1. For each of the following cases, determine the characteristic cones and characteristic surfaces.
   a) Wave equation with wave speed $c > 0$: $u_{xx} + u_{yy} = c^{-2}u_{tt}$.
   b) Tricomi-type equation: $u_{xx} + yu_{yy} = 0$.
   c) Ultrahyperbolic “wave” equation: $u_{xx} + u_{yy} = u_{zz} + u_{tt}$.

2. Consider the Cauchy problem for the Laplace equation:
   \[ u_{xx} + u_{tt} = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = \phi(x). \]
   For given $\varepsilon > 0$ and an integer $k > 0$, construct an initial datum $\phi$ such that
   \[ \| \phi \|_\infty + \ldots + \| \phi^{(k)} \|_\infty < \varepsilon, \]
   and
   \[ \| u(\cdot, \varepsilon) \|_\infty > 1/\varepsilon. \]

3. Solve
   \[ xu_x + 2yu_y + u_z = 3u, \quad u(x, y, 0) = g(x, y). \]

4. Prove that if $\beta \in \mathbb{R}$ and $u \in C^1(\mathbb{R}^2)$ is a solution of $u_t + \beta u_x = 0$, then
   \[ \{(x, t) : u \in C^k \text{ on a neighbourhood of } (x, t)\} \]
   is a union of rays.

5. Consider the equation
   \[ xyu_x + (2y^2 - x^6)u_y = 0, \quad x > 0, y > 0. \]
   Determine and sketch the characteristics. For $n \in \mathbb{N}$ and $\alpha > 0$, consider the initial condition
   \[ u(x, \alpha x^n) = x^2. \]
   For which $\alpha > 0$ does the problem have a solution? Give an explicit expression for the solution. For which $\alpha > 0$ is the solution uniquely determined? Answers may depend on $n$ (Try $n = 1$ and $n = 2$ etc first).

6. Consider the initial value problem
   \[ u_t + uu_x + \gamma u = 0, \quad u(x, 0) = f(x), \]
   for $(x, t) \in \mathbb{R} \times [0, \infty)$, where $\gamma \geq 0$ is a constant. Make a rough sketch of the characteristics on an $xt$ diagram and show that wave breaking occurs only if $f'(x) < -\gamma$ for some $x$.

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Date: Fall 2011.
7. Consider the Burgers equation $u_t + uu_x = 0$ with initial data

$$u(x, 0) = 0 \text{ if } |x| \geq 1 \quad \text{and} \quad u(x, 0) = 1 - |x| \text{ if } |x| \leq 1.$$ 

By sketching the characteristics, describe the entropy solution. Clearly indicate on your sketch of the characteristics where the shock is. Is the shock a line? What is the equation of the shock? What happens to $u(\cdot, t)$ as $t \to \infty$?