MATH 580 ASSIGNMENT 1

DUE THURSDAY SEPTEMBER 15

1. Let $\alpha > 0$, and let
$$f(x) = \begin{cases} \exp(-x^{-\alpha}) & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x \leq 0. \end{cases}$$
Prove that $f \in C^\infty(\mathbb{R})$, but f is not real analytic at 0.
2. (Analyticity of multiplicative inverse) Let f be a function real analytic at $a \in \mathbb{R}$, and suppose that $f(a) \neq 0$. Show that $1/f$ is real analytic at a.
3. Multi-indices are defined as n-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$. We define the operations $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$ or $x = \partial$. We also define $\alpha \pm \beta$ and $\alpha \leq \beta$, etc., acting componentwise. Prove the following relations.
   (a) $\alpha! \leq |\alpha|! \leq n^{|\alpha|}\alpha!$.
   (b) $(x_1 + \ldots + x_n)^m \leq \displaystyle\sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha$.
   (c) $\partial^\beta x^\alpha = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} & \text{if} \quad \alpha \geq \beta, \\ 0 & \text{otherwise.} \end{cases}$.
4. Consider the system
$$\partial_n^{q_i} u_i = F_i(x, u, \ldots, \partial^\alpha u, \ldots), \qquad i = 1, \ldots, p,$$
where each F_i is analytic, and all α that occur in F_i satisfy $|\alpha| \leq q_i$ and $\alpha_n < q_i$. Here x varies in $\mathbb{R}^n$, and $u = (u_1, \ldots, u_p)$ is a function in $\mathbb{R}^n$ taking values in $\mathbb{R}^p$. Such a system is called of *Kovalevskaya type*. We adjoin the initial conditions
$$\partial_n^k u_i(\xi, 0) = \phi_{i,k}(\xi), \qquad \xi \in \mathbb{R}^{n-1}, \quad 0 \leq k \leq q_i - 1, \quad 0 \leq i \leq p,$$
where $\phi_{i,k}$ are analytic functions of $n-1$ variables. Prove the Cauchy-Kovalevskaya theorem in this general case, i.e., prove that the above Cauchy problem has a unique analytic solution in a neighbourhood of the origin in $\mathbb{R}^n$.
5. Supposing that the solution of the heat equation $\partial_t u = \partial_x^2 u$, with initial data
$$u(x, 0) = \sum_{j=0}^{\infty} a_j x^j,$$

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can be written in the form

$$u(x,t) = \sum_{j,k=0}^{\infty} b_{j,k} x^j t^k,$$

determine the coefficients $b_{j,k}$. Now assuming that $a_j = j^{-\alpha j}$ with a constant $\alpha \in (0, \frac{1}{2})$, show that the radius of convergence of $u(x,0)$ is equal to ∞. On the other hand, show that the radius of convergence of

$$u(0,t) = \sum_{k=0}^{\infty} b_{0,k} t^k,$$

is equal to 0. In other words, if it exists, the solution is not analytic in any neighbourhood of $(0,0)$. Explain why this example does not contradict the Cauchy-Kovalevskaya theorem.

6. Consider the Laplace equation $\Delta u = 0$ on the unit disk, given in polar coordinates by $\mathbb{D} = \{(r,\theta) : r < 1\}$. Specify the Cauchy data

$$u(1,\theta) = f(\theta), \qquad \partial_r u(1,\theta) = g(\theta),$$

where f and g are 2π-periodic real analytic functions. Then show that a real analytic solution exists for all $\theta \in \mathbb{R}$ and $|r-1|$ sufficiently small. Investigate what happens to the solution as $r \to 0$ and $r \to \infty$, if f and g are of the form

$$a_0 + \sum_{n=1}^{m} a_n \cos n\theta + b_n \sin n\theta,$$

i.e., trigonometric polynomials.

