

ANALYTIC FUNCTIONS

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1. UNIFORM CONVERGENCE

In this section, we fix a nonempty set $G \subset \mathbb{C}$, and consider sequences of complex valued functions defined on G . By a *sequence of functions* we mean simply an assignment of a function $f_n : G \rightarrow \mathbb{C}$ to each index n , with the latter usually having positive integers as values. We denote this sequence by $\{f_1, f_2, \dots\}$ or $\{f_n\}$, and consider it also as a collection of functions. So for example, $\{f_n\} \subset \mathcal{C}(G)$ would mean that every function in the sequence is continuous.

Definition 1. We say that a sequence $\{f_n\}$ *converges pointwise in G* to a function $f : G \rightarrow \mathbb{C}$, if for each $z \in G$, the number sequence $\{f_n(z)\}$ converges to $f(z)$, i.e., $f_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$.

Example 2. Let $f_n(x) = \arctan(nx)$. Since $\arctan x \rightarrow \pm \frac{\pi}{4}$ as $x \rightarrow \pm \infty$, the sequence $\{f_n\}$ converges pointwise in \mathbb{R} to f , where

$$f(x) = \begin{cases} \frac{\pi}{4} & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\frac{\pi}{4} & \text{for } x < 0. \end{cases} \quad (1)$$

Pointwise convergence is a very weak kind of convergence. For instance, as we have seen in the preceding example, the pointwise limit of a sequence of continuous functions is *not* necessarily continuous. The notion of uniform convergence is a stronger type of convergence that remedies this deficiency.

Definition 3. We say that a sequence $\{f_n\}$ *converges uniformly in G* to a function $f : G \rightarrow \mathbb{C}$, if for any $\varepsilon > 0$, there exists N such that $|f_n(z) - f(z)| \leq \varepsilon$ for any $z \in G$ and all $n \geq N$.

Remark 4. Let us introduce the *uniform norm*

$$\|g\|_G = \sup_{z \in G} |g(z)| \quad \text{for } g : G \rightarrow \mathbb{C}. \quad (2)$$

Then $f_n \rightarrow f$ uniformly in G if and only if $\|f_n - f\|_G \rightarrow 0$ as $n \rightarrow \infty$.

Exercise 5. Show that uniform convergence implies pointwise convergence.

Example 6. The sequence $\{f_n\}$ from [Example 2](#) does *not* converge uniformly to f , since by continuity of the arctangent function, for any given n , however large, and for any given $\delta > 0$, there exists $x > 0$ small enough such that $\arctan(nx) < \delta$. So for instance, for any given n , there is $x > 0$ such that $|f(x) - f_n(x)| > \frac{\pi}{8}$.

Example 7. The sequence $\{f_n\}$ with $f_n(x) = \frac{1}{n} \sin x$ converges uniformly in \mathbb{R} to 0.

Example 8. Let $f_n(z) = 1 + z + z^2 + \dots + z^n$. Then the standard argument

$$f_n(z) - zf_n(z) = 1 + z + z^2 + \dots + z^n - (z + z^2 + \dots + z^n + z^{n+1}) = 1 - z^{n+1}, \quad (3)$$

implies that

$$f_n(z) = \frac{1 - z^{n+1}}{1 - z}. \quad (4)$$

We make a guess that $f_n(z)$ converges to $f(z) = \frac{1}{1-z}$, and compute

$$f(z) - f_n(z) = \frac{1}{1-z} - \frac{1 - z^{n+1}}{1-z} = \frac{z^{n+1}}{1-z}, \quad (5)$$

which leads to

$$|f(z) - f_n(z)| = \frac{|z^{n+1}|}{|1-z|} \leq \frac{|z|^{n+1}}{|1-z|}. \quad (6)$$

Now if $|z| \leq r$ for some $r < 1$, then $|1-z| \geq 1-|z| \geq 1-r$, and hence

$$|f(z) - f_n(z)| \leq \frac{r^{n+1}}{1-r}. \quad (7)$$

This shows that f_n converges to f uniformly in $\bar{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$, for any fixed $r < 1$.

