SOLUTIONS TO PROBLEMS FROM WRITTEN ASSIGNMENT 2

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Contents

Problem 1

Statement. Find all solutions of the equation

$$y \cos y \, \mathrm{d}x + x(\cos y + y \sin y - y \cos y) \, \mathrm{d}y = 0. \tag{1}$$

Solution. The equation is of the form

$$a(x,y) \, \mathrm{d}x + b(x,y) \, \mathrm{d}y = 0, \tag{2}$$

where

$$a(x,y) = y \cos y, \qquad \text{and} \qquad b(x,y) = x(\cos y + y \sin y - y \cos y). \tag{3}$$

We have

$$\frac{\partial a}{\partial y} = \cos y - y \sin y, \qquad \text{and} \qquad \frac{\partial b}{\partial x} = \cos y + y \sin y - y \cos y, \tag{4}$$

so the equation is *not* exact.

We know from class that an important component in deciding whether or not we can find an integrating factor of the form $\mu = \mu(x)$ or of the form $\mu = \mu(y)$ is if any of the expressions

$$A = \frac{\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}}{a}, \qquad \text{and} \qquad B = \frac{\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}}{b}, \tag{5}$$

depends either only on x or only on y. It is easy to see that while the quantity

$$A = \frac{-2y \sin y + y \cos y}{y \cos y} = -2 \frac{\sin y}{\cos y} + 1, \tag{6}$$

depends only on y, the quantity

$$B = \frac{-2y \sin y + y \cos y}{x(\cos y + y \sin y - y \cos y)}, \tag{7}$$

depends on both x and y.

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Thus we need to look for an integrating factor of the form $\mu = \mu(y)$, that is, a function $\mu(y)$ so that the equation

$$\mu(y)a(x,y)\mathrm{d}x + \mu(y)b(x,y)\mathrm{d}y = 0, \tag{8}$$

is exact. We have

$$\frac{\partial(\mu a)}{\partial y} = \frac{\partial \mu}{\partial y}a + \mu\frac{\partial a}{\partial y}, \qquad \text{and} \qquad \frac{\partial(\mu b)}{\partial x} = \underbrace{\frac{\partial \mu}{\partial x}}_{0} b + \mu\frac{\partial b}{\partial x} = \mu\frac{\partial b}{\partial x}, \tag{9}$$

and from the exactness condition $\frac{\partial(\mu a)}{\partial y} = \frac{\partial(\mu b)}{\partial x}$, we derive the equation

$$\frac{\partial \mu}{\partial y}a + \mu\frac{\partial a}{\partial y} = \mu\frac{\partial b}{\partial x}. \tag{10}$$

This implies

$$\mu' = \frac{\partial \mu}{\partial y} = \frac{\mu\frac{\partial b}{\partial x} - \mu\frac{\partial a}{\partial y}}{a} = -\mu A = (2\frac{\sin y}{\cos y} - 1)\mu, \tag{11}$$

which is an ODE to be solved for the function $\mu(y)$.

The above equation for $\mu(y)$ is easily solved by separation of variables. Since

$$\int (2\frac{\sin y}{\cos y} - 1)\mathrm{d}y = -2\log|\cos y| - y + C = -\log\cos^2 y - y + C, \tag{12}$$

we find a solution

$$\mu(y) = e^{-y}e^{-\log\cos^2 y} = \frac{e^{-y}}{\cos^2 y}. \tag{13}$$

Now we multiply the original equation (1) by $\mu(y)$, and get

$$\frac{ye^{-y}}{\cos y}\,\mathrm{d}x + \frac{xe^{-y}}{\cos^2 y}(\cos y + y\sin y - y\cos y)\,\mathrm{d}y = 0, \tag{14}$$

which is of the form

$$\tilde{a}(x,y)\mathrm{d}x + \tilde{b}(x,y)\mathrm{d}y = 0, \tag{15}$$

with

$$\tilde{a}(x,y) = \frac{ye^{-y}}{\cos y}, \qquad \text{and} \qquad \tilde{b}(x,y) = \frac{xe^{-y}}{\cos^2 y}(\cos y + y\sin y - y\cos y). \tag{16}$$

We have

$$\frac{\partial\tilde{a}}{\partial y} = \frac{(e^{-y} - ye^{-y})\cos y + ye^{-y}\sin y}{\cos^2 y}, \tag{17}$$

and

$$\frac{\partial\tilde{b}}{\partial x} = \frac{e^{-y}}{\cos^2 y}(\cos y + y\sin y - y\cos y), \tag{18}$$

so the equation (14) is *exact*.

