Lecture 5: Poisson equation in 3D

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Gauss' law

If instead of point charges, charge density $\rho(x)$ is given, the electric field is

$$E(x) = C \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|^2} \frac{x-y}{|x-y|} \mathrm{d}y,$$

For any closed surface S, we have

$$\int_S \frac{(x-y)\cdot \mathrm{d}S_x}{|x-y|^3} = \begin{cases} 4\pi & \text{if } y \text{ is inside } S, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, with V the volume enclosed by S, and Q the total charge in V

$$\int_S E(x)\cdot \mathrm{d}S_x = C \int_S \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|^2} \frac{(x-y)\cdot \mathrm{d}S_x}{|x-y|} \mathrm{d}y = 4\pi C \int_V \rho(y) \mathrm{d}y = 4\pi C Q.$$

The divergence theorem gives

$$\int_S E(x)\cdot \mathrm{d}S_x = \int_V \nabla \cdot E(y)\, \mathrm{d}y.$$

Since V is abritrary, we infer *Gauss' law*: $\nabla \cdot E = 4\pi C \rho$.

On the other hand, we have

$$E(x) = \int_{\mathbb{R}^3} \frac{C\rho(y)}{|x-y|^2} \frac{x-y}{|x-y|} \mathrm{d}y = -\int_{\mathbb{R}^3} C\rho(y) \nabla \frac{1}{|x-y|} \mathrm{d}y = -\nabla \int_{\mathbb{R}^3} \frac{C\rho(y)}{|x-y|} \mathrm{d}y$$

So the scalar potential

$$\varphi(x) = \int_{\mathbb{R}^3} \frac{C\rho(y)}{|x-y|} \mathrm{d}y,$$

satisfies

$$E = -\nabla\varphi, \qquad \text{and so} \qquad -\Delta\varphi = 4\pi C\rho.$$

If $4\pi C\rho(x) = \delta(x-z)$, then

$$\varphi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\delta(y-z)}{|x-y|} \mathrm{d}y = \frac{1}{4\pi|x-z|}.$$

Fundamental solution

The *fundamental solution* of the Laplace operator is

$$\Phi(x,z) = \frac{1}{4\pi|x-z|}, \qquad \text{which satisfies} \qquad -\Delta\Phi(x,z) = \delta(x-z).$$

If f is continuous and has bounded support, then

$$\varphi(x) = \int_{\mathbb{R}^3} \Phi(x,z) f(z) \mathrm{d}z,$$

satisfies

$$-\Delta\varphi = f.$$