MATH 566: FINAL PROJECT

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Complex analysis is a standard part of any math curriculum. Less known is the intense connection between the pure complex analysis and fluid dynamics. In this project we try to give an insight into some of the interesting applications that exist.

1. The Joukowsky airfoil

1.1. **Introduction.** The theory of complex variables comes in naturally in the study of fluid phenomena. Let us define

$$w = \phi + i\psi$$

with ϕ the velocity potential and ψ the Lagrange stream function. Then w turns out to be analytic because the Cauchy-Riemann conditions

$$v_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

are exactly the natural flow conditions that have to be satisfied. The function w is called the *complex velocity potential*. Also note that

$$\frac{\mathrm{d}w}{\mathrm{d}z} = v_x - iv_y.$$

Basic examples of velocity potentials are the *uniform stream* $w = \bar{U}z$ where U represents flow speed and the *vortex* $w = (-i\Gamma/2\pi)\ln(z)$ where Γ represents rotation speed counted counterclockwise. An important role is dedicated to transformations F that are angle-preserving or *conformal*. Because compositions of analytic functions are analytic, we can use transforms to map simple cases onto harder ones.

1.2. **Flow past a cylinder.** Let us include circulation and look at the circular cylindrical case. We omit viscosity and other effects. One can write

$$w = z\bar{U} + \frac{a^2}{z}U + \frac{i\Gamma}{2\pi}\ln\left(\frac{z}{a}\right)$$

for the velocity potential of a flow past a circular cylinder. Here a is the diameter of the cylinder, U the flow speed and Γ a vortex related parameter as discussed earlier. The latter term was added to account for lift effects. It is put in artificially here but will come naturally later on.

It is easy to check that $w(a\exp(i\theta))$ is real so that $\psi = 0$ on the circle and that $\mathrm{d}w/\mathrm{d}z$ goes to $\bar{U}$ for z at infinity. Stagnation points for the flow are at the point z such that $\mathrm{d}w/\mathrm{d}z = 0$. If G is sufficiently small, we can define $\sin(\beta) = G = \Gamma/4\pi a|U|$ and $U = |U|\exp(-i\alpha)$ so that the stagnation points are located at

$$z_{\text{downstream}} = a\exp(-i(\alpha+\beta)), \quad z_{\text{upstream}} = a\exp(-i(\alpha-\beta-\pi)).$$

One can prove the force on the cylinder per unit distance lengthwise is

$$F = i\rho\Gamma\bar{U}.$$

This is perpendicular to flow direction.

A conformal map called the *Joukowsky transform* can be used to find the flow past an elliptical cylinder. It is defined by

$$z' = z + \frac{b^2}{z}.$$

Here b is a real number. Applying this transform to a circle radius a will get an ellipse with major axis $a + b^2/a$ and minor axis $a - b^2/a$. The flow problem can be solved more easily using this transform and our knowledge about the circular case.

1.3. **The Joukowsky airfoil.** Using a conformal mapping one can map the circle to a specific kind of airfoil shape called the *Joukowsky airfoil*. The general airfoil shape is round in the front and sharp in the back. At the sharp trailing edge the mapping will not be conformal. So to obtain the desired shape one has to produce a sequence of transformations. If we start off with a circle z in the complex plane we first translate it away from the origin by z'_c by writing. Then we transform to

$$z'' = z' + \frac{b^2}{z'}$$

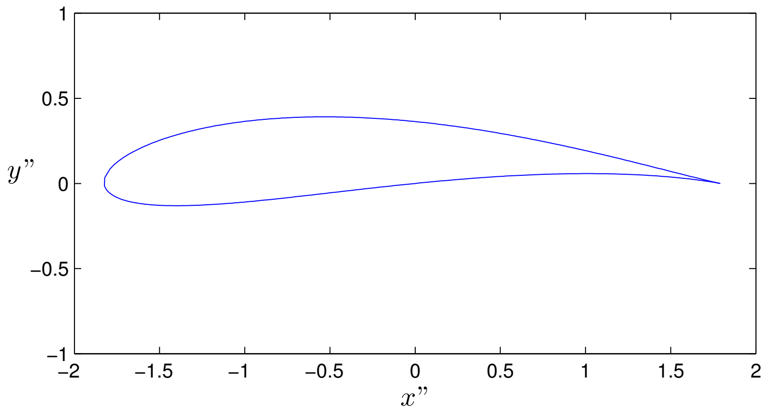
using the Joukowsky transform. The resulting effect on the derivative is

