MATH 579 ASSIGNMENT 4

DUE TUESDAY APRIL 14, 18:00 EDT

1. Let V and Q be reflexive Banach spaces, and $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ be bounded bilinear forms, with a satisfying
$$a(u,v) = a(v,u) \quad \text{and} \quad a(u,u) \geq 0 \quad \text{for all } u, v \in V.$$
Suppose that $(u,p) \in V \times Q$ satisfy
$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle \qquad \forall v \in V, \\ b(u,q) &= \langle g, q\rangle \qquad \forall q \in Q, \end{aligned}$$
with some $f \in V'$ and $g \in Q'$. Show that
$$J(u,q) \leq J(u,p) \leq J(v,p) \qquad \text{for all } (v,q) \in V \times Q,$$
where
$$J(v,q) = \tfrac{1}{2}a(v,v) + b(v,q) - \langle f, v\rangle - \langle g, q\rangle.$$
2. Let V and Q be reflexive Banach spaces, and $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ be bounded bilinear forms, satisfying
$$a(u,u) \geq \alpha \|u\|^2 \quad \text{for all } u \in V,$$
and
$$\sup_{v \in V} \frac{b(v,p)}{\|v\|} \geq \beta \|p\| \quad \text{for all } p \in Q,$$
where $\alpha > 0$ and $\beta > 0$ ate constants. Let $f \in V'$ and $g \in Q'$, and suppose that $(u,p) \in V \times Q$ satisfy
$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle \qquad \forall v \in V, \\ b(u,q) &= \langle g, q\rangle \qquad \forall q \in Q. \end{aligned}$$
Now let $\hat{V} \subset V$ and $\hat{Q} \subset Q$ be closed subspaces, satisfying the discrete inf-sup condition
$$\sup_{v \in \hat{V}} \frac{b(v,p)}{\|v\|} \geq \hat{\beta} \|p\| \quad \text{for all } p \in \hat{Q}.$$
   (a) Show that the Galerkin problem
$$\begin{aligned} a(\hat{u},v) + b(v,\hat{p}) &= \langle f, v\rangle \qquad \forall v \in \hat{V}, \\ b(\hat{u},q) &= \langle g, q\rangle \qquad \forall q \in \hat{Q}, \end{aligned}$$
   has a unique solution $(\hat{u},\hat{p}) \in \hat{V} \times \hat{Q}$.
   (b) Establish the Céa-type estimate
$$\|u - \hat{u}\| + \|p - \hat{p}\| \leq C \inf_{(v,q) \in \hat{V} \times \hat{Q}} \big(\|u - v\| + \|p - q\|\big),$$
   directly, i.e., without going through the general theory of Petrov-Galerkin methods. How does the constant C depend on the constants α, β, $\hat{\beta}$, and the norms of a and b?
   (c) Give an example of a situation where this framework is naturally applicable and an example where it is non-applicable.

Date: Winter 2020.

3. Given $f \in L^2(\Omega)$ and $\alpha \in \mathbb{R}$, consider the problem of finding $(u, \lambda) \in H_0^1(\Omega) \times \mathbb{R}$ satisfying
$$-\Delta u + \lambda = f \quad \text{in } \Omega, \qquad \text{and} \qquad \int_\Omega u = \alpha. \tag{$*$}$$
Here Ω is a bounded Lipschitz domain.
   (a) Prove that $(*)$ is well posed.
   (b) Design a finite element method for $(*)$, and show that it converges.
4. Consider the following boundary value problem:
$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$**$}$$
   (a) Propose a mixed formulation of $(**)$ that does not *a priori* require more than H^1-regularity for the unknown functions, and prove its well posedness.
   (b) Design a finite element method for the proposed formulation, and show that it converges. In particular, you will need to prove a discrete inf-sup condition.
5. Prove the inf-sup stability of the pair $(P_{1,h/2}, P_{0,h})$ for the stationary Stokes problem. Here the pressure space $P_{0,h}$ is the space of (possibly discontinuous) piecewise constants on a mesh, and the velocity space $P_{1,h/2}$ is the space of globally continuous, piecewise affine vector fields on the mesh that is one level higher than the pressure mesh (that is, we used one step of the red refinement procedure on every element).