MATH 579 ASSIGNMENT 1

DUE THURSDAY JANUARY 30

1. Consider the piecewise quadratic finite elements on the interval $I = [0, 1]$:
$$\hat{X} = \{u \in C(I) : u|_{(x_i, x_{i+1})} \in \mathbb{P}_2 \forall i\}.$$
   a) Show that $\hat{X}$ has a nodal basis, where the set of nodes consists of the points $\{x_i\}$ as well as the midpoints $\frac{1}{2}(x_i + x_{i+1})$.
   b) Calculate the stiffness matrix for these elements, corresponding to the two-point boundary value problem $-u'' = f$ with $u(0) = u(1) = 0$.
   c) Estimate the H^1-error of the Galerkin solution, in terms of $|u|_{H^3}$ and the mesh width $h = \max_i |x_{i+1} - x_i|$.
2. For measurable functions u and w defined on the interval $I = (0, 1)$, we say that w is a *weak derivative* of u if
$$\int_I u\varphi' = -\int_I w\varphi, \qquad \text{for all} \quad \varphi \in C_c^1(I).$$
Moreover, for $u, v \in L^p(I)$, we say that v is a *strong L^p derivative* of u if there exists a sequence $\{u_k\} \subset C^1(I)$ such that
$$u_k \to u \quad \text{and} \quad u_k' \to v \quad \text{both in } L^p, \quad \text{as} \quad k \to \infty.$$
Show that each of the two derivatives is unique, if it exists, up to modifications on sets of measure zero. Prove that for $u \in L^p(I)$, both concepts coincide. In other words, show that each of the following statements implies the other.
   a) $v \in L^p(I)$ is a weak derivative of u.
   b) $v \in L^p(I)$ is a strong L^p derivative of u.
3. In this exercise we will study Sobolev spaces on the interval $I = (0, 1)$. Let $1 \le p < \infty$, and define the norm
$$\|u\|_{1,p} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p\right)^{1/p},$$
for $u \in C^1(\overline{I})$. Then we define $H^{1,p}(I)$ to be the completion of $C^1(\overline{I})$ with respect to the norm $\|\cdot\|_{1,p}$, and let
$$W^{1,p}(I) = \{u \in L^p(I) : u' \in L^p(I)\},$$
where u' is understood in the weak sense (or in the strong sense, since they are equivalent).
   a) Prove the *Meyers-Serrin theorem*: $H^{1,p}(I) = W^{1,p}(I)$.

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b) Prove the *Sobolev inequality*

$$\|u\|_{L^\infty} \leq 2^{1-1/p}\|u\|_{1,p}, \qquad u \in W^{1,p}(I).$$

c) Recall that an element of $W^{1,p}(I)$ is an equivalence class of functions that differ on sets of measure zero. So one can change the values of a function on a set of measure zero, and it would still correspond to the same element in $W^{1,p}(I)$. Make sense of, and prove the statement that the elements of $W^{1,p}(I)$ are continuous functions.

d) Prove the *Friedrichs inequality*

$$\|u\|_{L^p}^p \leq 2^{p-1}\|u'\|_{L^p}^p + 2^{p-1}|u(\xi)|^p, \qquad u \in W^{1,p}(I), \quad \xi \in [0,1].$$

e) Prove the *Poincaré inequality*

$$\|u\|_{L^p}^p \leq 2^{p-1}\|u'\|_{L^p}^p + 2^{p-1}\Big|\int_I u\Big|^p, \qquad u \in W^{1,p}(I).$$

f) Let $W_0^{1,p}(I)$ be the closure of $C_c^1(I)$ in $W^{1,p}(I)$. Show that

$$W_0^{1,p}(I) = \{u \in W^{1,p}(I) : u(0) = u(1) = 0\}.$$

g) Show that $\{u \in C^1(\bar{I}) : u'(0) = u'(1) = 0\}$ is dense in $W^{1,p}(I)$.

4. Define the function $f \in L^1(B_1)$ by $f(x) = |x|^r$ where $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ and $r > 1 - n$ is a constant. Determine the values of $p \geq 1$ such that $f \in W^{1,p}(B_1)$.
5. Let $u \in C^\infty(\Omega)$ be given in polar coordinates by $u(r,\theta) = r^a \sin(a\theta)$ with

$$\Omega = \{(r,\theta) : r < 1,\, 0 < \theta < \pi/a\},$$

where $a \geq \frac{1}{2}$ is a constant. Determine the values of $p \geq 1$ such that $u \in W^{2,p}(\Omega)$.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.

Hand in your work in class, before the lecture on the due date. Email submissions are accepted only if you type your solutions in LaTeX. Please do not use any other means without discussing it with the instructor (In particular, we no not have a homework box). Please try not to use extensions but if you need, you can get individual extensions by sending me an email preferably well in advance of the due date stating your proposed new deadline.