$$\frac{\mathrm{d}w}{\mathrm{d}z''} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}z'}\frac{\mathrm{d}z'}{\mathrm{d}z''} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{z'^2}{z'^2 - b^2}.$$

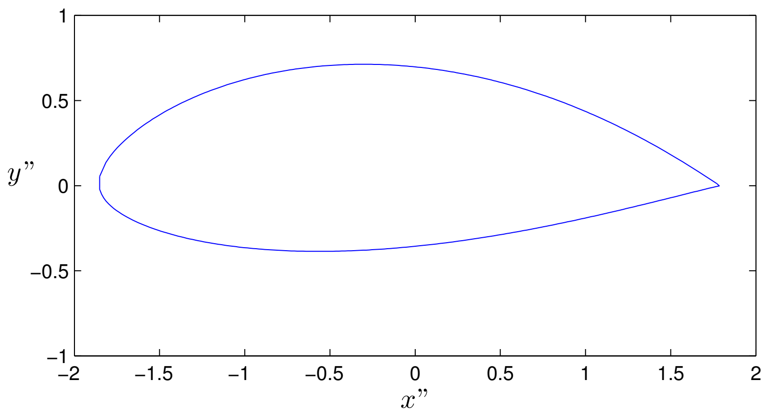
Note that the velocity in the z'' coordinate system will be singular when $z' = \pm b$. The only way for this to be prevented is by guaranteeing either of these points to be a stagnation point. Let us say that

$$z'_c + a\exp(i(\alpha-\beta)) = b.$$

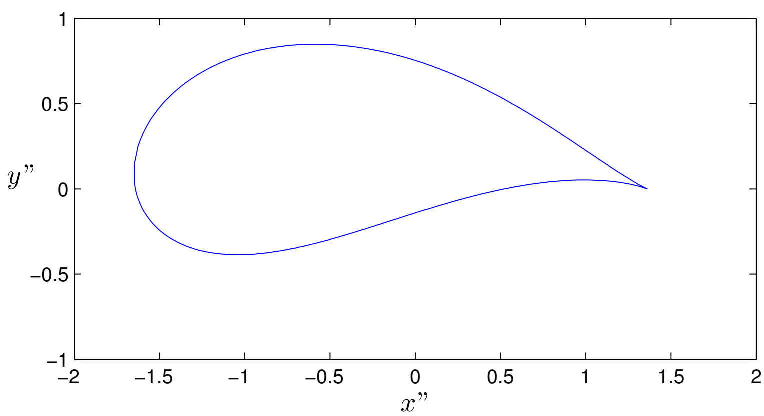
Additionally require the other point $-b$ to lie inside the circle.



(a) $n = 2, x'_c = -0.1, y'_c = 0.1$



(b) $n = 1.75, x'_c = -0.1, y'_c = 0.1$



(c) $n = 2, x'_c = -0.3, y'_c = 0.2$

FIGURE 1. A Karman-Trefftz family of airfoils.

The general shape of the Joukowsky airfoil can now be determined. It turns out α expresses attack angle, x'_c thickness and y'_c camber[1] of the airfoil. The parameters a and $|U|$ can be normalized. One can compute given these numbers

$$\beta = \alpha + \sin^{-1}(y'_c), \quad b = x'_c + \sqrt{1 - y'^2_c}$$

and

$$\Gamma = 4\pi \sin\beta.$$

Plots can be found in figures 1(a) and 1(c). Furthermore, the behavior at the trailing edge leads to important results in fluid mechanics about circulation of which one is known as the *Kutta condition*. The Kutta condition is responsible for the circulation that is necessary for the wing to create lift