This means that there is a function $F(x,y)$ such that

$$\frac{\partial F(x,y)}{\partial x} = \tilde{a}(x,y), \qquad \text{and} \qquad \frac{\partial F(x,y)}{\partial y} = \tilde{b}(x,y). \tag{19}$$

The first condition gives

$$F(x,y) = \int \tilde{a}(x,y)\mathrm{d}x = \frac{xye^{-y}}{\cos y} + f(y), \tag{20}$$

and substituting this into the second condition, we get

$$\frac{\partial F(x,y)}{\partial y} = x\frac{(e^{-y} - ye^{-y})\cos y + ye^{-y}\sin y}{\cos^2 y} + f'(y) = \tilde{b}(x,y). \tag{21}$$

Recalling $\tilde{b}(x,y)$ as given in (16), we see that $f(y)=0$ is a valid choice, which yields

$$F(x,y)=\frac{xye^{-y}}{\cos y}. \tag{22}$$

Therefore, we can write the solution of (14) in implicit form as

$$\frac{xye^{-y}}{\cos y}=C, \tag{23}$$

with C an arbitrary constant.

Problem 2

Statement. Show that every solution of the constant coefficient equation

$$y''+py'+qy=0, \tag{24}$$

tends to 0 as $x\to\infty$ if, and only if, the real parts of the roots of the characteristic equation

$$r^2+pr+q=0, \tag{25}$$

are negative. *Clarification*: Each root of the characteristic equation (25) is a complex number, which of course can be written as $r=k+\omega i$, with k and ω real numbers. Then the *real part* of r is the number k, and this fact is written as $k=\operatorname{Re} r$. "The real parts of the roots of the characteristic equation (25) are negative" means that $\operatorname{Re} r_1<0$ and $\operatorname{Re} r_2<0$ for both roots r_1 and r_2 of (25).

Solution. We know that every solution of (24) can be written as

$$y(x)=Ay_1(x)+By_2(x), \tag{26}$$

for some constants A and B, where $y_1(x)$ and $y_2(x)$ are defined, depending on the behaviour of the characteristic equation (25), as follows. Let

$$r_{1,2}=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}, \tag{27}$$

be the roots of (25). There are three cases to consider.

Case A. If $D=\frac{p^2}{4}-q>0$, then we have two real roots, and we define

$$y_1(x)=e^{r_1x}, \qquad \text{and} \qquad y_2(x)=e^{r_2x}. \tag{28}$$

Let r be a real number, and let us study the behaviour of the function e^{rx} as $x\to\infty$. It is clear that

$$\begin{aligned} e^{rx}\to\infty \quad &\text{as} \quad x\to\infty, \qquad \text{if} \quad r>0,\\ e^{rx}\to 1 \quad &\text{as} \quad x\to\infty, \qquad \text{if} \quad r=0,\\ e^{rx}\to 0 \quad &\text{as} \quad x\to\infty, \qquad \text{if} \quad r<0, \end{aligned} \tag{29}$$

which in particular implies that

$$e^{rx}\to 0 \quad \text{as} \quad x\to\infty, \qquad \text{if and only if} \quad r<0. \tag{30}$$

Indeed, the "if" part of (30) is the third line of (29). In the other direction, assume that $e^{rx}\to 0$ as $x\to\infty$. Then we cannot have $r\geq 0$, since $r\geq 0$ would imply, by the first and second lines of (29), that $e^{rx}\not\to 0$ as $x\to\infty$, which leaves the only possibility $r<0$. We conclude that $y_1(x)\to 0$ and $y_2(x)\to 0$ as $x\to\infty$ if and only if $r_1<0$ and $r_2<0$. This property, and its analogues for the other two cases, will be used later.

Case B. If $D < 0$, then we have two complex roots

$$r_{1,2} = k \pm i\omega, \qquad \text{where} \quad k = -\frac{p}{2}, \quad \text{and} \quad \omega = \sqrt{|D|}, \tag{31}$$

and we define

$$y_1(x) = e^{kx}\cos(\omega x), \qquad \text{and} \qquad y_2(x) = e^{kx}\sin(\omega x). \tag{32}$$

Note that k is equal to the real part of each of r_1 and r_2. If $k \geq 0$, then both $y_1(x)$ and $y_2(x)$ do not have limit as $x \to \infty$, because $\cos(\omega x)$ and $\sin(\omega x)$ oscillate between -1 and 1. On the other hand, if $k < 0$, then $y_1(x) \to 0$ and $y_2(x) \to 0$ as $x \to \infty$, since $\cos(\omega x)$ and $\sin(\omega x)$ cannot become larger than 1 in magnitude. Hence it is clear that $y_1(x) \to 0$ and $y_2(x) \to 0$ as $x \to \infty$ if and only if $k < 0$.

