PROBLEMS AND SOLUTIONS 3

Problem 3.1:1

Statement. Let $u(x,t)$ be a solution of $u_t = ku_{xx}$. Show that the following facts hold.

(a) For constants a, x_0 and t_0, the function $v(x,t) = u(ax - x_0, a^2t - t_0)$ satisfies $v_t = kv_{xx}$.

(b) For any constant k', the function $v(x,t) = u(x, (\frac{k'}{k})t)$ satisfies $v_t = k'v_{xx}$.

(c) The function $v(x,t) = t^{-\frac{1}{2}} \exp(-\frac{x^2}{4kt}) \cdot u(\frac{x}{t}, -\frac{1}{t})$ satisfies $v_t = kv_{xx}$.

Solution. (a) By a direct computation we have

$$v_t(x,t) = a^2u_t(ax - x_0, a^2t - t_0), \qquad v_{xx}(x,t) = a^2u_{xx}(ax - x_0, a^2t - t_0),$$

which confirms

$$v_t(x,t) = a^2u_t(ax - x_0, a^2t - t_0) = a^2ku_{xx}(ax - x_0, a^2t - t_0) = kv_{xx}(x,t),$$

where we have used the fact that $u_t = ku_{xx}$.

(b) Similarly to the preceding case, we have

$$v_t(x,t) = \frac{k'}{k}u_t(x, (\frac{k'}{k})t), \qquad v_{xx}(x,t) = u_{xx}(x, (\frac{k'}{k})t),$$

and so

$$v_t(x,t) = \frac{k'}{k}u_t(x, (\frac{k'}{k})t) = \frac{k'}{k}ku_{xx}(x, (\frac{k'}{k})t) = k'v_{xx}(x,t).$$

(c) Let $w(x,t) = t^{-\frac{1}{2}} \exp(-\frac{x^2}{4kt})$, so that $v(x,t) = w(x,t)u(\frac{x}{t}, -\frac{1}{t})$. We have

$$w_t(x,t) = -\frac{1}{2}t^{-\frac{3}{2}} \exp(-\frac{x^2}{4kt}) + t^{-\frac{1}{2}} \exp(-\frac{x^2}{4kt}) \cdot \frac{x^2}{4kt^2},$$
$$w_x(x,t) = t^{-\frac{1}{2}} \exp(-\frac{x^2}{4kt}) \cdot (-\frac{x}{2kt}),$$

and

$$w_{xx}(x,t) = t^{-\frac{1}{2}} \exp(-\frac{x^2}{4kt}) \cdot \frac{x^2}{4k^2t^2} + t^{-\frac{1}{2}} \exp(-\frac{x^2}{4kt}) \cdot (-\frac{1}{2kt}).$$

We note that

$$w_t = kw_{xx}, \qquad \text{and} \qquad w_x(x,t) = -\frac{x}{2kt}w(x,t). \tag{1}$$

Let us calculate v_t and v_{xx} as

$$v_t(x,t) = w_t(x,t)u(\frac{x}{t}, -\frac{1}{t}) + w(x,t)\left(-\frac{x}{t^2}u_x(\frac{x}{t}, -\frac{1}{t}) + \frac{1}{t^2}u_t(\frac{x}{t}, -\frac{1}{t})\right),$$
$$v_x(x,t) = w_x(x,t)u(\frac{x}{t}, -\frac{1}{t}) + w(x,t)\frac{1}{t}u_x(\frac{x}{t}, -\frac{1}{t}),$$

and

$$v_{xx}(x,t) = w_{xx}(x,t)u(\frac{x}{t}, -\frac{1}{t}) + 2w_x(x,t)\frac{1}{t}u_x(\frac{x}{t}, -\frac{1}{t}) + w(x,t)\frac{1}{t^2}u_{xx}(\frac{x}{t}, -\frac{1}{t}).$$

Now by comparing the expressions for v_t and v_{xx}, and taking (1) into account, we conclude $v_t = kv_{xx}$.

Date: Winter 2016.

Problems 3.1:6bd

Statement. Solve the problem

$$u_t = u_{xx}, \qquad (t \geq 0),$$

with the initial condition $u(x,0) = f(x)$, where the functions $u(x,t)$ and $f(x)$ are assumed to be 2π-periodic in the x variable. The function f is given in each case by

(b) $f(x) = \frac{1}{2} + \cos(2x) - 6\sin(2x)$,
(d) $f(x) = 6\sin(x) - 7\cos(3x) - 7\sin(3x)$.