Theorem 9 (Weierstrass 1861). *Suppose that $\{f_n\} \subset \mathcal{C}(G)$ converges uniformly in G to a function $f : G \rightarrow \mathbb{C}$. Then $f \in \mathcal{C}(G)$.*

Proof. We will use the sequential criterion of continuity. Let $z \in G$, and let $\{z_k\} \subset G$ be a sequence converging to z . Then we have

$$f(z_k) - f(z) = f(z_k) - f_n(z_k) + f_n(z_k) - f_n(z) + f_n(z) - f(z), \quad (8)$$

for any n , and hence

$$|f(z_k) - f(z)| \leq |f(z_k) - f_n(z_k)| + |f_n(z_k) - f_n(z)| + |f_n(z) - f(z)|, \quad (9)$$

by the triangle inequality.

Let $\varepsilon > 0$ be given. Then by uniform convergence, there is N such that

$$|f(w) - f_n(w)| \leq \varepsilon, \quad \text{for all } w \in G, \quad \text{and for all } n \geq N. \quad (10)$$

In particular,

$$|f(z_k) - f(z)| \leq 2\varepsilon + |f_n(z_k) - f_n(z)|, \quad \text{for all } k, \quad \text{and for all } n \geq N. \quad (11)$$

Now we fix one such n , for example, put $n = N$, and use the continuity of f_n to imply the existence of K with the property that

$$|f_n(z_k) - f_n(z)| \leq \varepsilon, \quad \text{for all } k \geq K. \quad (12)$$

Finally, this means that

$$|f(z_k) - f(z)| \leq 3\varepsilon, \quad \text{for all } k \geq K, \quad (13)$$

and since $\varepsilon > 0$ was arbitrary, we infer that f is continuous at z . \square

Exercise 10. Find a mistake in the following purported proof.

Claim: If $\{f_n\} \subset \mathcal{C}(G)$ converges pointwise in G to a function $f : G \rightarrow \mathbb{C}$, then f is continuous in G .

Proof: Let $z \in G$, and let $\{z_k\} \subset G$ be a sequence converging to z . Then as in the preceding proof, we have

$$|f(z_k) - f(z)| \leq |f(z_k) - f_n(z_k)| + |f_n(z_k) - f_n(z)| + |f_n(z) - f(z)|. \quad (14)$$

Since f_n converges pointwise to f , both $|f(z_k) - f_n(z_k)|$ and $|f_n(z) - f(z)|$ tend to 0 as $n \rightarrow \infty$. Furthermore, $|f_n(z_k) - f_n(z)| \rightarrow 0$ as $k \rightarrow \infty$, because the function f_n is continuous. Hence by choosing k and n large enough, we can make the right hand side of (14) arbitrarily small, which means that f is continuous at z .

The definition of uniform convergence involves the limit function f , so we cannot apply this definition without having a candidate for the limit f . However, we will want to construct new functions as the limits of convergent sequences, and in most cases there will not be any limit candidates available. Therefore it is of interest to develop tools that can turn some easily verifiable properties of a sequence into the existence of a limit. In complex analysis, Cauchy's criterion is the main device of such kind.

Definition 11. A sequence $\{f_n\}$ is called *uniformly Cauchy in G* , if for any $\varepsilon > 0$, there exists N such that $|f_n(z) - f_m(z)| \leq \varepsilon$ for all $z \in G$ and all $n, m \geq N$.

Remark 12. In terms of the uniform norm, the sequence $\{f_n\}$ being uniformly Cauchy in G is equivalent to the assertion that $\|f_n - f_m\|_G \rightarrow 0$ as $n, m \rightarrow \infty$.

Exercise 13. Show that if a sequence is uniformly convergent then it is uniformly Cauchy.