Case C. Finally, if $D = 0$, then we have only one real root at $k = -\frac{p}{2}$, and we define

$$y_1(x) = e^{kx}, \qquad \text{and} \qquad y_2(x) = xe^{kx}. \tag{33}$$

We already know that $y_1(x) \to 0$ as $x \to \infty$ if and only if $k < 0$. As for y_2, it is clear that if $k \geq 0$ then $y_2(x) \to \infty$ as $x \to \infty$. What remains is to investigate the limit xe^{kx} as $x \to \infty$ when $k < 0$. Writing

$$xe^{kx} = \frac{x}{e^{-kx}} = \frac{x}{e^{|k|x}}, \tag{34}$$

the question is which of x and $e^{|k|x}$ grows faster as $x \to \infty$. We know from calculus that $e^{|k|x}$ is faster, but we can also see it by expanding $e^{|k|x}$ into the Taylor series

$$e^{|k|x} = 1 + |k|x + \frac{(|k|x)^2}{2} + \ldots + \frac{(|k|x)^n}{n!} + \ldots, \tag{35}$$

and observing that each term in the right hand side is positive when $x > 0$, implying that

$$e^{|k|x} \geq \frac{(|k|x)^2}{2}, \qquad \text{for} \quad x > 0. \tag{36}$$

This gives

$$xe^{kx} = \frac{x}{e^{|k|x}} \leq \frac{2x}{(|k|x)^2} = \frac{2}{|k|^2 x}, \tag{37}$$

which clearly goes to 0 as $x \to \infty$. We conclude that $y_2(x) \to 0$ as $x \to \infty$ if and only if $k < 0$.

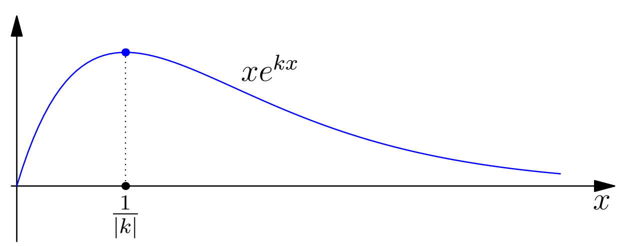

FIGURE 1. The behaviour of xe^{kx} with constant $k < 0$.

Conclusion. In all three cases, we concluded that $y_1(x) \to 0$ and $y_2(x) \to 0$ as $x \to \infty$ if and only if $\operatorname{Re} r_1 < 0$ and $\operatorname{Re} r_2 < 0$, where $\operatorname{Re} z$ denotes the real part of z. So if every solution of (24) tends to 0 as $x \to \infty$, this is true in particular for the solutions y_1 and y_2, and hence $\operatorname{Re} r_1 < 0$ and $\operatorname{Re} r_2 < 0$. Conversely, if $\operatorname{Re} r_1 < 0$ and $\operatorname{Re} r_2 < 0$, then $y_1(x)$ and $y_2(x)$ tend to 0 as $x \to \infty$. This means that any solution of (24), which can be written as $y(x) = Ay_1(x) + By_2(x)$ with some constants A and B, tends to 0 as $x \to \infty$.

Problem 3

Statement. Suppose that y_1 and y_2 are linearly independent solutions of the constant coefficient equation

$$y'' + py' + qy = 0, \tag{38}$$

and let $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ be the Wronskian of y_1 and y_2. Show that W is constant if and only if $p = 0$.

Solution. Recall the Abel formula

$$W(x) = W(0)e^{px}. \tag{39}$$

If $p = 0$, it is obvious that $W(x)$ would be constant. It remains to show that if $W(x)$ is constant, then $p = 0$. Assume that $W(x)$ is constant. This constant cannot be 0, because $W(x) = 0$ would imply that y_1 and y_2 are linearly dependent. In particular, $W(0) \neq 0$. Now we take the derivative of the Abel formula, which yields

$$W'(x) = pW(0)e^{px}. \tag{40}$$

Since $W(x)$ is constant, $W'(x) = 0$ everywhere. Moreover, we have $W(0) \neq 0$, and $e^{px} \neq 0$, implying that $p = 0$.

Problem 4

Statement. Solve the initial value problem

$$y'' - 6y' + 8y = 3e^x + 2x^2, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{41}$$

Solution. *First*, let us find the general solution to the complementary equation

$$y'' - 6y' + 8y = 0. \tag{42}$$

The characteristic equation is

$$r^2 - 6r + 8 = 0, \tag{43}$$

and its roots are

$$r_{1,2} = 3 \pm \sqrt{9-8} = 3 \pm 1. \tag{44}$$

So the general solution of (42) is

$$y_c(x) = Ae^{2x} + Be^{4x}, \tag{45}$$

where A and B are arbitrary constants.