Solution. We know that the solution of the above problem with the initial condition

$$f(x) = A_0 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)),$$

is given by

$$u(x,t) = A_0 + \sum_{n=1}^{N} e^{-n^2 t}(a_n \cos(nx) + b_n \sin(nx)).$$

A direct application of this formula gives

(b) $u(x,t) = \frac{1}{2} + e^{-4t}\cos(2x) - 6e^{-4t}\sin(2x)$,
(d) $u(x,t) = 6e^{-t}\sin(x) - 7e^{-9t}\cos(3x) - 7e^{-9t}\sin(3x)$.

Problem 3.1:8

Statement.

(a) Consider the problem

$$\begin{aligned} u_t &= k u_{xx}, \qquad (x \geq 0,\ t \geq 0), \\ u(0,t) &= \cos(\omega t), \qquad (t \geq 0). \end{aligned} \tag{2}$$

This is a heat conduction problem for a semi-infinite rod ($x \geq 0$) whose end (at $x = 0$) is subjected to a periodic temperature variation $u(0,t) = \cos(\omega t)$. Use the particular solutions

$$u(x,t) = Ae^{\lambda x}\cos(\lambda x + 2k\lambda^2 t) + Be^{\lambda x}\sin(\lambda x + 2k\lambda^2 t), \tag{3}$$

to find a solution of this problem which has both of the additional properties:

(P1) $u(x,t) \to 0$ as $x \to \infty$,
(P2) $u(x, t + \frac{2\pi}{\omega}) = u(x,t)$.

(b) Show that the solution of (2) is not unique, if either (P1) or (P2) is omitted.
(c) Assuming that $\omega = \frac{\pi}{2}$ and $k = \frac{\pi}{4}$, roughly sketch the graph of the temperature distribution in the xu-plane when $t = 0, 1, 2, 3, 4$, paying attention to where $u(x,t) = 0$.
(d) Show that at any fixed time t, the distance between consecutive local maxima, say x_1 and x_2, of $u(x,t)$ is $2\pi\sqrt{\frac{2k}{\omega}}$, and show that the ratio $u(x_2,t)/u(x_1,t)$ is $e^{-2\pi} \approx 0.00187$, regardless of the positive values of k and ω.

Solution. (a) In view of (3), the boundary condition $u(0,t) = \cos(\omega t)$ gives

$$u(0,t) = A\cos(2k\lambda^2 t) + B\sin(2k\lambda^2 t) = \cos(\omega t), \qquad t \geq 0,$$

implying that $A = 1$, $B = 0$, and $\lambda = \pm\sqrt{\frac{\omega}{2k}}$, i.e., we have the solution

$$u(x,t) = \exp(\pm\sqrt{\tfrac{\omega}{2k}}x)\cos(\pm\sqrt{\tfrac{\omega}{2k}}x + \omega t). \tag{4}$$

In order to satisfy (P1) we need to choose the minus sign in $\pm\sqrt{\frac{\omega}{2k}}$, so we finally have

$$u(x,t) = \exp(-\sqrt{\tfrac{\omega}{2k}}x)\cos(-\sqrt{\tfrac{\omega}{2k}}x + \omega t). \tag{5}$$

It is clear that this solution satisfies (P2).

(b) If (P1) is omitted, we can choose either plus or minus sign in (4), which means that the solution is not unique. On the other hand, if (P2) is dropped, we can add any $v(x,t)$ satisfying

$$\begin{aligned} v_t &= kv_{xx}, \qquad (x \geq 0,\, t \geq 0), \\ v(0,t) &= 0, \qquad (t \geq 0), \end{aligned}$$

to $u(x,t)$. For example, we can take

$$v(x,t) = \frac{1}{\sqrt{t+1}} \left(\exp\left(-\frac{(x-1)^2}{4k(t+1)} \right) - \exp\left(-\frac{(x+1)^2}{4k(t+1)} \right) \right).$$

(c) The time snapshots are depicted in Figure 1. To give a better idea of how the solution looks like, a spacetime graph of the solution is shown in Figure 2.

