Problem 1. *Compute the integrals $\int_{C_r} z^n \, dz$ for $n \in \mathbb{Z}$, where $C_r = \partial D_r$ and $r > 0$. Recall that C_r has the counter-clockwise orientation.*

Solution: We will use the Residue Theorem to solve this one. We could instead use other (perhaps more elementary methods) but this is a good warm-up. Below is a brief review of the theorems which we will use.

Theorem 1 (Residue Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be an open set, and let $z_1, \cdots, z_m \in \Omega$. Suppose that γ is a null-homotopic closed loop in Ω, and does not pass through any of the points $z_1, \cdots, z_m$. Let f be a holomorphic function in $\Omega - \{ z_1, \cdots, z_m \}$. Then we have*

$$\int_\gamma f = 2\pi i \sum_{j=1}^{m} \mathrm{Ind}(\gamma, z_j)\mathrm{Res}(f, z_j)$$

Recall that

$$\mathrm{Ind}(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-c} \qquad \mathrm{Res}(f, z_j) = \frac{1}{2\pi i}\int_{\partial D_\epsilon(z_j)} f(z)dz$$

We will use the properties proved in class

(a) If a_{-1} is the -1st coefficient in the Laurent series of f around c, then $\mathrm{Res}(f, c) = a_{-1}$

(b) If $\mathrm{Ord}(f, c) \geq 0$ (removable singularity) then $\mathrm{Res}(f, c) = 0$

(c) If $\mathrm{Ord}(f, c) = -k$ (a k-pole), then

$$\mathrm{Res}(f, c) = \lim_{z \to c} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z-c)^k f(z)$$

(d) If $\mathrm{Ord}(f, c) \geq 0$ and $\mathrm{Ord}(g, c) = 1$, then

$$\mathrm{Res}\left(\frac{f}{g}, c\right) = \frac{f(c)}{g'(c)}$$

(e) $\mathrm{Res}(\cdot, c)$ is $\mathbb{C}$-linear in $\mathcal{O}(D_r^x(c))$.

We changed a few of these from their statements in the notes. Most importantly, we changed property (c) and (d). The new property (c) is easy to prove, and it is left to the reader.

Proof of the new property (d). The case $\mathrm{Ord}(f, c) = 0$ is proved in the notes. When $\mathrm{Ord}(f, c) > 0$, then $f(z) = (z-c)\widetilde{f}(z)$ for some holomorphic $\widetilde{f}$. Then we have

$$\frac{f(z)}{g(z)} = \frac{\widetilde{f}(z)}{g(z)/(z-c)}$$

it follows that $g(z)/(z-c)$ has order 0 at c, and so the above expression has order 0 or greater, and thus it's residue is 0. However, we also have $f(c) = 0$ because f has order greater than 0 at c. It follows that the formula remains true. $\square$

Now to solve the problem using the residue theorem, we conclude

$$\int_{\partial D_r} z^n dz = 2\pi i \operatorname{Res}(z^n, 0)$$

Since z^n is holomorphic in D_r^x, and ∂D_r clearly has index 1 around 0. It is clear that $\operatorname{Ord}(z^n, 0)$ is n, and so by property (b) above, we conclude that

$$\int_{\partial D_r} z^n dz = 0 \qquad \text{if } n \geq 0$$

Next, by applying property (d) with $f(z) = 1$ and $g(z) = z$, we have

$$\operatorname{Res}(\frac{1}{z}, 0) = \frac{1}{1} = 1$$

and so

$$\int_{\partial D_r} z^{-1} = 2\pi i.$$

Now by property (c), we conclude that

$$\operatorname{Ord}(z^{-n}, 0) = \lim_{z \to 0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z^n \cdot z^{-n}) = 0 \qquad \text{for } n > 1,$$

(we remark that we could have used this to find $\operatorname{Res}(z^{-1}, 0)$, but we are practicing using all the properties) and so we obtain

$$\int_{\partial D_r} z^{-n} = 0 \qquad \text{for } n > 1.$$

Problem 2. *Evaluate the integrals*

$$\int_0^{+\infty} \frac{x^m dx}{1 + x^n} \qquad \textit{and} \qquad \int_0^{\pi} \frac{d\theta}{a + \sin^2 \theta}$$

where m, n are integers satisfying $1 \leq m \leq n - 2$ and $a > 0$ is real.