Second, we shall find a particular solution of

$$y'' - 6y' + 8y = 3e^x + 2x^2. \tag{46}$$

Following the method of undetermined coefficients, let us look for a particular solution in the form

$$y_p(x) = c_1e^x + c_2x^2 + c_3x + c_4. \tag{47}$$

We compute

$$y_p'(x) = c_1e^x + 2c_2x + c_3, \qquad y_p''(x) = c_1e^x + 2c_2, \tag{48}$$

and

$$\begin{aligned} y_p'' - 6y_p' + 8y_p &= c_1e^x + 2c_2 - 6(c_1e^x + 2c_2x + c_3) + 8(c_1e^x + c_2x^2 + c_3x + c_4) \\ &= 3c_1e^x + 8c_2x^2 + (8c_3 - 12c_2)x + 2c_2 - 6c_3 + 8c_4. \end{aligned} \tag{49}$$

This must be equal to $3e^x + 2x^2$, which yields

$$c_1 = 1, \qquad 8c_2 = 2, \qquad 8c_3 - 12c_2 = 0, \qquad 2c_2 - 6c_3 + 8c_4 = 0. \tag{50}$$

Solving for c_3 and c_4, we have

$$c_2 = \frac{1}{4}, \qquad c_3 = \frac{12c_2}{8} = \frac{3}{8}, \qquad c_4 = \frac{6c_3 - 2c_2}{8} = \frac{7}{32}, \tag{51}$$

and thus

$$y_p(x) = e^x + \frac{x^2 + 3x}{8} + \frac{7}{32}, \tag{52}$$

is a particular solution of (46).

As a penultimate step, combining (45) and (52), we conclude that the general solution of (46) is given by

$$y(x) = Ae^{2x} + Be^{4x} + e^x + \frac{x^2 + 3x}{8} + \frac{7}{32}, \tag{53}$$

where A and B are arbitrary constants.

Finally, we need to impose the initial conditions $y(0) = 1$ and $y'(0) = 0$ to pin down the values of A and B. From (53), we have

$$y(0) = A + B + 1 + \frac{7}{32},$$

and

$$y'(x) = 2Ae^{2x} + 4Be^{4x} + e^x + \frac{2x + 3}{8}, \tag{54}$$

the latter giving

$$y'(0) = 2A + 4B + 1 + \frac{3}{8}. \tag{55}$$

Imposing the initial conditions, we get

$$\begin{aligned} A + B &= -\frac{7}{32}, \\ 2A + 4B &= -\frac{11}{8}. \end{aligned} \tag{56}$$

Multiplying the first equation by 2, and subtracting the result from the second one, we get

$$4B - 2B = -\frac{11}{8} + \frac{2 \cdot 7}{32} = -\frac{22}{16} + \frac{7}{16} = -\frac{15}{16}, \tag{57}$$

which yields $B = -\frac{15}{32}$ and $A = -\frac{7}{32} - B = \frac{8}{32} = \frac{1}{4}$. Substituting these values into (53), we conclude that the solution to the initial value problem (41) is given by

$$y(x) = \frac{1}{4}e^{2x} - \frac{15}{32}e^{4x} + e^x + \frac{x^2 + 3x}{8} + \frac{7}{32}. \tag{58}$$

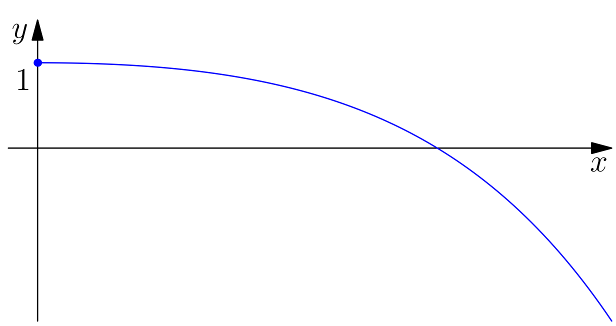

FIGURE 2. The graph of $y(x)$ from (58).

Problem 5

Statement. Find all solutions of the equation

$$x^2y'' - 5xy' + 9y = x^3, \qquad \text{for} \quad x > 0. \tag{59}$$

Solution. We immediately recognize that (59) is a Cauchy-Euler equation with a nonzero right hand side. As we have seen, a systematic method for dealing with Cauchy-Euler equations is the introduction of the new independent variable t by

$$x = e^t, \qquad \text{or equivalently,} \qquad t = \log x. \tag{60}$$

Writing $u(t) = y(e^t)$ and so $y(x) = u(\log x)$, and applying the chain rule, we have

$$y'(x) = u'(\log x) \cdot \frac{1}{x}, \qquad \text{and} \qquad y''(x) = u''(\log x) \cdot \frac{1}{x^2} - u'(\log x) \cdot \frac{1}{x^2}. \tag{61}$$

Then the left hand side of (59) becomes

$$\begin{aligned} x^2y'' - 5xy' + 9y &= u''(\log x) - u'(\log x) - 5u'(\log x) + 9u(\log x) \\ &= u''(t) - 6u'(t) + 9u(t), \end{aligned} \tag{62}$$

while the right hand side of (59) becomes $x^3 = e^{3t}$. Therefore the equation (59) in the new variables is

$$u'' - 6u' + 9u = e^{3t}. \tag{63}$$

This is a second order equation with constant coefficients, and we see that the right hand side is so that it can be handled by the method of undetermined coefficients.

