SOLUTIONS TO SELECTED PROBLEMS

1. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^x}, \tag{$*$}$$

where x is a *real* variable. Determine all possible intervals in which the series converges absolutely uniformly. Show that there is a continuous function f defined on $(1, \infty)$, to which the series converges locally uniformly, but *not* uniformly, in $(1, \infty)$.

Solution: We know that this series converges pointwise for $x > 1$ and diverges for $x \leq 1$. This defines a function $f(x)$ for $x \in (1, \infty)$. Let I be an interval, whose left endpoint we denote by $a > 1$. Since n^{-x} is a decreasing function of x, we have $|n^{-x}| \leq n^{-a}$ for $x \geq a$. Therefore, taking into account

$$\sum_{n=1}^{\infty} \frac{1}{n^a} < \infty,$$

and invoking the Weierstrass M-test, we infer that the series $(*)$ converges absolutely uniformly in I. Incidentally, it also shows that f is continuous in I, and since $(0, \infty)$ can be covered by, say, intervals of the form (a, ∞) with $a > 1$, we conclude that f is continuous in $(1, \infty)$. By recalling the definition of locally uniform convergence, we see that $(*)$ converges locally uniformly in $(1, \infty)$. Anticipating a contradiction, suppose that the convergence is *uniform in* $(1, \infty)$. This implies that the sequence of partial sums of the series $(*)$ is uniformly Cauchy, in the sense that given any $\varepsilon > 0$, there exists N such that

$$\sum_{n=m}^{k} \frac{1}{n^x} \leq \varepsilon \qquad \text{for all } x > 1 \text{ and } k > m \geq N. \tag{1}$$

However, given any m, no matter how large it is, we can choose k large enough and $x > 1$ close enough to 1, so as to make

$$\sum_{n=m}^{k} \frac{1}{n^x} \geq 1, \tag{2}$$

which is a contradiction.

2. Show that for every $r > 0$ there exists N such that for any $n \geq N$ the polynomial

$$p_n(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!},$$

does not have any zeroes in $D_r = \{z \in \mathbb{C} : |z| < r\}$.

Solution: We know that given any $r > 0$, the sequence $\{p_n\}$ converges uniformly in D_r to the complex exponential. The complex exponential is continuous, so $f(z) = |\exp z|$ is a real-valued continuous function in the closed disk $\bar{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$. Then by the Weierstrass theorem, f takes its minimum in $\bar{D}_r$. Since the complex exponential vanishes nowhere, the minimum of f over $\bar{D}_r$ must be strictly positive. Let us denote this minimum by $\varepsilon > 0$. Thus we have $|\exp z| \geq \varepsilon$ for all $z \in D_r$. Now by uniform convergence, we can choose N so large that

$$|p_n(z) - \exp z| \leq \frac{\varepsilon}{2} \qquad \text{for all } z \in D_r \text{ and } n \geq N,$$

This means that

$$|p_n(z)| \geq |\exp z| - |p_n(z) - \exp z| \geq \frac{\varepsilon}{2} \qquad \text{for all } z \in D_r \text{ and } n \geq N,$$

by the triangle inequality.

3. Show that if a power series $\sum a_n z^n$ converges to some function $f : \mathbb{C} \to \mathbb{C}$ *uniformly in* $\mathbb{C}$, then $a_n = 0$ for all but finitely many n, and hence f must be a polynomial.

   Solution: Uniform convergence implies that the partial sums f_n of $\sum a_n z^n$ form a sequence that is uniformly Cauchy in $\mathbb{C}$. In particular, this means that the sequence $\{a_n z^n\}$ must go to 0 uniformly in $\mathbb{C}$. That is, for any given $\varepsilon > 0$, there is N such that $|a_n z^n| \leq \varepsilon$ for all $z \in \mathbb{C}$ and $n \geq N$. But if $a_n \neq 0$ and $n \geq 1$ then there is no constant M such that $|a_n z^n| \leq M$ for all $z \in \mathbb{C}$. Hence $a_n = 0$ for all sufficiently large n.