$$C_p = 1 - \left| \frac{U}{U_\infty} \right|^2.$$

1.4. **The Karman-Trefftz airfoil and others.** The Joukowsky transform above can be rewritten by completing the square. Obtained is

$$z'' + 2b = \frac{(z'+b)^2}{z'}, \quad z'' - 2b = \frac{(z'-b)^2}{z'}$$

or equivalently

$$\frac{z''+2b}{z''-2b} = \left(\frac{z'+b}{z'-b}\right)^2.$$

Von Karman and Trefftz looked at a more general transformation satisfying

$$\frac{z''+2b}{z''-2b} = \left(\frac{z'+b}{z'-b}\right)^n.$$

It has most of the characteristic of the Joukowsky but the added parameter n allows to adjust the trailing edge angle to $(2-n)\pi$. An example with trailing edge angle 45° is found in figure 1(b). The somewhat artificial trailing edge shape of the Joukowsky airfoil is resolved.

Interesting other generalizations are the Jones-McWilliams airfoil and the NACA family of airfoils.

[1] Camber expresses the measure of asymmetry between the upper and lower surface of the airfoil.

2. Vortex shedding and the Von Karman street

Let us consider the flow of a fluid around an object in the plane. In general vortices will be shed from the edges of the object into its wake. This chain of shed vortices is called a *vortex street*. The pattern of the vortices in the street usually starts off symmetrical but shifts to be alternating. An interesting topic of study is the mechanism behind this phenomenon.

2.1. **Von Karman vortex street.** Let us model the vortex street by two rows of vortices spaced b apart in the y-direction. The vortices at $y = b/2$ are rotating counterclockwise and located at

$$\dots, -2a, -a, 0, a, 2a, \dots$$

in the x-direction. The lower row of vortices is rotating clockwise and is offset μa from the upper one. So each core is located at

$$\dots, -2a+\mu a, -a+\mu a, \mu a, a+\mu a, 2a+\mu a, \dots$$

This simple model is known as the *Von Karman vortex street*. The variable μ is a *offset parameter*. The case $\mu = 0$ represents symmetry while $\mu = 1/2$ represents the alternating pattern. The complex velocity potential can be found from the elementary vortex potential defined in the introduction. A vortex in the upper row can be expressed as

$$\frac{-i\Gamma}{2\pi} \ln\left(z - \left(na + i\frac{b}{2}\right)\right)$$

with $n \in \mathbb{Z}$. The lower row is expressed similarly by

$$\frac{i\Gamma}{2\pi} \ln\left(z - \left(na + \mu a - i\frac{b}{2}\right)\right).$$

The vortex street potential then equals

$$w = \frac{-i\Gamma}{2\pi} \sum_{n=1}^{\infty} \left[\ln\left(z - \left(na + i\frac{b}{2}\right)\right) - \ln\left(z - \left(na + \mu a - i\frac{b}{2}\right)\right)\right].$$

The advantage of using complex variables is immediately clear. We can rewrite the infinite sum of logarithms as a logarithm of an infinite product. Thus we obtain

$$w = \frac{-i\Gamma}{2\pi} \left[\ln\left(\sin\frac{\pi\left(z - i\frac{b}{2}\right)}{a}\right) - \ln\left(\sin\frac{\pi\left(z - \mu a + i\frac{b}{2}\right)}{a}\right)\right].$$

The associated complex conjugate velocity potential is

$$\begin{aligned}\bar{V} = \frac{\mathrm{d}w}{\mathrm{d}z} &= \frac{-i\Gamma}{2a}\left[\cot\frac{\pi\left(z - i\frac{b}{2}\right)}{a} - \cot\frac{\pi\left(z - \mu a + i\frac{b}{2}\right)}{a}\right] \\ &= \bar{V}_{\text{upper}} + \bar{V}_{\text{lower}}.\end{aligned}$$

The velocity in the fluid caused by the upper and lower row of vortices is denoted with V_{upper} and V_{lower} respectively.

