# Lecture 27: Rectangular problems

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# Laplace eigenproblem on a rectangle

Let us solve the eigenproblem

$$\Delta v = \lambda v \qquad \text{on} \quad (0,\pi)\times(0,\pi),$$

with the homogeneous Dirichlet boundary condition. Considering $v(x,y)$ for any fixed $y$ as a function of $x$, we can write

$$v(x,y) = \sum_{j=1}^{\infty} \omega_j(y)\sin(jx) = \sum_{j=1}^{\infty} \omega_j(y)\, v_j(x).$$

This leads to

$$\sum_{j=1}^{\infty} \left(-j^2\omega_j + (\omega_j)_{yy}\right) v_j = \sum_{j=1}^{\infty} \lambda\omega_j v_j \qquad \Rightarrow \qquad (\omega_j)_{yy} = (\lambda + j^2)\omega_j,$$

with $\omega_j(0) = \omega_j(\pi) = 0$. We know that the only solutions are

$$\omega(y) = \sin(ky), \qquad \text{provided} \qquad \lambda + j^2 = -k^2,$$

for some positive integer $k$.

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We see that the eigenvalues are of the form

$$\lambda_{jk} = -(j^2 + k^2),$$

with the corresponding eigenfunctions

$$v_{jk}(x, y) = v_j(x)\, v_k(y) = \sin(jx)\sin(ky).$$

These eigenfunctions are pairwise orthogonal:

$$\left\langle v_{jk}, v_{j'k'} \right\rangle = (\pi/2)^2 \delta_{jj'} \delta_{kk'},$$

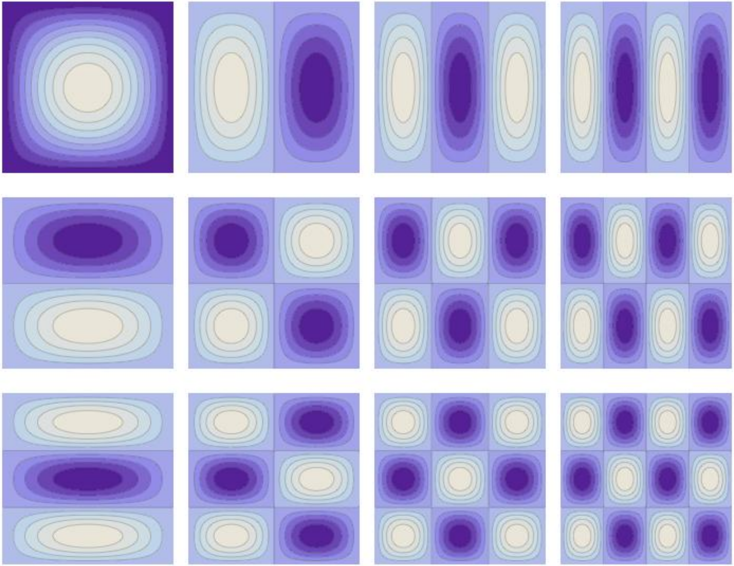
where

$$\langle v, w \rangle = \int_0^\pi \int_0^\pi v(x, y)\, w(x, y) \mathrm{d}x \mathrm{d}y.$$

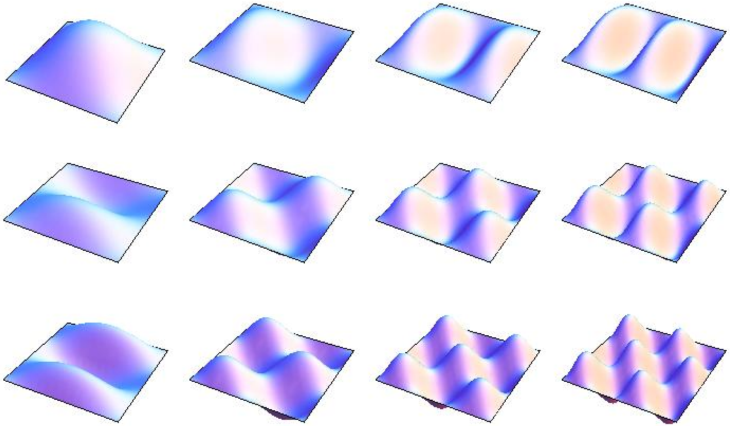
Some eigenfunctions share the same eigenvalues, e.g., $\lambda_{1,2} = -5 = \lambda_{2,1}$ and $\lambda_{7,6} = -85 = \lambda_{9,2}$. The number $N(\mu)$ of eigenvalues with magnitude below given $\mu > 0$, and the $n$-th eigenvalue $\lambda_n$ are roughly

$$N(\mu) \sim \pi\mu/4, \qquad \lambda_n \sim -4n/\pi.$$

# Laplace eigenfunctions on a rectangle



# Laplace eigenfunctions on a rectangle



# Rectangular problems

Let $f$ and $g$ be functions defined on $(0,\pi)^2$. Then with homogeneous Dirichlet boundary conditions, consider

- The Poisson problem $\Delta u = f$
- The heat equation $u_t = \Delta u$, with $u(x,y,0) = f(x,y)$
- Wave $u_{tt} = \Delta u$, with $u(x,y,0) = f(x,y)$ and $u_t(x,y,0) = g(x,y)$

We can write

$$u(x,y,t) = \sum_{j,k=1}^{\infty} \xi_{jk}(t) v_{jk}(x,y), \qquad f = \sum_{j,k=1}^{\infty} \beta_{jk} v_{jk}, \qquad g = \sum_{j,k=1}^{\infty} \gamma_{jk} v_{jk},$$

with $u$ (and so $\xi_{jk}$) not depending on $t$ for the Poisson case. Then

- for Poisson $\xi_{jk} = -\beta_{jk} / \left(j^2 + k^2\right)$
- for heat $\xi_{jk}(t) = e^{-\left(j^2+k^2\right)t} \beta_{jk}$
- for wave $\xi_{jk}(t) = \beta_{jk} \cos \omega_{jk} t + \frac{\gamma_{jk}}{\omega_{jk}} \sin \omega_{jk} t$, with $\omega_{jk} = \sqrt{-\left(j^2 + k^2\right)}$