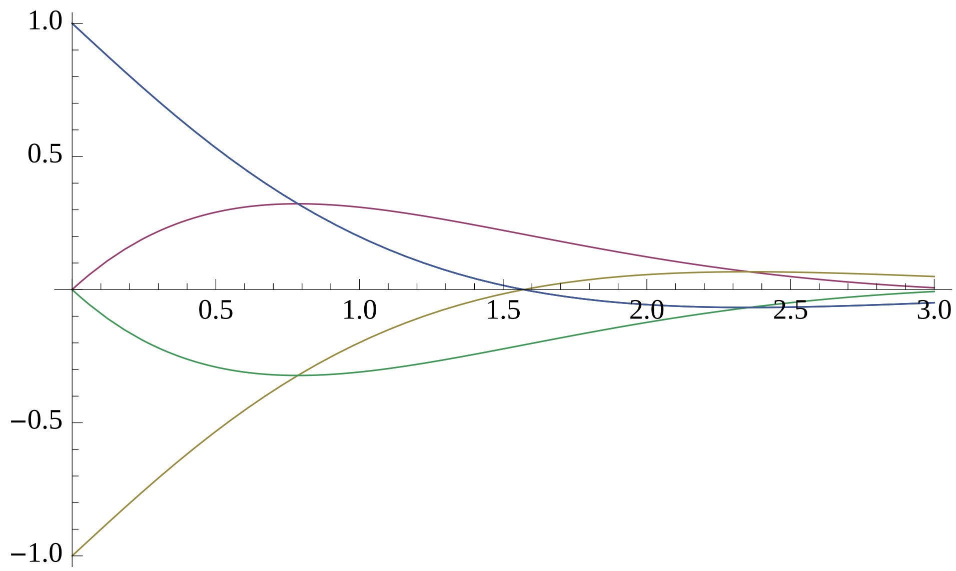

FIGURE 1. Time snapshots of the solution for 3.1:8c. Legend: $t = 0$ blue, $t = 1$ red, $t = 2$ yellow, $t = 3$ green, $t = 4$ blue again.

(d) The x-derivative of (5) is

$$\begin{aligned} u_x(x,t) &= -\sqrt{\tfrac{\omega}{2k}} \exp(-\sqrt{\tfrac{\omega}{2k}}x) \left(\cos(-\sqrt{\tfrac{\omega}{2k}}x + \omega t) - \sin(-\sqrt{\tfrac{\omega}{2k}}x + \omega t)\right) \\ &= -\sqrt{\tfrac{\omega}{k}} \exp(-\sqrt{\tfrac{\omega}{2k}}x) \cos(-\sqrt{\tfrac{\omega}{2k}}x + \omega t + \tfrac{\pi}{4}). \end{aligned}$$

Since $\exp(-\sqrt{\frac{\omega}{2k}}x) \neq$ for all x, the zeroes of $u_x(x,t)$ coincide with the zeroes of $\cos(-\sqrt{\frac{\omega}{2k}}x + \omega t + \frac{\pi}{4})$. The latter function is periodic in x with period $2\pi\sqrt{\frac{2k}{\omega}}$. This implies that the distance between consecutive local maxima is $2\pi\sqrt{\frac{2k}{\omega}}$ (there are two zeroes of u_x in one period, but one of the zeroes corresponds to a local minumum). As for the ratio of the values, we have

$$\frac{u(x_2,t)}{u(x_1,t)} = \frac{\exp(-\sqrt{\frac{\omega}{2k}}x_2)\cos(-\sqrt{\frac{\omega}{2k}}x_2 + \omega t + \frac{\pi}{4})}{\exp(-\sqrt{\frac{\omega}{2k}}x_1)\cos(-\sqrt{\frac{\omega}{2k}}x_1 + \omega t + \frac{\pi}{4})} = \exp(-\sqrt{\tfrac{\omega}{2k}} \cdot 2\pi\sqrt{\tfrac{2k}{\omega}}) = e^{-2\pi},$$

where we have taken into account the periodicity of cosine and the fact that $x_2 - x_1 = 2\pi\sqrt{\frac{2k}{\omega}}$.