Solution: For the first one we will integrate along the contour shown below

$$\gamma_1(t) = t,\ t \in [0, R] \qquad \gamma_2(t) = t \exp(2\pi i/n),\ t \in [0, R] \qquad \gamma_3(t) = R \exp(it),\ t \in [0, 2\pi i/n]$$

It is clear that the only zero of $1 + z^n$ inside the interior of our contours is at $z = \exp(\pi i/n)$, and thus by the Residue theorem, we can conclude that

$$\int_{\gamma_1 + \gamma_3 - \gamma_2} \frac{z^m}{1 + z^n} dz = 2\pi i \operatorname{Res}\left(\frac{z^m}{1 + z^n}, \exp(\pi i/n)\right)$$

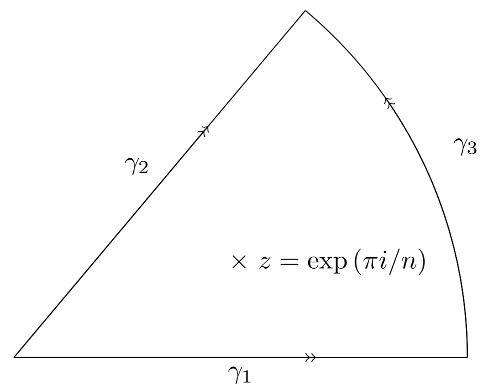

Figure 1: Contour used in the evaluation of the first integral

To find the residue we expand $1 + z^n$ around $z = u + \exp(\pi i/n)$, where we obtain an expansion beginning with the term $n\exp(\pi i/n)u$. This proves that $\mathrm{Ord}(1+z^n, \exp(\pi i/n)) = 1$, and so we can use property (c) of residues above to conclude that

$$\mathrm{Res}\left(\frac{z^m}{1+z^n}, \exp(\pi i/n)\right) = \lim_{z\to\exp(\pi i/n)} \frac{z^m}{nz^{n-1}} = \frac{z^{m-n+1}}{n} = -\frac{z^{m+1}}{n} = -\frac{1}{n}\exp\left(i\pi\frac{m+1}{n}\right)$$

Now we note that on γ_2 we have $z^m \to \exp(2\pi i(m/n))t^m$, $1 + z^n = 1 + t^n$ and $dz \to \exp(2\pi i/n)dt$, and so we have

$$\int_{-\gamma_2} \frac{z^m}{1+z^n} = -\exp\left(2\pi i\frac{m+1}{n}\right)\int_{\gamma_1}\frac{z^m}{1+z^n}$$

Combining these, we obtain

$$\left(1 - \exp\left(2\pi i\frac{m+1}{n}\right)\right)\int_{\gamma_1}\frac{z^m}{1+z^n}dz + \int_{\gamma_3}\frac{z^m}{1+z^n}dz = -\frac{2\pi i}{n}\exp\left(i\pi\frac{m+1}{n}\right)$$

When we take the limit $R \to \infty$, we can conclude that the integral over γ_3 vanishes, since the integrand decays *at least* like z^{-2}, and the length of the curves grows only like z, so the integral decays like z^{-1}. We also note that the integral we wish to evaluate is $\int_{\gamma_1} f(z)dz$ as $R \to \infty$. Dividing by $-2i\exp(i\pi(m+1)/n)$, we obtain

$$\frac{1}{2i}\left(\exp\left(\pi i\frac{m+1}{n}\right) - \exp\left(-\pi i\frac{m+1}{n}\right)\right)\lim_{R\to\infty}\int_0^R \frac{t^m}{1+t^n}dt = \frac{\pi}{n}$$