Theorem 14 (Cauchy's criterion). *If $\{f_n\}$ is uniformly Cauchy in G , then there exists a function $f : G \rightarrow \mathbb{C}$, such that $f_n \rightarrow f$ uniformly in G .*

Proof. Let $z \in G$, and let $f_n(z) = a_n + ib_n$, with a_n and b_n real. Since f_n is uniformly Cauchy in G , for any $\varepsilon > 0$, there exists N such that $|f_n(z) - f_m(z)| \leq \varepsilon$ whenever $n, m \geq N$. In other words, the complex number sequence $\{f_n(z)\}$ is Cauchy. Then the identity

$$|f_n(z) - f_m(z)|^2 = |a_n - a_m|^2 + |b_n - b_m|^2, \quad (15)$$

implies that both $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences of numbers, and hence there exist two real numbers a and b , such that $a_n \rightarrow a$ and $b_n \rightarrow b$ as $n \rightarrow \infty$. We set $f(z) := a + ib$. Since $z \in G$ was arbitrary, this procedure defines a function $f : G \rightarrow \mathbb{C}$, and by construction, $f_n \rightarrow f$ pointwise in G . It remains to show that this convergence is uniform.

Let $\varepsilon > 0$. Then there exists N such that $|f_n(z) - f_m(z)| \leq \varepsilon$ for all $z \in G$ and all $n, m \geq N$. Now for any given $z \in G$, by pointwise convergence, there exists $m \geq N$ so large that $|f_m(z) - f(z)| \leq \varepsilon$. Therefore we have

$$|f_n(z) - f(z)| \leq |f_n(z) - f_m(z)| + |f_m(z) - f(z)| \leq 2\varepsilon, \quad (16)$$

whenever $n \geq N$ and $z \in G$, implying that $f_n \rightarrow f$ uniformly in G . \square

Example 15. Consider $f_n(z) = 1 + z + \frac{z^2}{2} + \dots + \frac{z^n}{n}$, which is a partial sum of the power series $\sum_n \frac{z^n}{n}$. Let us apply Cauchy's criterion to $\{f_n\}$. For $n \geq m$, we have

$$f_n(z) - f_m(z) = \frac{z^{m+1}}{m+1} + \frac{z^{m+2}}{m+2} + \dots + \frac{z^n}{n}, \quad (17)$$

leading to the bound

$$|f_n(z) - f_m(z)| \leq \frac{|z|^{m+1}}{m+1} + \frac{|z|^{m+2}}{m+2} + \dots + \frac{|z|^n}{n} \leq |z|^{m+1}(1 + |z| + \dots + |z|^{n-m-1}). \quad (18)$$

Now if we fix some $r < 1$, and assume that $|z| \leq r$, then

$$|f_n(z) - f_m(z)| \leq r^{m+1}(1 + r + \dots + r^{n-m-1}) \leq \frac{r^{m+1}}{1-r}, \quad (19)$$

which shows that $\{f_n\}$ is uniformly Cauchy in the disk \bar{D}_r , and hence there is a function $f : \bar{D}_r \rightarrow \mathbb{C}$ such that $f_n \rightarrow f$ uniformly in \bar{D}_r . Finally, since each f_n is continuous in \bar{D}_r , by [Theorem 9](#), we conclude that $f \in \mathcal{C}(\bar{D}_r)$.

Recall that $D_r(z) = \{w \in \mathbb{C} : |w - z| < r\}$ is the open disk of radius r with the centre z .

Definition 16. A sequence $\{f_n\}$ of functions $f_n : G \rightarrow \mathbb{C}$ is called *locally uniformly convergent* in G if for each $z \in G$ there is $r > 0$ such that $\{f_n\}$ converges uniformly in $G \cap D_r(z)$. Similarly, $\{f_n\}$ is called *locally uniformly Cauchy*, if for each $z \in G$ there is $r > 0$ such that $\{f_n\}$ is uniformly Cauchy in $G \cap D_r(z)$.

Example 17. The series $\sum_n \frac{z^n}{n}$ converges locally uniformly in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Exercise 18. Prove the following.

- (a) If $\{f_n\}$ is locally uniformly convergent in G , then there is a function $f : G \rightarrow \mathbb{C}$ such that $f_n \rightarrow f$ pointwise in G .
- (b) If $\{f_n\} \subset \mathcal{C}(G)$ converges to $f : G \rightarrow \mathbb{C}$ locally uniformly in G , then $f \in \mathcal{C}(G)$.
- (c) If $\{f_n\}$ locally uniformly Cauchy in G , then $\{f_n\}$ converges locally uniformly in G .