First, we need to find the general solution to the complementary equation

$$u'' - 6u' + 9u = 0. \tag{64}$$

The characteristic equation is

$$r^2 - 6r + 9 = 0, \tag{65}$$

and its roots are

$$r_{1,2} = 3 \pm \sqrt{9-9} = 3. \tag{66}$$

So we have repeated roots at $r = 3$, and the general solution of (64) is

$$u_c(t) = Ae^{3t} + Bte^{3t}, \tag{67}$$

where A and B are arbitrary constants.

Next, we shall find a particular solution of (63). The initial step of the method of undetermined coefficients suggests that we look for a particular solution in the form $u_p(t) = e^{3t}$. However, as seen from (67), the function e^{3t} is among the solutions of the complementary equation. We need to find the smallest integer n so that t^ne^{3t} is not among the the solutions of the complementary equation. Clearly, we have $n = 2$, and so we look for a particular solution in the form

$$u_p(t) = ct^2e^{3t}. \tag{68}$$

We compute

$$u_p'(t) = 2cte^{3t} + 3ct^2e^{3t}, \qquad u_p''(t) = 2ce^{3t} + 12cte^{3t} + 9ct^2e^{3t}, \tag{69}$$

and

$$\begin{aligned} u_p'' - 6u_p' + 9u_p &= 2ce^{3t} + 12cte^{3t} + 9ct^2e^{3t} - 6(2cte^{3t} + 3ct^2e^{3t}) + 9ct^2e^{3t} \\ &= 2ce^{3t}. \end{aligned} \tag{70}$$

This must be equal to e^{3t}, which immediately yields $c = \frac{1}{2}$, and thus

$$u_p(t) = \frac{t^2e^{3t}}{2}, \tag{71}$$

is a particular solution of (63).

Finally, by combining (67) and (71), we write the general solution of (63) as

$$u(t) = Ae^{3t} + Bte^{3t} + \frac{t^2 e^{3t}}{2}, \tag{72}$$

where A and B are arbitrary constants. Changing back to the original variables x and y, we conclude that the general solution of (59) is

$$y(x) = u(\log x) = Ax^3 + Bx^3 \log x + \frac{x^3(\log x)^2}{2}, \tag{73}$$

where A and B are arbitrary constants.

Problem 6

Statement. Find all solutions of the equation

$$y'' - 2y' + y = \frac{e^x}{4+x^2}. \tag{74}$$

Solution. *First*, let us find the general solution to the complementary equation

$$y'' - 2y' + y = 0. \tag{75}$$

The characteristic equation is $r^2 - 2r + y = 0$, and its roots are

$$r_{1,2} = 1 \pm \sqrt{1-1} = 1. \tag{76}$$

So we have repeated roots at $r = 1$, and the general solution of (75) is

$$y_c(x) = Ae^x + Bxe^x, \tag{77}$$

where A and B are arbitrary constants. In other words, the functions

$$y_1(x) = e^x, \qquad \text{and} \qquad y_2(x) = xe^x, \tag{78}$$

are a linearly independent set of solutions of (75).

Second, we shall find a particular solution of (74), that is,

$$y'' - 2y' + y = f(x), \qquad \text{with} \qquad f(x) = \frac{e^x}{4+x^2}. \tag{79}$$

Since $f(x)$ is not in a form amenable to the method of undetermined coefficients, we follow the variation of parameters method. This method gives a particular solution in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x), \tag{80}$$

where $u_1(x)$ and $u_2(x)$ are functions satisfying

$$u_1' = -\frac{y_2 f}{W}, \qquad \text{and} \qquad u_2' = \frac{y_1 f}{W}, \tag{81}$$

with $W = y_1 y_2' - y_1' y_2$ the Wronskian of y_1 and y_2. Note that since y_1, y_2, f, and W are known, solving (81) means direct integrations. Let us compute the Wronskian

$$W(x) = (e^x)(e^x + xe^x) - (e^x)(xe^x) = e^{2x}. \tag{82}$$

Then the two integrals needed for solving (81) can be computed as

$$\int \frac{y_2(x)f(x)}{W(x)}\mathrm{d}x = \int \frac{xe^x}{e^{2x}} \cdot \frac{e^x}{4+x^2}\mathrm{d}x = \int \frac{x\mathrm{d}x}{4+x^2} = \frac{1}{2}\int \frac{\mathrm{d}(x^2+4)}{4+x^2} = \frac{1}{2}\log(4+x^2) + C, \tag{83}$$

and

$$\int \frac{y_1(x)f(x)}{W(x)}\mathrm{d}x = \int \frac{e^x}{e^{2x}} \cdot \frac{e^x}{4+x^2}\mathrm{d}x = \int \frac{\mathrm{d}x}{4+x^2} = \frac{1}{2}\int \frac{\mathrm{d}(\frac{x}{2})}{1+(\frac{x}{2})^2} = \frac{1}{2}\arctan\frac{x}{2} + C_1. \tag{84}$$