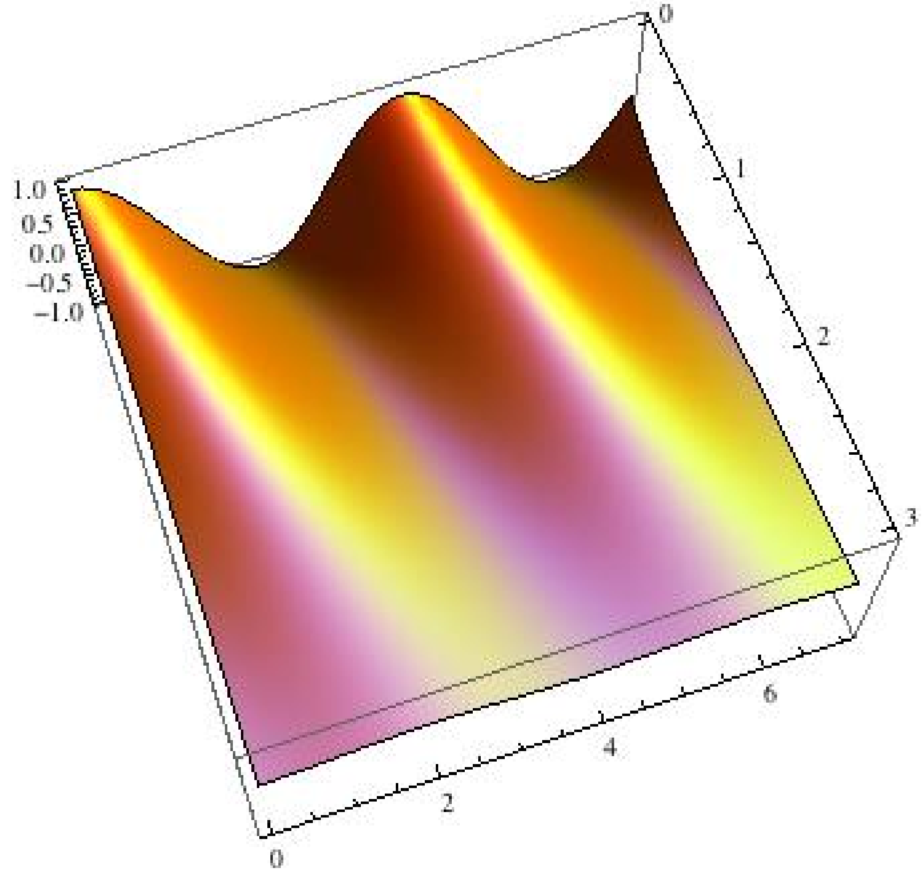

FIGURE 2. Spacetime graph of the solution for 3.1:8c. The t-axis is the one from left to right, the x-axis is from top to bottom, and the u-axis is directed towards the reader.

PROBLEM 3.2:1

Statement.

(a) Let $v(x,t)$ be any C^2 solution of $v_t = kv_{xx}$ ($0 \le x \le L$, $t \ge 0$), which satisfies the boundary conditions $v(0,t) = 0$ and $v(L,t) = 0$ (without initial condition). Show that for any t_1, t_2, with $t_2 \ge t_1 \ge 0$,

$$\int_0^L [v(x,t_2)]^2 \mathrm{d}x \le \int_0^L [v(x,t_1)]^2 \mathrm{d}x. \tag{6}$$

(b) Explain why the conclusion (6) still holds when the boundary conditions are replaced by any of the following pairs of boundary conditions:
   (i) $v_x(0,t) = v_x(L,t) = 0$,
   (ii) $v_x(0,t) = v(L,t) = 0$,
   (iii) $v_x(0,t) = h \cdot v(0,t)$ and $v(L,t) = 0$, where $h > 0$.

Solution. Let us define the function

$$E(t) = \int_0^L [v(x,t)]^2 \mathrm{d}x,$$

which can be called *energy*. Then (6) can be rephrased as

$$E(t_2) \le E(t_1), \qquad \text{for} \quad t_2 \ge t_1 \ge 0.$$

In other words, we have to show that E is a nondecreasing function of t. Let us calculate the time derivative of E as

$$\begin{aligned} E'(t) &= \int_0^L 2v(x,t)v_t(x,t)\mathrm{d}x = \int_0^L 2v(x,t)kv_{xx}(x,t)\mathrm{d}x \\ &= 2kv(L,t)v_x(L,t) - 2kv(0,t)v_x(0,t) - 2k\int_0^L |v_x(x,t)|^2\mathrm{d}x \\ &\leq 2kv(L,t)v_x(L,t) - 2kv(0,t)v_x(0,t). \end{aligned} \tag{7}$$

We will show below that $E'(t) \leq 0$ in various cases, which will then imply that E is nondecreasing. Note that each case requires a slightly different reasoning.

(a) We have

$$E'(t) \leq 2k\underbrace{v(L,t)}_{=0}v_x(L,t) - 2k\underbrace{v(0,t)}_{=0}v_x(0,t) = 0.$$

(b)(i) Similarly, we have

$$E'(t) \leq 2kv(L,t)\underbrace{v_x(L,t)}_{=0} - 2kv(0,t)\underbrace{v_x(0,t)}_{=0} = 0.$$

(b)(ii) We have

$$E'(t) \leq 2k\underbrace{v(L,t)}_{=0}v_x(L,t) - 2kv(0,t)\underbrace{v_x(0,t)}_{=0} = 0.$$

(b)(iii) We have

$$E'(t) \leq 2k\underbrace{v(L,t)}_{=0}v_x(L,t) - 2kv(0,t)\underbrace{v_x(0,t)}_{=hv(0,t)} = -2kh|v(0,t)|^2 \leq 0.$$

Problem 3.2:2

Statement. State and prove a uniqueness theorem for the problem

$$u_t = u_{xx},$$

with the boundary conditions $u_x(0,t) = a(t)$ and $u_x(L,t) = b(t)$, and the initial condition $u(x,0) = f(x)$.