2. ABSOLUTELY UNIFORM CONVERGENCE

Given a sequence $\{f_m\}$ of functions $f_m : G \rightarrow \mathbb{C}$, by defining $g_0 = f_0$ and $g_n = f_n - f_{n-1}$ for $n = 1, 2, \dots$, we can write

$$f_m = f_0 + (f_1 - f_0) + \dots + (f_m - f_{m-1}) = \sum_{n=0}^m g_n. \quad (20)$$

This leads to the *function series*

$$\sum_{n=0}^{\infty} g_n, \quad (21)$$

which can be understood as an overloaded notation for both

- the sequence $\{f_m\}$ (or, what is the same, the sequence $\{g_n\}$), and
- the limit $\lim_{m \rightarrow \infty} f_m = \lim_{m \rightarrow \infty} \sum_{n=0}^m g_n$, if it exists.

Thus when one says that the series $\sum_n g_n$ converges uniformly in G , one is referring to the sequence $\{f_m\}$. On the other hand, the equality $\sum_n g_n = f$, or the statement that the sum of the series $\sum_n g_n$ is f , would be referring to the limit $\lim_{m \rightarrow \infty} f_m$.

Definition 19. We say that a series $\sum_n g_n$ of functions $g_n : G \rightarrow \mathbb{C}$ *converges absolutely uniformly in G* if the series $\sum_n |g_n|$ converges uniformly in G , that is, if the sequence $\{\tilde{f}_n\}$ with the terms $\tilde{f}_n = |g_1| + \dots + |g_n|$ converges uniformly in G .

Lemma 20. *If a series is absolutely uniformly convergent, then it converges uniformly.*

Proof. Suppose that $\sum_n g_n$ is a series of functions $g_n : G \rightarrow \mathbb{C}$, converging absolutely uniformly in G . Let $f_n = g_1 + g_2 + \dots + g_n$ and $\tilde{f}_n = |g_1| + |g_2| + \dots + |g_n|$. We start with the inequality

$$|f_m(z) - f_n(z)| \leq |g_{n+1}(z)| + \dots + |g_m(z)| = \tilde{f}_m(z) - \tilde{f}_n(z), \quad (22)$$

which is true for all $z \in G$ and $m \geq n$. By hypothesis, $\tilde{f}_n \rightarrow \tilde{f}$ uniformly in G for some $\tilde{f} : G \rightarrow \mathbb{C}$. Now let $\varepsilon > 0$, and let N be such that $|\tilde{f}_n(z) - \tilde{f}(z)| \leq \varepsilon$ for all $n \geq N$. Then for all $z \in G$ and $n, m \geq N$, we have

$$\tilde{f}_m(z) - \tilde{f}_n(z) = |\tilde{f}_m(z) - \tilde{f}(z)| + |\tilde{f}(z) - \tilde{f}_n(z)| \leq 2\varepsilon, \quad (23)$$

which implies that

$$|f_m(z) - f_n(z)| \leq 2\varepsilon, \quad (24)$$

for all $z \in G$ and $m \geq n \geq N$. This shows that the sequence $\{f_n\}$ is uniformly Cauchy, hence $f_n \rightarrow f$ uniformly in G for some $f : G \rightarrow \mathbb{C}$. \square

Remark 21. In the notation of the preceding proof, we have the estimate

$$|f(z) - f_n(z)| \leq \tilde{f}(z) - \tilde{f}_n(z) = \sum_{k=n+1}^{\infty} |g_k(z)|, \quad (25)$$

for any $z \in G$ and any n . Indeed, let $\varepsilon > 0$ be arbitrary, and let m be a large integer so that $|f_m(z) - f(z)| \leq \varepsilon$ and $|\tilde{f}_m(z) - \tilde{f}(z)| \leq \varepsilon$. Then we have

$$\begin{aligned} |f_n(z) - f(z)| &\leq |f_n(z) - f_m(z)| + |f_m(z) - f(z)| \leq |\tilde{f}_n(z) - \tilde{f}_m(z)| + \varepsilon \\ &\leq |\tilde{f}_n(z) - \tilde{f}(z)| + |\tilde{f}(z) - \tilde{f}_m(z)| + \varepsilon \leq \tilde{f}(z) - \tilde{f}_n(z) + 2\varepsilon, \end{aligned} \quad (26)$$

