1. Solve \((-2y^2 \sin x + 3y^3) \, dx + (4y \cos x + 9xy^2) \, dy = 0\).

**Solution:** The equation is of the form

\[ a(x, y) \, dx + b(x, y) \, dy = 0, \]

where

\[ a(x, y) = -2y^2 \sin x + 3y^3, \quad \text{and} \quad b(x, y) = 4y \cos x + 9xy^2. \]

We have

\[ \frac{\partial a}{\partial y} = -4y \sin x + 9y^2, \quad \text{and} \quad \frac{\partial b}{\partial x} = -4y \sin x + 9y^2, \]

so the equation is *exact*. This means that there is a function \(F(x, y)\) such that

\[ \frac{\partial F(x, y)}{\partial x} = a(x, y), \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = b(x, y). \]

The latter condition gives

\[ F(x, y) = \int b(x, y) \, dy = \int (4y \cos x + 9xy^2) \, dy = 2y^2 \cos x + 3xy^3 + f(x), \]

and substituting this into the former, we get

\[ \frac{\partial F(x, y)}{\partial x} = -2y^2 \sin x + 3y^3 + f'(x) = a(x, y) = -2y^2 \sin x + 3y^3. \]

From this it is clear that we can choose \(f(x) = 0\), and we get \(F(x, y) = 2y^2 \cos x + 3xy^3\).

We conclude that the solution written in implicit form is

\[ 2y^2 \cos x + 3xy^3 = A, \]

with \(A\) an arbitrary constant.

2. Solve the initial value problem \(y' - 2xy = xe^x y^2\) with \(y(0) = 0\).

**Solution:** A quick solution would be to observe that the function \(y(x) = 0\) satisfies the equation as well as the initial condition \(y(0) = 0\).

The more systematic way is of course to recognize the equation as a Bernoulli equation \(y' + \alpha y = \beta y^k\), with \(k = 2\), and recall that a good substitution is \(u = y^{1-k} = y^{-1}\) or \(y = \frac{1}{u}\). If one does not remember the exact exponent \(q\) in the substitution \(y = u^q\), then it is also possible to work with an unknown exponent \(q\) until a point when the equation itself reveals which value of \(q\) would lead to a linear equation.

Getting back to solving the equation, if \(y = \frac{1}{u}\), then \(y' = -\frac{u'}{u^2}\), and so

\[ -\frac{u'}{u^2} - \frac{2x}{u} = \frac{xe^x}{u^2}, \quad \text{or} \quad u' + 2xu = -xe^x. \]
Since \((x^2)' = 2x\), a suggested integrating factor is \(\mu(x) = e^{x^2}\). Let us compute
\[
(e^{x^2}u(x))' = e^{x^2}u'(x) + e^{x^2}(2x)u(x) = e^{x^2}(u'(x) + xu(x)).
\]
This must be equal to \(e^{x^2}(-xe^{x^2})\), if \(u\) were to satisfy \(u' + 2xu = -xe^{x^2}\). So we have
\[
(e^{x^2}u(x))' = -xe^{2x^2}.
\]
A direct integration gives
\[
e^{x^2}u(x) = -\int xe^{x^2} \, dx = -\frac{1}{2} \int e^{x^2} \, dx^2 = -\frac{e^{2x^2}}{4} + C,
\]
and thus
\[
u(x) = -\frac{e^{x^2}}{4} + Ce^{-x^2}.
\]
By introducing the constant \(c = -C/4\), we can rewrite it as \(u(x) = -\frac{e^{x^2} + ce^{-x^2}}{4}\), which gives the following convenient expression
\[
y(x) = \frac{1}{u(x)} = -\frac{4}{e^{x^2} + ce^{-x^2}}.
\]  \((*)\)

Now we would like to impose the initial condition \(y(0) = 0\), and we face the problem that the value
\[
y(0) = -\frac{4}{1+c},
\]
is nonzero for any value of the constant \(c \in \mathbb{R}\). The only way to achieve \(y(0) = 0\) would be to formally put \(c = \infty\), because \(\frac{4}{1+c} \to 0\) as \(c \to \infty\). Rather than putting \(c = \infty\) directly into \((*)\), a better approach is to send \(c \to \infty\) and examine how the value \(y(x)\) behaves. We see that for any fixed \(x \in \mathbb{R}\), the value \(y(x)\) tends to 0 as \(c \to \infty\) (or even as \(c \to -\infty\)). This suggests that the function \(y(x) = 0\) may be a solution of the original problem \(y' - 2xy = xe^{x^2}y^2\) with \(y(0) = 0\), and as we have observed in the beginning, it is easy to verify that it is indeed the case. The existence of such a solution (called a singular solution) that cannot be obtained from \((*)\) with any number \(c \in \mathbb{R}\), is an interesting property of nonlinear equations. The term “singular” refers to the fact that the solution stays by itself, “isolated” from the family \((*)\). As a function, there is nothing “singular” about a singular solution; it is a perfectly regular function. Formally, the singular solution corresponds to the value \(c = \pm \infty\), and therefore the set of all solutions to the equation \(y' - 2xy = xe^{x^2}y^2\) forms a circle, in the sense that the singular solution at \(c = \infty\) can be reached from two sides: by increasing the value of \(c\) towards \(+\infty\), and by decreasing it towards \(-\infty\). The point \(c = \infty\) can be thought of as being diagonally opposite to the point \(c = 0\). In contrast, we know that all solutions of a first order linear equation can be obtained by varying a parameter \(c\) over \(\mathbb{R}\), hence the set of solutions in this case forms a line. In general, the solutions of some nonlinear equations may form lines, that of others may form shapes more complicated than a circle.

Note: The initial condition \(y(0) = 0\) leads to the singular solution, which was not really intended. You will receive a full mark if you have reached the point \((*)\), and have shown an attempt to find \(c\).