Solution. We will prove that any two C^2 solutions u_1 and u_2 must be equal to each other.

Supposing that u_1 and u_2 are two C^2 solutions of our problem, let us define $v = u_1 - u_2$. Then by subtracting the equations satisfied by u_2 from the corresponding ones for u_1, we see that v satisfies $v_t = v_{xx}$ with the boundary conditions $v_x(0,t) = v_x(L,t) = 0$, and the initial condition $v(x,0) = 0$. We want to show that v is zero everywhere. From Part (b)(i) of the previous problem, we have $E(t) \leq E(0)$ for all $t \geq 0$, that is

$$E(t) = \int_0^L [v(x,t)]^2\mathrm{d}x \leq E(0) = \int_0^L [v(x,0)]^2\mathrm{d}x = 0.$$

Since $v(x,t)$ is a continuous function of x, this implies that $v(x,t) = 0$ for all $0 \leq x \leq L$, and as $t \geq 0$ was arbitrary, we conclude that $v = 0$ everywhere.

Problem 3.2:3

Statement. Use maximum/minimum principles to deduce that the solution u of the problem

$$\begin{aligned} u_t &= k u_{xx}, \qquad (0 \le x \le \pi,\ t \ge 0), \\ u(0,t) = u(\pi,t) &= 0, \qquad (t \ge 0), \\ u(x,0) &= \sin x + \tfrac{1}{2}\sin 2x, \qquad (0 \le x \le \pi), \end{aligned}$$

satisfies $0 \le u(x,t) \le \frac{3}{4}\sqrt{3}$ for all $0 \le x \le \pi$ and $t \ge 0$.

Solution. We will show that $0 \le \sin x + \frac{1}{2}\sin 2x \le \frac{3}{4}\sqrt{3}$ for all $0 \le x \le \pi$, which would then imply by the maximum and minimum principles the desired bounds for the solution u. First of all, the representation

$$f(x) = \sin x + \tfrac{1}{2}\sin 2x = \sin x + \sin x \cos x = (1 + \cos x)\sin x,$$

reveals that $f(x) \ge 0$ for $0 \le x \le \pi$. Let us find the maximum of $f(x)$. We calculate

$$f'(x) = \cos x + \cos 2x = 2\cos^2 x + \cos x - 1,$$

whose zeros are given by $\cos x = \frac{-1 \pm 3}{4}$. This implies $x = \frac{2\pi}{3}$ and $x = \pi$. The point $x = \pi$ is clearly not a maximum because $f(\pi) = 0$. The other candidate gives

$$f(\tfrac{2\pi}{3}) = (1 + \cos\tfrac{2\pi}{3})\sin\tfrac{2\pi}{3} = \tfrac{3}{2}\cdot\tfrac{\sqrt{3}}{2}.$$

It is easy to see from the behaviour of the function $f'(x)$ or from an inspection of $f''(x)$ that $x = \frac{2\pi}{3}$ is the only maximum point in the interval $0 \le x \le \pi$.

Problem 3.3:5

Statement. Solve

$$\begin{cases} u_t = 5u_{xx}, \quad 0 \le x \le 10,\ t \ge 0, \\ u_x(0,t) = 2, \quad u_x(10,t) = 3, \\ u(x,0) = \frac{1}{20}x^2 + 2x + \cos(\pi x). \end{cases}$$

Solution. First of all, we need to shift the unknown function so that the boundary conditions are homogeneous. Looking for a polynomial $p(x) = Ax^2 + Bx$ satisfying $p'(0) = 2$ and $p'(10) = 3$, we find $A = \frac{1}{20}$ and $B = 2$. Now define the new unknown $v = u - p$, so that $u = p + v$. Then since $u_t = v_t$ and $u_{xx} = v_{xx} + 2A$, we see that v must satisfy

$$\begin{cases} v_t = 5v_{xx} + \frac{1}{2}, \quad 0 \le x \le 10,\ t \ge 0, \\ v_x(0,t) = 0, \quad v_x(10,t) = 0, \\ v(x,0) = u(x,0) - p(x) = \cos(\pi x). \end{cases}$$

Next, we want to get rid of the inhomogeneous term $(+\frac{1}{2})$ in the right hand side of the heat equation. This can be achieved by modifying v by a linear function of t. Namely, if we put $v(x,t) = w(x,t) + \frac{1}{2}t$, then $v_t = w_t + \frac{1}{2}$ and $v_{xx} = w_{xx}$, so that w satisfies

$$\begin{cases} w_t = 5w_{xx}, \quad 0 \le x \le 10,\ t \ge 0, \\ w_x(0,t) = 0, \quad w_x(10,t) = 0, \\ w(x,0) = \cos(\pi x). \end{cases}$$