where we have taken into account that $|\tilde{f}_n(z) - \tilde{f}(z)| = \tilde{f}(z) - \tilde{f}_n(z)$, because $\{\tilde{f}_n(z)\}$ is a nondecreasing sequence. This shows that $|f(z) - f_n(z)| \leq \tilde{f}(z) - \tilde{f}_n(z)$. Finally, the equality in (25) follows from

$$\begin{aligned} \sum_{k=n+1}^{\infty} |g_k(z)| &= \lim_{m \rightarrow \infty} \sum_{k=n+1}^m |g_k(z)| = \lim_{m \rightarrow \infty} (\tilde{f}_m(z) - \tilde{f}_n(z)) \\ &= \left(\lim_{m \rightarrow \infty} \tilde{f}_m(z) \right) - \tilde{f}_n(z) = \tilde{f}(z) - \tilde{f}_n(z), \end{aligned} \quad (27)$$

where we have used the additive property of the limits of real number sequences.

Example 22. Absolutely uniform convergence is strictly stronger than the combination of uniform convergence and pointwise absolute convergence. For example, take the series

$$\sum_n \frac{(-1)^n x^n}{n} \quad \text{for } x \in (0, 1). \quad (28)$$

Obviously, it converges absolutely for each $x \in (0, 1)$. Let $f_n(x)$ be the n -th partial sum of (28). Then from the alternating nature of the series, we have

$$|f_m(x) - f_n(x)| \leq \frac{x^{n+1}}{n+1} < \frac{1}{n+1} \quad \text{for } x \in (0, 1), \quad \text{and for } m \geq n, \quad (29)$$

which shows that the series is uniformly convergent in $(0, 1)$. However, (28) does *not* converge absolutely uniformly in $(0, 1)$, because

$$\sum_{n=m}^k \frac{|(-1)^n x^n|}{n} = \sum_{n=m}^k \frac{x^n}{n} \rightarrow \sum_{n=m}^k \frac{1}{n} \quad \text{as } x \rightarrow 1, \quad (30)$$

and the sum in the right hand side grows unboundedly with k , regardless of the value of m .

Theorem 23 (Weierstrass M-test, majorant test, or comparison test). *Let the functions $g_n : G \rightarrow \mathbb{C}$ satisfy $|g_n(z)| \leq a_n$ for all $z \in G$ and for each n , where $\sum_n a_n$ is a convergent series of real numbers. Then the function series $\sum_n g_n$ converges absolutely uniformly in G . In other words, if $\sum_n \|g_n\|_G < \infty$ then $\sum_n g_n$ converges absolutely uniformly in G .*

Proof. With $\tilde{f}_n = |g_1| + \dots + |g_n|$, for $n > m$ and $z \in G$, we have

$$\tilde{f}_n(z) - \tilde{f}_m(z) = |g_{m+1}(z)| + \dots + |g_n(z)| \leq a_{m+1} + \dots + a_n \leq \sum_{k=m+1}^{\infty} a_k, \quad (31)$$

which tends to 0 when $m \rightarrow \infty$. Note that the bound on the right hand side does not depend on z . So $\{\tilde{f}_n\}$ is uniformly Cauchy in G , hence converges uniformly in G . \square

Example 24. Since $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, and $\sum n^{-2} < \infty$, the series $\sum_n \frac{\sin(n^2 x)}{n^2}$ converges absolutely uniformly in \mathbb{R} .

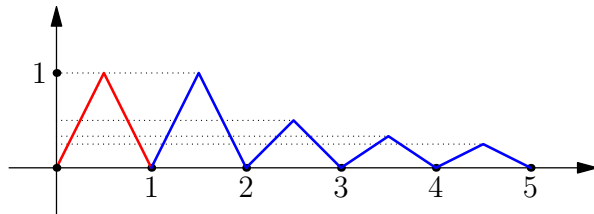


FIGURE 1. The blue curve is the graph of the 4-th partial sum of the series $\sum_n g_n$ from [Example 25](#). Then each “blue tooth” represents an individual term g_n . The function ϕ is represented by the red curve in the interval $(0, 1)$, outside of which ϕ vanishes.