We note that this implies

$$\sin\left(\pi\frac{m+1}{n}\right)\int_0^\infty \frac{t^m}{1+t^n}dt = \frac{\pi}{n} \implies \boxed{\int_0^\infty \frac{t^m}{1+t^n}dt = \frac{\pi}{n}\csc\left(\pi\frac{m+1}{n}\right)}$$

For the second integral, we note by symmetry of sin that

$$\int_0^\pi \frac{d\theta}{a+\sin^2\theta} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{a+\sin^2\theta}$$

Now let $z = \exp(i\theta)$, whereby we obtain $dz = iz\,d\theta$, and

$$\sin\theta = \frac{z - 1/z}{2i} \implies \sin^2\theta = \frac{2 - z^2 - z^{-2}}{4}$$

We also note that when we integrate θ from 0 to 2π, we are integrating the variable z over the unit disk. We obtain

$$\frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + \sin^2\theta} = -2i\int_{\partial D} \frac{dz}{4az + 2z - z^3 - z^{-1}} = 2i\int_{\partial D} \frac{z\,dz}{z^4 - (4a+2)z^2 + 1}$$

Now to find the residues, we must find the zeroes of $z^4 - (4a+2)z^2 + 1$. Completing the square,

$$z^4 - (4a+2)z^2 + 1 = (z^2 - (2a+1))^2 - 4a^2 - 4a$$

Which implies that the set of zeroes is the four roots

$$r_{\pm,\pm} = \pm\sqrt{2a + 1 \pm 2\sqrt{a(a+1)}}$$

Since $a > 0$, we can conclude that $r_{\pm,+} > 0$, and so it lies outside of ∂D. Since $2a + 1 - 2\sqrt{a(a+1)} > 1$ implies $4a^2 + 4a + 1 \geq 4a^2 + 4a + 1 + 4\sqrt{a(a+1)}$ and $a \neq 0$, we conclude that $r_{\pm,-} \in \partial D$. Finally we note that $2a + 1 - 2\sqrt{a(a+1)} = 0$ implies that $1 = 0$, a contradiction. This means that $z^4 - (4a+2)z^2 + 1$ has order 1 at each of the roots, and so we can apply property (c) of residues to conclude

$$\operatorname{Res}\left(\frac{z}{z^4 - (4a+2)z^2 + 1}, r_{\pm,-}\right) = \frac{1}{4(r_{\pm,-})^2 - 4(2a+1)} = \frac{1}{-8\sqrt{a(a+1)}}$$

We conclude that the integral is

$$2i\int_{\partial D} \frac{z\,dz}{z^4 - (4a+2)z^2 + 1} = 2i \times 2\pi i \sum_{\pm} \frac{1}{-8\sqrt{a(a+1)}} = \boxed{\frac{\pi}{\sqrt{a(a+1)}}}$$

Problem 3. *Calculate the residues of* $\tan(\pi z)$ *and* $\cot(\pi z)$ *at their poles.*

Solution: Using the fact that $\tan\pi z = \sin\pi z/\cos\pi z$, we can conclude that whenever $\tan\pi z$ has a pole, we have $\cos\pi z = 0$. As we proved in assignment 2, $\cos z = 0$ implies that $z = \pi/2 + \pi n$ for some $n \in \mathbb{Z}$. Since $\cos'(\pi/2 + \pi n) = \pm 1$ we conclude that cos has order 1 at it's zeroes. We can therefore apply property (c) of residues and conclude that

$$\operatorname{Res}\left(\frac{\sin\pi z}{\cos\pi z}, r\right) = -\frac{1}{\pi}\frac{\sin z}{\sin z} = \frac{-1}{\pi}$$

Therefore tan has residue $\frac{-1}{\pi}$ at each of it's poles. Applying the same exact analysis to cot, we obtain

$$\operatorname{Res}\left(\frac{\cos z}{\sin z}, r\right) = \frac{\cos z}{\pi\cos z} = \frac{1}{\pi}$$

And so cot has residue $\frac{1}{\pi}$ at each of it's poles.