2.2. **Evolution and stability.** Because of symmetry, vortices in the upper row do not produce a net force on each other and vortices in the lower row do not produce a net force on other vortices in the lower row. However, there is an effect on the vortices in the opposite row. The effect of the lower row of vortices on the vortex at $z = i\frac{b}{2}$ for example is described by

$$\bar{V}_{\text{lower}}\left(i\frac{b}{2}\right) = \frac{i\Gamma}{2a}\cot\pi\left(i\frac{b}{a} - \mu\right).$$

Note that this velocity is in general not pointed purely in the horizontal direction. Consequently, the shed vortex rows will tend to drift laterally towards or away from each other. Experimentally one can observe this drifting effect to be nearly nonexistent. Therefor we will choose to kill off the drifting effect by selecting μ such that $\bar{V}_{\text{lower}}\left(i\frac{b}{2}\right) \in \mathbb{R}$. The relevant values of μ can be computed by rewriting the cotangent as a quotient of cosine and sine and expressing these on their turn in exponentials. The outcome is that the velocity can be real only when

$$\sin 2\mu\pi = 0$$

or equivalently when $\mu = 0$ or $\mu = 1/2$. This means that the symmetrical and alternating patterns arise naturally. A next logical step is to look at the stability of both of these cases. The complete calculation is not done here because it is quite lengthy, but it relies heavily on complex analysis. The symmetrical arrangement is found to be unstable for at least some possible perturbations. The $\mu = 1/2$ case is more stable as long as $b/a \approx 0.28055$. This defines a stable ratio for the distance between the vortex rows and the vortices per row.

3. Christoffel transformation

The *Riemann Mapping Theorem* proves existence of holomorphic maps from random domains to the unit circle. A practical construction for these maps however is not given. For polygonal domains one can explicitly construct a mapping. This mapping is known as the *Schwarz-Christoffel tranformation* and it maps the interior of a closed polygon to the upper or lower half plane. A classical example of a fluid mechanical problem that profits from a treatment with the Christoffel transform is the flow in a channel over a step.

Assume we have some region D in the complex plane $\mathbb{C}$ bounded by a polygon with vertices at $w_1, w_2, \ldots, w_n$ (in counter clockwise direction) and respective interior angles $\varphi_1\pi, \varphi_2\pi, \ldots, \varphi_n\pi$. Let f be a conformal mapping from the upper half plane to D. We call $z_k = f^{-1}(w_k)$ the *pre-vertex* of w_k. All z_k are real. Furthermore, let us assume that $z_n = \infty$.

Note that if this is not the case, one can simply add the corresponding vertex with angle π to the list of w_k It turns out that under these assumptions

$$f(z) = \alpha \int_{z_0}^{z} (\zeta - z_1)^{\varphi_1 - 1} \dots (\zeta - z_n)^{\varphi_n - 1} \, \mathrm{d}\zeta + \beta.$$

Here α and β are free parameters and as a consequence z_0 is chosen at random. Evidently

$$f'(z) = \alpha (z - z_1)^{\varphi_1 - 1} \dots (z - z_n)^{\varphi_n - 1}.$$

There remains an obvious problem however. Because f is not known, the pre-vertices z_k are in general not known either. It turns out that in most regular cases the pre-vertices can be computed quite easily. For example for the channel with a single step one has four w_k. Without loss of generality $z_4 = \infty$. We can choose $z_1 = 0$ and $z_3 = 1$ as we wish. Only z_2 then needs to be determined and the parameter α. This can be done by using what extra information is known about the geometrical properties of the region D. For the problem at hand z_2 can be found after about a page of work.

4. References

The main sources used were *Elementary fluid mechanics* by Acheson, *Advanced fluid mechanics* by Graebel and *Complex analysis in Mathematica* by Shaw. Our knowledge of the theories presented has significantly improved in the last fifty years or so. For example the Von Karman model is now considered obsolete. Moreover, while reading up for this project I came across several interesting topics in complex analysis somewhat relevant to my own domain of research. So the time was very well spent.