Example 25. The Weierstrass M-test is only a sufficient condition for absolutely uniform convergence. For example, let $\phi(x) = \max\{0, \min\{2x, 2 - 2x\}\}$, and let $g_n(x) = \frac{1}{n}\phi(x - n)$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then the series $\sum_n g_n$ converges absolutely uniformly in \mathbb{R} , because

$$0 \leq \sum_{n=m}^k g_n(x) \leq \frac{1}{m} \quad \text{for any } x \in \mathbb{R} \quad \text{and any } k \geq m. \quad (32)$$

However, we have $\max_{x \in \mathbb{R}} g_n(x) = \frac{1}{n}$, hence the M-test is not applicable here (see [Figure 1](#)).

Remark 26. Despite the observation in the preceding example, the Weierstrass M-test is applicable to all frequently occurring series in complex analysis. In fact, there is a notion of convergence, called *normal convergence*, that is by definition equivalent to the hypothesis of the Weierstrass M-test. Normal convergence is stronger than absolutely uniform convergence, hence simplifies many proofs. On the other hand, we do not lose generality, because complex analysis is basically built on normally convergent series anyways. Thus it appears that normal convergence is the optimal notion of convergence in complex analysis. In these notes we will not formally define normal convergence but will use it implicitly in some places.

Theorem 27 (Dirichlet 1837). *If $\sum_n g_n$ converges absolutely uniformly in G , then all its rearrangements converge (absolutely uniformly in G) to the same limit.*

Proof. Let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection, so that $\sum_n g_{\sigma(n)}$ is a rearrangement of $\sum_n g_n$. Fix $\varepsilon > 0$. Then there exists N such that

$$|g_n(z)| + \dots + |g_{n+k}(z)| \leq \varepsilon \quad \text{for all } z \in G, \quad (33)$$

whenever $n \geq N$ and $k > 0$. Now let $f_n = g_1 + \dots + g_n$ and $F_n = g_{\sigma(1)} + \dots + g_{\sigma(n)}$. By the hypothesis of the theorem, $f_n \rightarrow f$ uniformly in G , for some $f : G \rightarrow \mathbb{C}$. Thus there is N' such that¹

$$|f_n(z) - f(z)| \leq \varepsilon \quad \text{for all } z \in G, \quad (34)$$

whenever $n \geq N'$. We want show that $F_m - f_m$ gets smaller as m grows. To this end, let

$$M = \max\{\sigma^{-1}(n) : n = 1, 2, \dots, N\}, \quad (35)$$

that is, the image of the interval $\{1, \dots, M\}$ under σ covers the entire interval $\{1, \dots, N\}$. Therefore for $m \geq M$, we have

$$|F_m(z) - f_m(z)| \leq |g_{N+1}(z)| + \dots + |g_{N+k}(z)| \leq \varepsilon \quad \text{for all } z \in G, \quad (36)$$

for some k large enough. In combination with (34), this shows that

$$|F_m(z) - f(z)| \leq 2\varepsilon \quad \text{for all } z \in G, \quad (37)$$

¹In fact, from the estimate (25) it follows that $N' = N$ would be sufficient.

whenever $m \geq \max\{M, N'\}$, and so $F_m \rightarrow f$ uniformly in G .

We have proved that if a series converges absolutely uniformly, then its rearrangement converges *uniformly*. To show that the rearrangement converges *absolutely uniformly*, we simply apply what we have proved to the series $\sum_n |g_{\sigma(n)}|$, which is a rearrangement of $\sum_n |g_n|$. Since the latter converges absolutely uniformly, the former converges uniformly, meaning that $\sum_n g_{\sigma(n)}$ converges absolutely uniformly. \square

The following is a simplification of the theorem we have proved in class. We impose a stronger hypothesis (which is related to normal convergence discussed in [Remark 26](#)), but the resulting theorem will nevertheless be sufficient for our purposes.

