MATH 579 ASSIGNMENT 2

DUE TUESDAY FEBRUARY 18

1. Let T be an infinite collection of triangles (in $\mathbb{R}^2$). For any triangle $\tau \in T$, we let $h_\tau = \operatorname{diam}(\tau)$, $|\tau|$ denote the area of τ, and let ρ_τ be the radius of the inscribed circle of τ. Show that the following conditions are equivalent.
   (a) The ratio $h_\tau^2/|\tau|$ is uniformly bounded.
   (b) The ratio h_τ/ρ_τ is uniformly bounded.
   (c) The minimum angle of τ is uniformly bounded away from 0.

   If any (hence all) of the preceding conditions holds for T , then we say that the collection T is *shape regular* (or *non-degenerate*).
2. Show that each of the following conditions is *not* equivalent to any of the conditions (a)–(c) in the preceding problem.
   (a) The ratio between the maximum and minimum edge lengths of τ is uniformly bounded.
   (b) The maximum angle of τ is uniformly bounded away from π.
3. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and let $\mathscr{P}$ be a family of *conforming* triangulations of Ω. We say that $\mathscr{P}$ is *graded* (or *locally quasi-uniform*, or has the *K-mesh property*), if
$$\sup\left\{\frac{h_\sigma}{h_\tau} : \sigma,\tau \in P,\, \overline{\sigma}\cap\overline{\tau} \neq \varnothing,\, P \in \mathscr{P}\right\} < \infty.$$
   Prove that if $\mathscr{P}$ is shape regular (as the collection $\bigcup_{P\in\mathscr{P}} P$), then it is graded. Note that shape regularity is defined at the end of Problem 1.
4. Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral domain. Show that the Lagrange finite element spaces are contained in $W^{1,p}(\Omega)$ for all $1 \le p \le \infty$, but *not* in $W^{2,p}(\Omega)$ for any $p \ge 1$.
5. Let $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ be a finite union of bounded star-shaped domains. By using the error bound for averaged Taylor polynomials in terms of truncated Riesz potentials as we have developed in class, prove the following *Sobolev inequality*
$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)},$$
   for $1 \le p \le q < \infty$, and $\frac{1}{p} < \frac{1}{q} + \frac{1}{n}$. *Hint*: Use the Young inequality
$$\|f * g\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^r(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)},$$
   where $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $1 \le p,q,r \le \infty$. (Note that the Sobolev inequality is true for the borderline case $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$ as well, which can be proved for instance by using

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the Hardy-Littlewood-Sobolev inequality for the Riesz potentials, or by the elementary method due to Gagliardo and Nirenberg.)

6. Let $\tau \subset \mathbb{R}^n$ be a simplex and let $I_\tau : C(\overline{\tau}) \to \mathbb{P}_{d-1}$ be the standard nodal interpolation onto the polynomials of order d. Derive a bound on the interpolation error $\|u - I_\tau u\|_{W^{k,\infty}(\tau)}$ in terms of $h = \operatorname{diam} \tau$ and Sobolev (semi) norms of u. Explicitly state what parameters (k, γ etc.) the constant may depend on.
7. Identify the classical boundary value problem corresponding to the following variational problem: Minimize
$$E(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \int_\Omega fu,$$
over $u \in H^2(\Omega)$, where Ω is a bounded domain with smooth boundary, and $f \in L^2(\Omega)$.