Problem 4. *Show that the sum of the residues of a rational function (together with the residue at ∞) is equal to zero. As part of this exercise you need to introduce a natural definition for the residue at ∞.*

Solution:

Definition 1. *The* residue at infinity *of a function $f \in \mathcal{O}D_{r,\infty}$, where $D_{r,\infty}$ is an annulus with an infinite outer radius, is defined to be*

$$\operatorname{Res}(f, \infty) = \int_{\partial D_{r,\infty}} f(z)dz$$

A few remarks:

1. *Since we require f to be holomorphic on $D_{r,\infty}$, we can not have any singularities in $D_{r,\infty}$.*
2. *By homotopy of $\partial D_{r,\infty}$* and *$\partial D_{\tilde{r},\infty}$, this definition does not depend on the r chosen.*
3. *By our choice of* orientation *of $\partial D_{r,R}$ [shown below] we conclude that $\partial D_{r,\infty} = -\partial D_r$.*

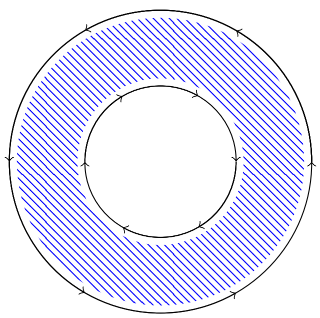

Figure 2: Orientation of ∂D_{r_R}

Let f be a rational function, then $f = p/q$, where p and q are polynomials. Clearly f is meromorphic, and it's isolated singularities are the roots of the polynomial q. Since q has finitely many roots, f has finitely many singularities $\{ z_1, \cdots, z_n \}$.

Let D_r be a disk so large that it contains all of $\{ z_1, \cdots, z_n \}$. Then by the residue theorem, we can conclude that

$$\frac{1}{2\pi i} \int_{\partial D_r} f(z)dz = \sum_{i=1}^{n} \operatorname{Res}(f, z_i) \tag{1}$$

However, in this sum we are omitting the residue at infinity. Since all of the singularities are in D_r we can conclude that f is holomorphic on $D_{r,\infty}$, and so

$$\operatorname{Res}(f, \infty) = \frac{1}{2\pi i} \int_{\partial D_{r,\infty}} f(z)dz = -\frac{1}{2\pi i} \int_{\partial D_R} f(z)dz \tag{2}$$

where we used $\partial D_r = -\partial D_{r,\infty}$. Adding (1) and (2), we conclude

$$\operatorname{Res}(f,\infty) + \sum_{i=1}^{n} \operatorname{Res}(f, z_i) = 0$$

Problem 5. *Let f be holomorphic and bounded in a punctured neighbourhood of 0. Show that*

$$g(z) = \begin{cases} z^2 f(z) & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

is holomorphic in a neighbourhood of 0.

Solution: For concreteness, let f be holomorphic on $D_r^\times(0)$. To prove that $g(z)$ is holomorphic on $D_r(0)$, we will simply prove that it is differentiable at $z = 0$. Since we already know that it is differentiable everywhere on $D_r^\times(0)$, we will have solved the problem.

To prove that g is differentiable at 0, consider the difference quotient:

$$\lim_{z\to 0} \frac{g(z) - g(0)}{z} = \lim_{z\to 0} \frac{z^2 f(z)}{z} = \lim_{z\to 0} z f(z)$$

Now since f is bounded, we can conclude that this limit goes to zero, and thus g is differentiable at 0. It follows that g is holomorphic everywhere. One consequence of the proof is that g has $g(0) = 0$ and $g'(0) = 0$, this allows us to conclude that $g(z) = z^2 h(z)$ for some holomorphic h. Notice that this result gives an alternate proof of the removable singularity theorem: since $f(z) = g(z)/z^2 = h(z)$ everywhere in $D_r^\times(0)$ we can use the identity theorem to conclude that f has a removable singularity at 0.