Since u_1 and u_2 are only required to satisfy (81), and no other conditions are imposed on them, we can pick any values for the constants C and C_1. So putting $C = C_1 = 0$, we have

$$u_1(x) = -\frac{1}{2}\log(4+x^2), \qquad \text{and} \qquad u_2(x) = \frac{1}{2}\arctan\frac{x}{2}, \tag{85}$$

leading to

$$y_p(x) = -\frac{e^x}{2}\log(4+x^2) + \frac{xe^x}{2}\arctan\frac{x}{2}. \tag{86}$$

Finally, combining (77) and (86), we conclude that the general solution of (74) is given by

$$y(x) = Ae^x + Bxe^x - \frac{e^x}{2}\log(4+x^2) + \frac{xe^x}{2}\arctan\frac{x}{2}, \tag{87}$$

where A and B are arbitrary constants.

Problem 7

Statement. Compute the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < \pi, \\ \sin 2t, & t \ge \pi. \end{cases} \tag{88}$$

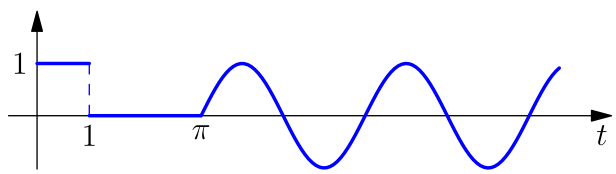

Figure 3. The graph of $f(t)$.

Solution. First, we write $f(t)$ as a "one line formula" involving the Heaviside step function:

$$f(t) = 1 - \theta(t-1) + \theta(t-\pi)\sin 2t. \tag{89}$$

Recall the shift rule

$$\mathscr{L}\{\theta(t-a)g(t-a)\} = e^{-as}\mathscr{L}\{g(t)\}, \qquad (a > 0). \tag{90}$$

Applying this with $a = 1$ and $g(t) = 1$ gives

$$\mathscr{L}\{\theta(t-1)\} = e^{-s}\mathscr{L}\{1\} = \frac{e^{-s}}{s}. \tag{91}$$

In order to find the Laplace transform of $\theta(t-\pi)\sin 2t$, we need to write $\sin 2t$ as $g(t-\pi)$ for some function g. What this means is that we have to write $\sin 2t$ as a function of the new variable $t' = t - \pi$, and once we have asked the correct question, the solution is easy:

$$\sin 2t = \sin 2(t'+\pi) = \sin(2t' + 2\pi) = \sin 2t', \tag{92}$$

where we have used $t = t' + \pi$. So as a function of $t' = t - \pi$, $\sin 2t$ is still $\sin 2t'$, as can be seen from the graph, or from the fact that $\sin 2t$ is periodic with period π. Note that this is a very special case. In general, for example, if we want to write t^2 as a function of $r = t - 1$, then we would have

$$t^2 = (r+1)^2 = r^2 + 2r + 1. \tag{93}$$

Getting back to the problem, applying (90) with $a = \pi$ and $g(t') = \sin 2t'$, we get

$$\mathscr{L}\{\theta(t-\pi)\sin 2t\} = \mathscr{L}\{\theta(t-\pi)\sin 2(t-\pi)\} = e^{-\pi s}\mathscr{L}\{\sin 2t\} \tag{94}$$

and taking into account that

$$\mathscr{L}\{\sin 2t\} = \frac{1}{2} \cdot \frac{1}{(\frac{s}{2})^2 + 1} = \frac{2}{s^2+4}, \tag{95}$$

we conclude

$$\mathscr{L}\{\theta(t-\pi)\sin 2t\} = e^{-\pi s}\frac{2}{s^2+4} = \frac{2e^{-\pi s}}{s^2+4}. \tag{96}$$

Finally, combining (91) and (96), we infer

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1\} - \mathscr{L}\{\theta(t-1)\} + \mathscr{L}\{\theta(t-\pi)\sin 2t\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{2e^{-\pi s}}{s^2+4}. \tag{97}$$

Note that this function has a finite value at $s = 0$, because we have $e^{-s} = 1 - s + \frac{s^2}{2} + \ldots$ near $s = 0$, see Figure 4.

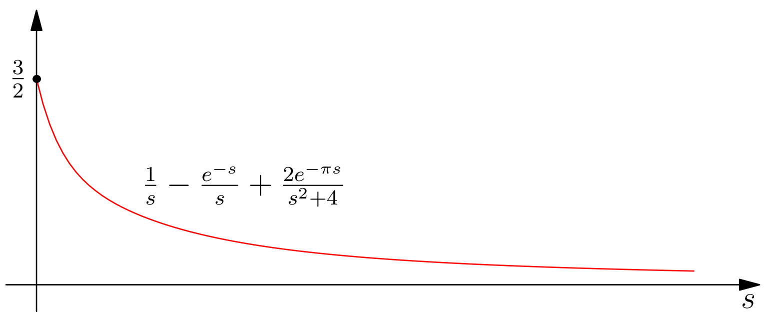

FIGURE 4. The Laplace transform of $f(t)$.