Note that we have used the properties $w_x(0,t) = v_x(0,t)$, $w_x(10,t) = v_x(10,t)$, and $w(x,0) = v(x,0)$. The problem for w can be solved easily, as

$$w(x,t) = e^{-5\pi^2 t}\cos(\pi x),$$

which yields

$$u(x,t) = w(x,t) + \frac{1}{2}t + p(x) = \frac{1}{2}t + \frac{1}{20}x^2 + 2x + e^{-5\pi^2 t}\cos(\pi x).$$

Problem 3.4:3

Statement. Solve

$$\begin{cases} u_t - u_{xx} = e^{-4t}\cos(t)\sin(2x), \quad 0 \le x \le \pi,\ t \ge 0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \\ u(x,0) = \sin(3x). \end{cases}$$

Solution. Let us decouple the effects of the initial condition and the inhomogeneity, by writing $u(x,t) = v(x,t) + w(x,t)$, where v satisfies

$$\begin{cases} v_t - v_{xx} = 0, \quad 0 \le x \le \pi,\ t \ge 0, \\ v(0,t) = 0, \quad v(\pi,t) = 0, \\ v(x,0) = \sin(3x), \end{cases}$$

and w satisfies

$$\begin{cases} w_t - w_{xx} = e^{-4t}\cos(t)\sin(2x), \quad 0 \le x \le \pi,\ t \ge 0, \\ w(0,t) = 0, \quad w(\pi,t) = 0, \\ w(x,0) = 0. \end{cases}$$

From separation of variables it is immediate that

$$v(x,t) = e^{-9t}\sin(3x).$$

To find w, for each fixed $t \ge 0$, we expand $w(x,t)$ as a function of $x \in [0,\pi]$, in terms of a Fourier sine series, as

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx).$$

Now our task is to find the (time-dependent) coefficients $a_n(t)$. If we also similarly expand the right hand side $f(x,t) = e^{-4t}\cos(t)\sin(2x)$ in the equation $w_t - w_{xx} = f$, then it is obvious that the coefficients $b_n(t)$ in

$$f(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(nx),$$

all vanish except for the case $n = 2$. The n-th coefficient of $w_t - w_{xx}$ is $a_n'(t) + n^2 a_n(t)$, which leads to the equations

$$\begin{cases} a_n'(t) + n^2 a_n(t) = 0 \qquad\qquad (n \ne 2), \\ a_2'(t) + 2^2 a_2(t) = e^{-4t}\cos(t). \end{cases}$$

Note that since since $u(x,0) = 0$, we have $a_n(0) = 0$ for all $n = 1, 2, 3, \ldots$. Hence for $n \ne 2$, we get $a_n(t) \equiv 0$. For $n = 2$, we have

$$(e^{4t}a_2)' = e^{4t}(a_2' + 4a_2) = \cos t,$$

and so

$$e^{4t}a_2(t) = a_2(0) + \sin t = \sin t,$$

or

$$a_2(t) = e^{-4t}\sin t.$$

Finally, we conclude

$$w(x,t) = e^{-4t}\sin(t)\sin(2x),$$

and thus

$$u(x,t) = e^{-9t}\sin(3x) + e^{-4t}\sin(t)\sin(2x).$$

Problem 3.4:9

Statement. Solve

$$\begin{cases} u_t - u_{xx} = x - x^2 + 2t + e^{-4\pi^2 t}\cos(2\pi x), & 0 \le x \le 1,\, t \ge 0, \\ u_x(0,t) = t, \quad u_x(\pi,t) = -t, \\ u(x,0) = 0. \end{cases}$$

Solution. First, we need to shift the unknown function so that the boundary conditions are homogeneous. The function $p(x,t) = (x - x^2)t$ satisfies $p_x(0,t) = t$ and $p_x(1,t) = -t$. Now define the new unknown $v = u - p$, so that $u = p + v$. Then since $u_t = v_t + x - x^2$ and $u_{xx} = v_{xx} - 2t$, we see that v must satisfy

$$\begin{cases} v_t - v_{xx} = e^{-4\pi^2 t}\cos(2\pi x), & 0 \le x \le 1,\, t \ge 0, \\ v_x(0,t) = 0, \quad v_x(1,t) = 0, \\ v(x,0) = 0. \end{cases}$$