Problem 6. *Let $c \in \mathbb{C}$ be an isolated essential singular point of f. Prove that for any given $\alpha \in \mathbb{R}$ and an arbitrary small $r > 0$, there exists $z \in D_r(c) - \{ c \}$ such that $\Re f(z) = \alpha$.*

Solution: We will use two theorems proved in the course notes:

Theorem (Casorati-Weierstrass Theorem)**.** *Let $f \in \mathcal{O}(D_r^x(c))$, and let c be an essential singularity of f. Then $f(D_r^x(c))$ is dense in $\mathbb{C}$.*

Theorem (Preservation of Domains)**.** *If $\Omega \subset \mathbb{C}$ is a connected open set and $f \in \mathcal{O}(\Omega)$ is nonconstant, then $f(\Omega)$ is also a connected open set*

Now suppose that there exists an $\alpha \in \mathbb{R}$ such that <u>there does not exist</u> a $z \in D_r^x(c)$ such that $\Re f(z) = \alpha$. Then the line $\ell = \{ z \in \mathbb{C} \mid \Re z = \alpha \}$ is disjoint from $f(D_r^x(c))$. Since $f(D_r^x(c))$ is dense, we can conclude that there exist points of $f(D_r^x(c))$ on either side of the line ℓ. Let $C_{-\alpha}$ and $C_{+\alpha}$ be the two open halves of the complex plane

$$\mathbb{C}_{-\alpha} = \{ z \in \mathbb{C} \mid \Re z < \alpha \} \qquad \text{and} \qquad \mathbb{C}_{+\alpha} = \{ z \in \mathbb{C} \mid \Re z > \alpha \}$$

It is clear that these disjoint open sets cover $f(D_r^x(c))$, and both $C_{-\alpha} \cap f(D_r^x(c))$ and $C_{+\alpha} \cap f(D_r^x(c))$ are non-empty. Therefore $f(D_r^x(c))$ can not be connected. However, by the Preservation of Domains theorem, and the fact that $D_r^x(c)$ is connected, we must have $f(D_r^x(c))$ connected. This is a contradiction, and so we must conclude the existence of some $z \in D_r^x(c)$ such that $\Re f(z) = \alpha$.

Problem 7. *Let $\gamma : [a,b] \to A$ be a piecewise differentiable curve, where $A \subset \mathbb{C}$ is an open set, and let $g : A \times \Omega \to \mathbb{C}$ be a countinuous function of two complex variables, where $\Omega \subset \mathbb{C}$ is also an open set. Assume that for any fixed $w \in A$, the function $z \mapsto g(w,z)$ is holomorphic in Ω. Then prove that*

$$f(z) = \int_\gamma g(w,z)dw$$

is holomorphic in Ω.

Solution: Let $c \in \Omega$. Since $g(w,\cdot)$ is holomorphic in Ω for each w, we can conclude that

$$g(w,z) = g(w,c) + h(w,z)(z-c)$$

For some $h(w,z)$ continous at c (in the second argument, holding the first one fixed). Furthermore, since

$$h(w,z) = \frac{g(w,z) - g(w,c)}{(z-c)}$$

we can conclude that h is also continuous in the first argument (holding the second one fixed). However, if a function is continuous in both arguments seperately, we can conclude that it is continuous[1].

Then we can conclude that

$$f(z) = \int_\gamma g(w,c) + h(w,z)(z-c)\,dw = f(c) + (z-c)\int_\gamma h(w,z) \tag{3}$$

It is clear that the integral $\displaystyle\int_\gamma h(w,z)dw$ is well defined by continuity of h. Furthermore, it is a basic fact of integration that $\displaystyle\int_\gamma h(w,z)dw$ is a continuous function of z. Therefore the equation (3) implies that f is differentiable at c. Since c was arbitary, we conclude that f is holomorphic on Ω.

Note: We could have also used Morera's theorem in the manner we have used to prove the Weierstrass convergence theorem (and the Schwarz reflection principle).

[1]proof: $h(w+\delta, z+\epsilon) - h(w,z) = h(w+\delta, z+\epsilon) - h(w, z+\epsilon) + h(w, z+\epsilon) - h(w,z)$