Math 315 Midterm Exam Winter 2014

Version 2

1. Solve $(-2y^2 \sin x + 3y^3)\, \mathrm{d}x + (4y \cos x + 9xy^2)\, \mathrm{d}y = 0$.

   Solution: The equation is of the form

   $$a(x, y)\, \mathrm{d}x + b(x, y)\, \mathrm{d}y = 0,$$

   where

   $$a(x, y) = -2y^2 \sin x + 3y^3, \qquad \text{and} \qquad b(x, y) = 4y \cos x + 9xy^2.$$

   We have

   $$\frac{\partial a}{\partial y} = -4y \sin x + 9y^2, \qquad \text{and} \qquad \frac{\partial b}{\partial x} = -4y \sin x + 9y^2,$$

   so the equation is *exact*. This means that there is a function $F(x, y)$ such that

   $$\frac{\partial F(x, y)}{\partial x} = a(x, y), \qquad \text{and} \qquad \frac{\partial F(x, y)}{\partial y} = b(x, y).$$

   The latter condition gives

   $$F(x, y) = \int b(x, y) \mathrm{d}y = \int (4y \cos x + 9xy^2) \mathrm{d}y = 2y^2 \cos x + 3xy^3 + f(x),$$

   and substituting this into the former, we get

   $$\frac{\partial F(x, y)}{\partial x} = -2y^2 \sin x + 3y^3 + f'(x) = a(x, y) = -2y^2 \sin x + 3y^3.$$

   From this it is clear that we can choose $f(x) = 0$, and we get $F(x, y) = 2y^2 \cos x + 3xy^3$. We conclude that the solution written in implicit form is

   $$2y^2 \cos x + 3xy^3 = A,$$

   with A an arbitrary constant.

2. Solve the initial value problem $y' - 2xy = xe^{x^2} y^2$ with $y(0) = 0$.

   Solution: A quick solution would be to observe that the function $y(x) = 0$ satisfies the equation as well as the initial condition $y(0) = 0$.

   The more systematic way is of course to recognize the equation as a Bernoulli equation $y' + \alpha y = \beta y^k$, with $k = 2$, and recall that a good substitution is $u = y^{1-k} = y^{-1}$ or $y = \frac{1}{u}$. If one does not remember the exact exponent q in the substitution $y = u^q$, then it is also possible to work with an unknown exponent q until a point when the equation itself reveals which value of q would lead to a linear equation.

   Getting back to solving the equation, if $y = \frac{1}{u}$, then $y' = -\frac{u'}{u^2}$, and so

   $$-\frac{u'}{u^2} - \frac{2x}{u} = \frac{xe^{x^2}}{u^2}, \qquad \text{or} \qquad u' + 2xu = -xe^{x^2}.$$

Since $(x^2)' = 2x$, a suggested integrating factor is $\mu(x) = e^{x^2}$. Let us compute

$$(e^{x^2}u(x))' = e^{x^2}u'(x) + e^{x^2}(2x)u(x) = e^{x^2}(u'(x) + xu(x)).$$

This must be equal to $e^{x^2}(-xe^{x^2})$, if u were to satisfy $u' + 2xu = -xe^{x^2}$. So we have

$$(e^{x^2}u(x))' = -xe^{2x^2}.$$

A direct integration gives

$$e^{x^2}u(x) = -\int xe^{2x^2}\,\mathrm{d}x = -\frac{1}{2}\int e^{2x^2}\,\mathrm{d}x^2 = -\frac{e^{2x^2}}{4} + C,$$

and thus

$$u(x) = -\frac{e^{x^2}}{4} + Ce^{-x^2}.$$

By introducing the constant $c = -C/4$, we can rewrite it as $u(x) = -\dfrac{e^{x^2} + ce^{-x^2}}{4}$, which gives the following convenient expression

$$y(x) = \frac{1}{u(x)} = -\frac{4}{e^{x^2} + ce^{-x^2}}. \tag{$*$}$$

Now we would like to impose the initial condition $y(0) = 0$, and we face the problem that the value

$$y(0) = -\frac{4}{1+c},$$

is nonzero for any value of the constant $c \in \mathbb{R}$. The only way to achieve $y(0) = 0$ would be to formally put $c = \infty$, because $\frac{4}{1+c} \to 0$ as $c \to \infty$. Rather than putting $c = \infty$ directly into $(*)$, a better approach is to send $c \to \infty$ and examine how the value $y(x)$ behaves. We see that for any fixed $x \in \mathbb{R}$, the value $y(x)$ tends to 0 as $c \to \infty$ (or even as $c \to -\infty$). This suggests that the function $y(x) = 0$ may be a solution of the original problem $y' - 2xy = xe^{x^2}y^2$ with $y(0) = 0$, and as we have observed in the beginning, it is easy to verify that it is indeed the case. The existence of such a solution (called a *singular solution*) that cannot be obtained from $(*)$ with any number $c \in \mathbb{R}$, is an interesting property of nonlinear equations. The term "singular" refers to the fact that the solution stays by itself, "isolated" from the family $(*)$. As a function, there is nothing "singular" about a singular solution; it is a perfectly regular function. Formally, the singular solution corresponds to the value $c = \pm\infty$, and therefore the set of all solutions to the equation $y' - 2xy = xe^{x^2}y^2$ forms a *circle*, in the sense that the singular solution at $c = \infty$ can be reached from two sides: by increasing the value of c towards $+\infty$, and by decreasing it towards $-\infty$. The point $c = \infty$ can be thought of as being diagonally opposite to the point $c = 0$. In contrast, we know that all solutions of a first order *linear* equation can be obtained by varying a parameter c over $\mathbb{R}$, hence the set of solutions in this case forms a *line*. In general, the solutions of some nonlinear equations may form lines, that of others may form shapes more complicated than a circle.

Note: The initial condition $y(0) = 0$ leads to the singular solution, which was not really intended. You will receive a full mark if you have reached the point $(*)$, and have shown an attempt to find c.