This can be solved by using the Duhamel principle. The coefficients of the expansion

$$v(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos(\pi n x),$$

must satisfy

$$\begin{cases} a_n'(t) + \pi^2 n^2 a_n(t) = 0 & (n \ne 2), \\ a_2'(t) + \pi^2 2^2 a_2(t) = e^{-4\pi^2 t}, \end{cases}$$

and $a_n(0) = 0$ for all $n = 0, 1, 2, \ldots$. Hence for $n \ne 2$, we get $a_n(t) \equiv 0$. For $n = 2$, we have

$$(e^{4\pi^2 t} a_2)' = e^{4\pi^2 t}(a_2' + 4\pi^2 a_2) = 1,$$

and so

$$e^{4\pi^2 t} a_2(t) = a_2(0) + t = t,$$

or

$$a_2(t) = t e^{-4\pi^2 t}.$$

Finally, we conclude

$$v(x,t) = t e^{-4\pi^2 t}\cos(2\pi x),$$

and thus

$$u(x,t) = t(x - x^2) + t e^{-4\pi^2 t}\cos(2\pi x).$$

Problem 4.3:9

Statement. (a) Find a formal solution of the problem

$$\begin{cases} u_t = k u_{xx}, & 0 \le x \le 1,\, t \ge 0, \\ u(0,t) = 0, \quad u(1,t) = 0, \\ u(x,0) = f(x), \end{cases}$$

where

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2}, \\ 1 - x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

(b) If $u_t(x,t)$ is formally computed by differentiating each term of the formal solution with respect to t, then show that $u_t(\frac{1}{2}, 0) = -\infty$ results. Provide a physical explanation of this observation by considering the flux of heat through the ends of a small interval centred at $x = \frac{1}{2}$.

Solution. (a) The formal solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n\pi^2 kt} \sin(n\pi x), \qquad \text{with} \quad b_n = 2\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x.$$

Note that $\sin(n\pi(1-x)) = \sin(n\pi - n\pi x) = \sin(n\pi x)$ when n is odd, and $\sin(n\pi - n\pi x) = -\sin(n\pi x)$ when n is even. Using this symmetry, and taking into account that $f(1-x) = f(x)$, we infer

$$b_n = 2\int_0^{\frac{1}{2}} f(x)\sin(n\pi x)\,\mathrm{d}x + 2(-1)^{n+1}\int_0^{\frac{1}{2}} f(x)\sin(n\pi x)\,\mathrm{d}x.$$

So $b_n = 0$ for even n, and

$$\begin{aligned} b_n = 4\int_0^{\frac{1}{2}} f(x)\sin(n\pi x)\,\mathrm{d}x &= -\frac{4}{n\pi}\cos(n\pi x)\Big|_0^{\frac{1}{2}} + \frac{4}{n\pi}\int_0^{\frac{1}{2}}\cos(n\pi x)\,\mathrm{d}x \\ &= \frac{4}{n\pi} + (-1)^m \frac{4}{n^2\pi^2}, \end{aligned}$$

for odd n, with $n = 2m+1$. Putting everything together, we conclude

$$u(x,t) = \sum_{m=0}^{\infty}\left(\frac{4}{n\pi} + (-1)^m\frac{4}{n^2\pi^2}\right)e^{-n\pi^2 kt}\sin(n\pi x),$$

where n depends on m as $n = 2m+1$.

(b) Formally, we compute

$$u_t(x,t) = -\sum_{m=0}^{\infty}\left(4\pi k + (-1)^m\frac{4k}{n}\right)e^{-n\pi^2 kt}\sin(n\pi x),$$

and so

$$\begin{aligned} u_t(\tfrac{1}{2},0) &= -4k\sum_{m=0}^{\infty}\left(\pi + \frac{(-1)^m}{2m+1}\right)\sin\frac{(2m+1)\pi}{2} \\ &= -4k\sum_{m=0}^{\infty}\left((-1)^m\pi + \frac{1}{2m+1}\right) = -\infty, \end{aligned}$$

since $k > 0$ and $\sum_{m=0}^{\infty}\frac{1}{2m+1} = \infty$. This result suggests that the temperature at the point $x = \frac{1}{2}$ drops infinitely fast for a very short (in fact infinitesimal) time near $t = 0$.

As suggested in the statement, we can also formally compute the flux as

$$\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} u_t(x,0)\,\mathrm{d}x = k\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} u_{xx}(x,0)\,\mathrm{d}x = ku_x(x,0)\Big|_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} = -2k,$$

where $\varepsilon > 0$ is small. We see that no matter how small the interval $[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$ is, the integral of u_t over it is a fixed negative number. Therefore, the function u_t must become negative infinity at $x = \frac{1}{2}$.

Problem 4.3:12

Statement. Find a formal solution of the problem

$$\begin{cases} u_t = ku_{xx}, \quad 0 \le x \le 10,\ t \ge 0, \\ u_x(0,t) = 2, \quad u_x(10,t) = 3, \\ u(x,0) = 0. \end{cases}$$

Solution. First of all, we need to shift the unknown function so that the boundary conditions are homogeneous. Looking for a polynomial $p(x) = Ax^2 + Bx$ satisfying $p'(0) = 2$ and $p'(10) = 3$, we find $A = \frac{1}{20}$ and $B = 2$. Now define the new unknown $v = u - p$, so that $u = p + v$. Then since $u_t = v_t$ and $u_{xx} = v_{xx} + 2A$, we see that v must satisfy

$$\begin{cases} v_t = kv_{xx} + 2kA, \quad 0 \leq x \leq 10,\, t \geq 0, \\ v_x(0,t) = 0, \quad v_x(10,t) = 0, \\ v(x,0) = -p(x) = -Ax^2 - Bx. \end{cases}$$

In order to use separation of variables, we assume

$$v(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos(\frac{n\pi x}{10}),$$

and formally substitute it into $v_t = kv_{xx} + 2kA$, to get

$$a_0'(t) = 2kA,$$

and

$$a_n'(t) = -\frac{n^2\pi^2 k}{100} a_n(t), \qquad n > 0.$$

Note that the cosine series of $2kA$ involves only the constant term, so it does not affect at all the equations for $n > 0$, which remain the same as the equation for the homogeneous case $v_t = kv_{xx}$. The equations are easily solved as

$$a_0(t) = a_0(0) + 2kAt, \qquad \text{and} \qquad a_n(t) = a_n(0) \exp(-\frac{n^2\pi^2 k}{100} t), \quad n > 0.$$

Obviously, $a_n(0)$ for $n \geq 0$ are the cosine series coefficients of the initial datum $v(x,0) = -p(x)$, which are given by

$$a_0(0) = -\frac{1}{10} \int_0^{10} p(x)\,\mathrm{d}x, \qquad \text{and} \qquad a_n(0) = -\frac{1}{5} \int_0^{10} p(x) \cos(\frac{n\pi x}{10})\,\mathrm{d}x, \quad n > 0.$$

Let us do the computation. We have

$$\int_0^{10} p(x)\,\mathrm{d}x = \left(\frac{Ax^3}{3} + \frac{Bx^2}{2} \right)\Big|_0^{10} = \frac{50}{3} + 100,$$

and

$$\begin{aligned} \int_0^{10} x \cos(\frac{n\pi x}{10})\,\mathrm{d}x &= \frac{10}{n\pi} x \sin(\frac{n\pi x}{10})\Big|_0^{10} - \frac{10}{n\pi} \int_0^{10} \sin(\frac{n\pi x}{10})\,\mathrm{d}x \\ &= \frac{100}{n^2\pi^2} \cos(\frac{n\pi x}{10})\Big|_0^{10} = ((-1)^n - 1)\frac{100}{n^2\pi^2}, \end{aligned}$$

as well as

$$\begin{aligned} \int_0^{10} x^2 \cos(\frac{n\pi x}{10})\,\mathrm{d}x &= \frac{10}{n\pi} x^2 \sin(\frac{n\pi x}{10})\Big|_0^{10} - \frac{20}{n\pi} \int_0^{10} x \sin(\frac{n\pi x}{10})\,\mathrm{d}x \\ &= \frac{200}{n^2\pi^2} x \cos(\frac{n\pi x}{10})\Big|_0^{10} - \frac{200}{n^2\pi^2} \int_0^{10} \cos(\frac{n\pi x}{10})\,\mathrm{d}x \\ &= (-1)^n \frac{2000}{n^2\pi^2}, \end{aligned}$$

leading to

$$a_0(0) = \frac{35}{3}, \qquad a_n(0) = -(-1)^n \frac{400}{n^2\pi^2} A + (1 - (-1)^n) \frac{20}{n^2\pi^2} B = \frac{40 + 60(-1)^{n+1}}{n^2\pi^2}.$$

Therefore, we conclude

$$v(x,t) = \frac{35}{3} + \frac{k}{10}t + \sum_{n=1}^{\infty} \frac{40 + 60(-1)^{n+1}}{n^2\pi^2} \exp(-\frac{n^2\pi^2 k}{100}t)\cos(\frac{n\pi x}{10}),$$

and so

$$u(x,t) = \frac{35}{3} + 2x + \frac{1}{20}x^2 + \frac{k}{10}t + \sum_{n=1}^{\infty} \frac{40 + 60(-1)^{n+1}}{n^2\pi^2} \exp(-\frac{n^2\pi^2 k}{100}t)\cos(\frac{n\pi x}{10}).$$