Theorem 28. *Let $f_{k,\ell} : G \rightarrow \mathbb{C}$ for $k, \ell \in \mathbb{N}$, and let $\sigma : \mathbb{N}^2 \rightarrow \mathbb{N}$ be a bijection. Define the sequence $\{g_n\}$ by $g_{\sigma(k,\ell)} = f_{k,\ell}$. Assume that $\{a_{k,\ell}\} \subset \mathbb{R}$ satisfy $|f_{k,\ell}(z)| \leq a_{k,\ell}$ for all $z \in G$, $k, \ell \in \mathbb{N}$, and let $\{b_n\}$ be such that $b_{\sigma(k,\ell)} = a_{k,\ell}$. Then the following are equivalent:*

- (a) $\sum_n b_n < \infty$.
- (b) $\sum_\ell a_{k,\ell} < \infty$ for each k , and moreover $\sum_k (\sum_\ell a_{k,\ell}) < \infty$.
- (c) $\sum_k a_{k,\ell} < \infty$ for each ℓ , and moreover $\sum_\ell (\sum_k a_{k,\ell}) < \infty$.

If any (hence all) of the above conditions is satisfied, then we have

$$\sum_\ell \left(\sum_k f_{k,\ell} \right) = \sum_k \left(\sum_\ell f_{k,\ell} \right) = \sum_n g_n. \quad (38)$$

Proof. First we prove the implication (a) \Rightarrow (b). Let $N = \sum_n b_n < \infty$. This obviously implies that for all $k \in \mathbb{N}$, $M_k = \sum_\ell a_{k,\ell} < \infty$. Let $\varepsilon > 0$ and let m_k be such that $\sum_{\ell > m_k} a_{k,\ell} \leq 2^{-k}\varepsilon$. So for any m we have

$$\sum_{k \leq m} \left(\sum_\ell a_{k,\ell} \right) \leq \sum_{k \leq m} \left(\sum_{\ell \leq m_k} a_{k,\ell} \right) + 2\varepsilon \leq N + 2\varepsilon. \quad (39)$$

For the other direction (b) \Rightarrow (a), we start with the definition of M_k and the condition $M = \sum_k M_k < \infty$. Then for any p we have

$$\sum_{n \leq p} b_n \leq \sum_{k \leq m} \sum_{\ell \leq m} a_{k,\ell} \leq M, \quad (40)$$

where m is such that $\{\ell \leq m\}^2 \supseteq \sigma^{-1}(\{n \leq p\})$. The equivalence of (a) and (c) can be proven analogously.

Now we shall prove that $g = \sum_n g_n$ is equal to $f = \sum_k (\sum_\ell f_{k,\ell})$. To this end, let $\varepsilon > 0$, and let m be such that $\sum_{k > m} (\sum_\ell a_{k,\ell}) \leq \varepsilon$. We also let m_k be such that $\sum_{\ell > m_k} a_{k,\ell} \leq 2^{-k}\varepsilon$, and let $\tilde{f}_\varepsilon = \sum_{k \leq m} \sum_{\ell \leq m_k} f_{k,\ell}$. Then for $z \in G$ we have

$$|f(z) - \tilde{f}_\varepsilon(z)| \leq \sum_{k > m} \left(\sum_\ell a_{k,\ell} \right) + \sum_{k \leq m} \left(\sum_{\ell > m_k} a_{k,\ell} \right) \leq 3\varepsilon. \quad (41)$$

Similarly, for sufficiently large p , the partial sum $\tilde{g}_p = \sum_{n \leq p} g_n$ satisfies

$$|\tilde{g}_p(z) - \tilde{f}_\varepsilon(z)| \leq \sum_{k > m} \left(\sum_\ell a_{k,\ell} \right) + \sum_{k \leq m} \left(\sum_{\ell > m_k} a_{k,\ell} \right) \leq 3\varepsilon, \quad (42)$$

and so we have

$$|f(z) - g(z)| \leq |g(z) - \tilde{g}_p(z)| + 6\varepsilon. \quad (43)$$

Since $\tilde{g}_p(z) \rightarrow g(z)$ and both ε and z are arbitrary, we conclude that $f = g$. \square

As an immediate application, we prove a result on the product of two series.

Corollary 29. *Let $f_k : G \rightarrow \mathbb{C}$ and $g_k : G \rightarrow \mathbb{C}$ satisfy $|f_k(z)| \leq a_k$ and $|g_k(z)| \leq b_k$ for all $z \in G$, and for $k \in \mathbb{N}$. