Math 315 Midterm Exam Winter 2014
Version 4

1. Solve \( x^2 e^{x^2+y} (2x^2 + 3) \, dx + (x^3 e^{x^2+y} + \sin y \cos y) \, dy = 0. \)

**Solution:** The equation is of the form
\[
a(x, y) \, dx + b(x, y) \, dy = 0,
\]
where
\[
a(x, y) = x^2 e^{x^2+y} (2x^2 + 3), \quad \text{and} \quad b(x, y) = x^3 e^{x^2+y} + \sin y \cos y.
\]
We have
\[
\frac{\partial a}{\partial y} = x^2 e^{x^2+y} (2x^2 + 3), \quad \text{and} \quad \frac{\partial b}{\partial x} = 3x^2 e^{x^2+y} + 2x^3 e^{x^2+y} + 0,
\]
so the equation is exact. This means that there is a function \( F(x, y) \) such that
\[
\frac{\partial F(x, y)}{\partial x} = a(x, y), \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = b(x, y).
\]
The latter condition gives
\[
F(x, y) = \int b(x, y) \, dy = \int (x^3 e^{x^2+y} + \sin y \cos y) \, dy = x^3 e^{x^2+y} + \frac{1}{2} \sin^2 y + f(x),
\]
and substituting this into the former, we get
\[
\frac{\partial F(x, y)}{\partial x} = 3x^2 e^{x^2+y} + 2x^4 e^{x^2+y} + f'(x) = a(x, y) = x^2 e^{x^2+y} (2x^2 + 3).
\]
From this it is clear that we can choose \( f(x) = 0 \), and we get
\[
F(x, y) = x^3 e^{x^2+y} + \frac{1}{2} \sin^2 y.
\]
Finally, we can write the solution in implicit form as
\[
x^3 e^{x^2+y} + \frac{1}{2} \sin^2 y = A,
\]
with \( A \) an arbitrary constant.

2. Solve the initial value problem \( y' + y \sin x = y^2 \sin x \) with \( y(\frac{\pi}{2}) = \frac{1}{2} \).

**Solution:** We recognize the equation as a Bernoulli equation \( y' + ay = by^k \), with \( k = 2 \), and recall that a good substitution is \( u = y^{1-k} = y^{-1} \) or \( y = \frac{1}{u} \). If one does not remember the exact exponent \( q \) in the substitution \( y = u^q \), then it is also possible to
work with an unknown exponent $q$ until a point when the equation itself reveals which value of $q$ would lead to a linear equation.

Getting back to solving the equation, if $y = \frac{1}{u}$, then $y' = -\frac{u'}{u^2}$, and so

$$-\frac{u'}{u^2} + \frac{\sin x}{u} = \frac{\sin x}{u^2}, \quad \text{or} \quad u' - u \sin x = -\sin x.$$  

Since $(\cos x)' = -\sin x$, a suggested integrating factor is $\mu(x) = e^{\cos x}$. Let us compute

$$(e^{\cos x}u(x))' = e^{\cos x}u'(x) + e^{\cos x}(-\sin x)u(x) = e^{\cos x}(u'(x) - u(x) \sin x).$$

This must be equal to $-e^{\cos x} \sin x$, if $u$ were to satisfy $u' - u \sin x = -\sin x$. So we have

$$(e^{\cos x}u(x))' = -e^{\cos x} \sin x.$$ 

A direct integration gives

$$e^{\cos x}u(x) = -\int e^{\cos x} \sin x \, dx = \int e^{\cos x} \, d \cos x = e^{\cos x} + C,$$

and thus

$$u(x) = 1 + Ce^{-\cos x}.$$ 

Before converting this into an expression for $y(x)$, we want to impose the initial condition $y(\frac{\pi}{2}) = \frac{1}{2}$. In terms of $u = \frac{1}{y}$, this initial condition is $u(\frac{\pi}{2}) = 2$. Since $\cos \frac{\pi}{2} = 0$, it leads to

$$u(\frac{\pi}{2}) = 1 + C = 2, \quad \text{that is,} \quad C = 1.$$ 

Finally, the solution to the original initial value problem is

$$y(x) = \frac{1}{u(x)} = \frac{1}{1 + e^{-\cos x}}.$$