PROBLEM 8

Statement. By using the Laplace transform method, solve the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, \qquad y'(0) = 1, \tag{98}$$

where $f(t)$ is given by

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < \pi, \\ \sin 2t, & t \ge \pi. \end{cases} \tag{99}$$

Sketch the graphs of $f(t)$ and the solution $y(t)$.

Solution. Recall the differentiation rule

$$\mathscr{L}\{y'(t)\} = s\mathscr{L}\{y(t)\} - y(0). \tag{100}$$

Applying this rule to the function $y'(t)$, we get

$$\mathscr{L}\{y''(t)\} = s\mathscr{L}\{y'(t)\} - y'(0) = s^2\mathscr{L}\{y(t)\} - sy(0) - y'(0). \tag{101}$$

Therefore if $y(t)$ satisfies (98), then $\mathscr{L}\{y''\} + 4\mathscr{L}\{y(t)\} = \mathscr{L}\{f(t)\}$, that is,

$$s^2\mathscr{L}\{y(t)\} - sy(0) - y'(0) + 4\mathscr{L}\{y(t)\} = \mathscr{L}\{f(t)\}. \tag{102}$$

Taking into account the initial conditions $y(0) = 1$ and $y'(0) = 1$, and introducing the notation $Y(s) = \mathscr{L}\{y(t)\}$, we have

$$s^2\, Y(s) - 1 + 4Y(s) = \mathscr{L}\{f(t)\}, \tag{103}$$

and so

$$Y(s) = \frac{\mathscr{L}\{f(t)\} + 1}{s^2+4}. \tag{104}$$

The Laplace transform of $f(t)$ has been computed in the preceding problem, as

$$\mathscr{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{2e^{-\pi s}}{s^2+4}, \tag{105}$$

which leads to

$$Y(s) = \frac{1}{s^2+4} + \frac{1}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)} + \frac{2e^{-\pi s}}{(s^2+4)^2}. \tag{106}$$

What remains to be done is to compute the inverse Laplace transform of $Y(s)$. From

$$\mathscr{L}\{\sin 2t\} = \frac{1}{2}\cdot\frac{1}{(\frac{s}{2})^2+1} = \frac{2}{s^2+4}, \tag{107}$$

we infer, for the first term in (106) that

$$\mathscr{L}^{-1}\{\frac{1}{s^2+4}\} = \frac{1}{2}\mathscr{L}^{-1}\{\frac{2}{s^2+4}\} = \frac{\sin 2t}{2}. \tag{108}$$

The second and third terms in (106) are similar to each other, and can be treated by partial fraction decompositions. The condition

$$\frac{1}{s(s^2+4)} = \frac{As+B}{s^2+4} + \frac{C}{s} = \frac{(As+B)s+C(s^2+4)}{s(s^2+4)} = \frac{(A+C)s^2+Bs+4C}{s(s^2+4)}, \tag{109}$$

implies that $A+C=0$, $B=0$, and $4C=1$, that is, $A=-\frac{1}{4}$, $B=0$ and $C=\frac{1}{4}$. Hence

$$\frac{1}{s(s^2+4)} = -\frac{1}{4}\cdot\frac{s}{s^2+4} + \frac{1}{4s}, \tag{110}$$

and recalling that

$$\mathscr{L}\{\cos 2t\} = \frac{1}{2}\cdot\frac{(\frac{s}{2})}{(\frac{s}{2})^2+1} = \frac{1}{s^2+4}, \qquad \text{and} \qquad \mathscr{L}\{1\} = \frac{1}{s}, \tag{111}$$

we infer, for the second term in (106) that

$$\mathscr{L}^{-1}\{\frac{1}{s(s^2+4)}\} = -\frac{\cos 2t}{4} + \frac{1}{4}. \tag{112}$$

For the third term in (106), we need to use the shift rule

$$\mathscr{L}^{-1}\{e^{-as}\mathscr{L}\{g(t)\}\} = \theta(t-a)g(t-a), \qquad (a>0), \tag{113}$$

which is just another form of

$$\mathscr{L}\{\theta(t-a)g(t-a)\} = e^{-as}\mathscr{L}\{g(t)\}, \qquad (a>0). \tag{114}$$

So for the third term in (106), we have

$$\mathscr{L}^{-1}\{\frac{e^{-s}}{s(s^2+4)}\} = \theta(t-1)\big(-\frac{\cos 2(t-1)}{4} + \frac{1}{4}\big). \tag{115}$$

