Math 315 Midterm Exam Winter 2014

Version 1

1. Solve $(\sin x \cos x + y^3)\mathrm{d}x + 3xy^2\mathrm{d}y = 0$.

Solution: The equation is of the form

$$a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y = 0,$$

where

$$a(x,y) = \sin x \cos x + y^3, \qquad \text{and} \qquad b(x,y) = 3xy^2.$$

We have

$$\frac{\partial a}{\partial y} = 0 + 3y^2, \qquad \text{and} \qquad \frac{\partial b}{\partial x} = 3y^2,$$

so the equation is *exact*. This means that there is a function $F(x,y)$ such that

$$\frac{\partial F(x,y)}{\partial x} = a(x,y), \qquad \text{and} \qquad \frac{\partial F(x,y)}{\partial y} = b(x,y).$$

The latter condition gives

$$F(x,y) = \int b(x,y)\mathrm{d}y = \int 3xy^2\mathrm{d}y = xy^3 + f(x),$$

and substituting this into the former, we get

$$\frac{\partial F(x,y)}{\partial x} = y^3 + f'(x) = a(x,y) = \sin x \cos x + y^3.$$

Now we need to find a function $f(x)$ such that $f'(x) = \sin x \cos x$. It can be done by a simple integration:

$$f(x) = \int \sin x \cos x \,\mathrm{d}x = \int \sin x \,\mathrm{d} \sin x = \frac{1}{2}\sin^2 x + C.$$

Since we just need one function $f(x)$ such that $f'(x) = \sin x \cos x$, we pick the simplest choice $f(x) = \frac{1}{2}\sin^2 x$, which yields

$$F(x,y) = \frac{1}{2}\sin^2 x + xy^3.$$

Finally, we can write the solution in implicit form as

$$\frac{1}{2}\sin^2 x + xy^3 = A,$$

with A an arbitrary constant.

2. Solve the initial value problem $y' - 2y\sin x = 2\sqrt{y}\sin x$ with $y(\frac{\pi}{2}) = 0$.

Solution: We recognize this as a Bernoulli equation $y' + \alpha y = \beta y^k$, with $k = \frac{1}{2}$, and recall that a suggested substitution is $u = y^{1-k} = \sqrt{y}$ or $y = u^2$. Alternatively, this substitution could be guessed just by looking at the equation, because the "trouble maker" is the term $\sqrt{y}$, and the substitution $y = u^2$ gets rid of the square root. If one does not remember the exact exponent q in the substitution $y = u^q$, then it is also possible to work with an unknown exponent q until a point when the equation itself reveals which value of q would lead to a linear equation.

Getting back to solving the equation, if $y = u^2$, then $y' = 2uu'$, and so

$$2uu' - 2u^2\sin x = 2u\sin x, \qquad \text{or} \qquad u' - u\sin x = \sin x.$$

Since $(\cos x)' = -\sin x$, a suggested integrating factor is $\mu(x) = e^{\cos x}$. Let us compute

$$(e^{\cos x}u(x))' = e^{\cos x}u'(x) + e^{\cos x}(-\sin x)u(x) = e^{\cos x}(u'(x) - u(x)\sin x).$$

This must be equal to $e^{\cos x}\sin x$, if u were to satisfy $u' - u\sin x = \sin x$. So we have

$$(e^{\cos x}u(x))' = e^{\cos x}\sin x.$$

A direct integration gives

$$e^{\cos x}u(x) = \int e^{\cos x}\sin x\,\mathrm{d}x = -\int e^{\cos x}\,\mathrm{d}\cos x = -e^{\cos x} + C,$$

and thus

$$u(x) = -1 + Ce^{-\cos x}.$$

Before converting this into an expression for $y(x)$, we want to impose the initial condition $y(\frac{\pi}{2}) = 0$. In terms of $u = \sqrt{y}$, this initial condition is still $u(\frac{\pi}{2}) = 0$. Since $\cos\frac{\pi}{2} = 0$, it leads to

$$u(\tfrac{\pi}{2}) = -1 + C = 0, \qquad \text{that is,} \qquad C = 1.$$

Finally, the solution to the original initial value problem is

$$y(x) = [u(x)]^2 = (e^{-\cos x} - 1)^2.$$