic in Ω.
4. Let $B_r^\times = B_r(0) \setminus \{0\}$ be the punctured ball of radius $r > 0$, and let $u \in C^2(B_r^\times)$ be harmonic in $B_r^\times$. Show that the singularity at 0 is removable if $u(x) = o(E(x))$ as $x \to 0$, where E is the radial fundamental solution we defined in class. Take space dimension to be $n \geq 2$.
5. For a function $u : \mathbb{R}^n \to \mathbb{R}$, its graph is defined to be the set $\{(x, u(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$. Prove that any tangent hyperplane to the graph of an entire harmonic function intersects the graph more than once.
6. (a) Why is the existence of a barrier for any specific point on the boundary $\partial\Omega$ of a domain Ω a local property of that point? In other words, if $z \in \partial\Omega$ is regular for the Dirichlet problem on Ω, and if Ω' is a domain that coincides with Ω in a neighbourhood of z (hence in particular $z \in \partial\Omega'$), then can you conclude that z is also regular for the Dirichlet problem on Ω'?
   (b) Show that any isolated boundary point in the plane is exceptional for the Dirichlet problem.

Date: Fall 2011.

7. (*This is a bonus problem and can be handed in anytime during the semester.*) Show that all the boundary points of a domain in $\mathbb{R}^2$ are regular for the Dirichlet problem if each connected component of the boundary contains more than one point.