7. [1] Let $f : \Omega \to \mathbb{R}$ be a function defined on some nonempty open set $\Omega \subset \mathbb{R}^2$. Then one can define the *directional derivative* $Df(z,X)$ of f at $z \in \Omega$ along the vector $X \in \mathbb{R}^2$, by considering the function $g(t) = f(z+tX)$ defined on some nonempty interval $(-\varepsilon, \varepsilon)$. So we define $Df(z,X) = g'(0)$ if the latter exists, or more explicitly, if there is a number $\lambda \in \mathbb{R}$ such that

$$f(z+tX) = f(z) + \lambda t + o(t), \qquad \text{for} \quad t \in \mathbb{R} \quad \text{with} \quad t \to 0,$$

then we say that f is differentiable at z in the direction X, and write $Df(z,X) = \lambda$. Assuming that $Df(z,X)$ exists for all $X \in \mathbb{R}^2$ and for all $z \in \Omega$, the totality of all possible directional derivatives of f defines a function $(z,X) \mapsto Df(z,X)$ on $\Omega \times \mathbb{R}^2$. There is no obvious *a priori* structure on this function, except to say that $Df(z,X)$ is homogeneous in X, i.e., $Df(z,tX) = tDf(z,X)$ for any $t \in \mathbb{R}$. This essentially means that at each $z \in \Omega$, the directional derivatives $Df(z,X)$ are completely determined by their values on the unit circle $S^1 = \{z \in \mathbb{R}^2 : |z| = 1\}$.

[1] Bonus problem: not counted in this assignment, but +1 point towards the final grade. At the minimum, it is important to read and understand the statement of the problem.

A substantial simplification occurs if we require the graph of f to be locally a plane, i.e., if we require that f can be locally approximated by linear functions of two variables. To be precise, if there is a vector $\Lambda \in \mathbb{R}^2$ such that

$$f(z+h) = f(z) + \Lambda \cdot h + o(|h|), \qquad \text{for} \quad h \in \mathbb{R}^2 \quad \text{with} \quad |h| \to 0,$$

then we say that f is *differentiable* at z, and write $Df(z) = \Lambda$. An obvious consequence of differentiability is that the directional derivative $Df(z,X)$ is now linear in X, and is given by $Df(z,X) = Df(z) \cdot X$. This means that at each $z \in \Omega$, the directional derivatives are completely determined by their values at two non-collinear vectors, that is, if $X = \alpha_1 X_1 + \alpha_2 X_2$, then we have $Df(z,X) = \alpha_1 Df(z,X_1) + \alpha_2 Df(z,X_2)$. Prove the following partial converse:

Let $X_1, X_2 \in \mathbb{R}^2$ be non-collinear vectors, and assume that the directional derivatives $Df(z,X_1)$ and $Df(z,X_2)$ exist and are continuous in Ω as functions of z. Then f is differentiable in Ω, and the derivative Df is continuous in Ω.

Note that this explains why we *can* simply focus on the partial derivatives $\partial_x f(z) = Df(z,e_1)$ and $\partial_y f(z) = Df(z,e_2)$, where $e_1 = (1,0)$ and $e_2 = (0,1)$, and their continuity, rather than the totality $Df(z,X)$ for $X \in \mathbb{R}^2$. There is nothing special about the directions e_1 and e_2; any non-collinear directions X_1 and X_2 would do.

For $X \in \mathbb{R}^2$ fixed, let us denote by D_X the operator sending the function f to the function $z \mapsto Df(z,X)$, i.e., $D_X f = Df(\cdot, X)$. Since $D_X f$ is again a function defined on $\mathbb{R}^2$ with values in $\mathbb{R}$, not only one can talk about its directional differentiability but the operators D_X can be applied recursively, so for instance, $D_X D_X f$, $D_Y D_X f$, or even $D_X^n f$ can be defined. Prove the following:

Let $X, Y \in \mathbb{R}^2$, and assume that $D_X D_Y f$ and $D_Y D_X f$ exist and are continuous in Ω. Then $D_X D_Y f = D_Y D_X f$ in Ω.

Note that this in particular justifies the formula $\partial_x \partial_y f = \partial_y \partial_x f$ under the condition that the participated derivatives are continuous.