MATH 579 ASSIGNMENT 3

DUE TUESDAY MARCH 31

1. Let $\Omega = (0,1)^2$ be the unit square, and for $j \in \mathbb{N}$, let P_j be the collection of 2^{2j} small squares of side length 2^{-j} tiling up Ω. Denote by $\bar{\mathbb{P}}_{d-1}$ the set of bivariate polynomials of the form $p(x_1)q(x_2)$ with $p, q \in \mathbb{P}_{d-1}$ single variable polynomials. Given $d \in \mathbb{N}$, we define the space S_j of *dyadic splines* as follows:
$$S_j^{d,r} = \{u \in C^r(\Omega) : u|_Q \in \bar{\mathbb{P}}_{d-1} \text{ for each cube } Q \in P_j\}.$$
We also define the *cardinal B-splines* on $\mathbb{R}$ by the recursive formula
$$N^d = N^{d-1} * N^1, \qquad d = 2, 3, \dots,$$
with $N^1 = \chi_{(0,1)}$ the characteristic function of the unit interval.
a) Show that $N^d \in C^{d-2}(\mathbb{R})$, $N^d|_{(k,k+1)} \in \mathbb{P}_{d-1}$ for $k \in \mathbb{Z}$, and $\operatorname{supp} N^d = [0, d]$.
b) We fix d, and define the dyadic cardinal B-splines
$$\phi_{j,k}(x) = N^d(2^j x - k), \qquad j \in \mathbb{N}_0,\ k \in \mathbb{Z},$$
and their tensor product version
$$\phi_{j,\alpha}(x,y) = \phi_{j,\alpha_1}(x)\phi_{j,\alpha_2}(y), \qquad j \in \mathbb{N}_0,\ \alpha \in \mathbb{Z}^2.$$
For $j \in \mathbb{N}_0$, let Φ_j be the collection of those $\phi_{j,\alpha}$ ($\alpha \in \mathbb{Z}^2$) whose supports nontrivially intersect the unit square Ω. Show that Φ_j is a basis of $S_j^{d,d-2}$.
c) From now on we will fix $d = 4$. For each $Q \in P_j$, we define the *Hermite interpolant* $v = H_Q u \in \bar{\mathbb{P}}_3$ for functions $u \in C^1(\overline{\Omega})$ by the following relations
$$\begin{aligned} v(x) &= u(x), \\ \partial_i v(x) &= \partial_i u(x), \quad (i = 1, 2), \\ \partial_1\partial_2 v(x) &= \partial_1\partial_2 u(x), \end{aligned}$$
where x runs over the corner points of Q. Since $\dim \bar{\mathbb{P}}_3 = 16$, the polynomial v is well defined. Let us define the global interpolant $H_j u$ by $(H_j u)|_Q = H_Q u$ for each $Q \in P_j$. Show that $H_j u \in S_j^{4,1}$ for $u \in C^1(\overline{\Omega})$.
d) Prove the error estimate
$$\|u - H_j u\|_{W^{k,p}(\Omega)} \le c\, 2^{-j(m-k)} |u|_{W^{m,p}(\Omega)},$$
for $0 \le k \le m \le 4$, $m > \frac{n}{p} + 1$ and $1 \le p \le \infty$. Why are there restrictions on m?
2. Identify the spaces $[C(\mathbb{T}), C^1(\mathbb{T})]_{\theta,q}$.
3. Show that $[W^{1,1}(\Omega), W^{1,\infty}(\Omega)]_{1-1/p,p} = W^{1,p}(\Omega)$.
4. Show that $B^1_{p,q}(\mathbb{T}) \ne W^{1,p}(\mathbb{T})$ unless $p = q = 2$.

Date: Winter 2020.