Now we turn to the fourth term in (106). Discarding the term $e^{-\pi s}$ for the moment, let us first try to compute $\mathscr{L}^{-1}\{\frac{1}{(s^2+4)^2}\}$. The presence of $(s^2+4)^2$ in the denominator suggests, in light of the differentiation rule

$$\mathscr{L}\{tg(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\{g(t)\}, \tag{116}$$

that $\mathscr{L}^{-1}\{\frac{1}{(s^2+4)^2}\}$ might involve terms such as $t\cos 2t$ or $t\sin 2t$. In any case, let us compute the Laplace transforms of $t\cos 2t$ and $t\sin 2t$. We have

$$\mathscr{L}\{t\sin 2t\} = -(\frac{2}{s^2+4})' = \frac{4s}{(s^2+4)^2}, \tag{117}$$

and

$$\mathscr{L}\{t\cos 2t\} = -\left(\frac{s}{s^2+4}\right)' = -\frac{s^2+4-2s^2}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}. \tag{118}$$

We can manipulate the latter as

$$\frac{s^2-4}{(s^2+4)^2} = \frac{s^2+4-8}{(s^2+4)^2} = \frac{1}{s^2+4} - \frac{8}{(s^2+4)^2}, \tag{119}$$

giving

$$\frac{1}{(s^2+4)^2} = \frac{1}{8}\cdot\frac{1}{s^2+4} - \frac{1}{8}\cdot\frac{s^2-4}{(s^2+4)^2}, \tag{120}$$

or

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} = \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\right\} - \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{s^2-4}{(s^2+4)^2}\right\} = \frac{\sin 2t}{16} - \frac{t\cos 2t}{8}. \tag{121}$$

Finally, invoking the shift rule (113), for the fourth term in (106), we infer

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{2e^{-\pi s}}{(s^2+4)^2}\right\} &= 2\theta(t-\pi)\left(\frac{\sin 2(t-\pi)}{16} - \frac{(t-\pi)\cos 2(t-\pi)}{8}\right) \\ &= \theta(t-\pi)\left(\frac{\sin 2t}{8} - \frac{(t-\pi)\cos 2t}{4}\right),\end{aligned} \tag{122}$$

where we have used the periodicity of cos and sin.

At this point, we have computed the inverse Laplace transform of each term in (106), and they are given in (108), (112), (115) and (122). By combining those formulas, we conclude

$$\begin{aligned}y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\ &= \frac{\sin 2t}{2} - \frac{\cos 2t}{4} + \frac{1}{4} + \theta(t-1)\left(\frac{\cos 2(t-1)}{4} - \frac{1}{4}\right) + \theta(t-\pi)\left(\frac{\sin 2t}{8} - \frac{(t-\pi)\cos 2t}{4}\right).\end{aligned}$$

Perhaps a bit more intelligible way to write this is to consider the three intervals $0 \leq t < 1$, $1 \leq t < \pi$, and $\pi \leq t < \infty$ one at a time, as follows.

$$y(t) = \begin{cases} \frac{\sin 2t}{2} - \frac{\cos 2t}{4} + \frac{1}{4}, & \text{for} \quad 0 \leq t < 1, \\ \frac{\sin 2t}{2} - \frac{\cos 2t}{4} + \frac{\cos 2(t-1)}{4}, & \text{for} \quad 1 \leq t < \pi, \\ \frac{5\sin 2t}{8} - \frac{\cos 2t}{4} + \frac{\cos 2(t-1)}{4} - \frac{(t-\pi)\cos 2t}{4}, & \text{for} \quad t \geq \pi. \end{cases} \tag{123}$$

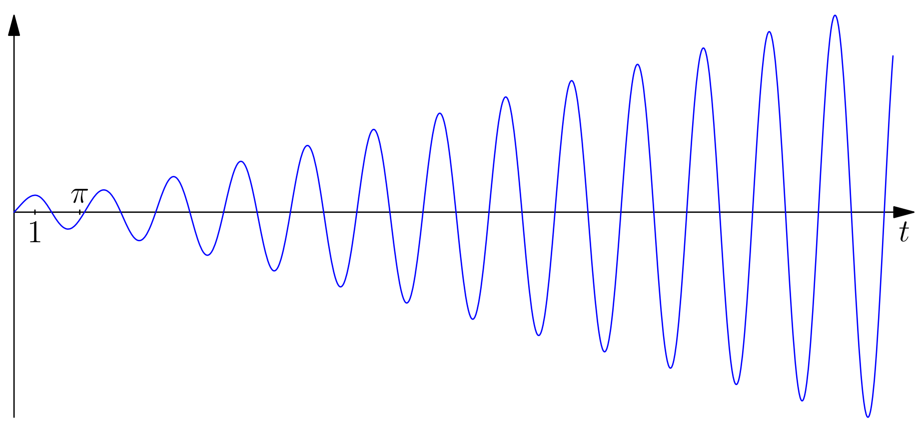

FIGURE 5. The graph of $y(t)$.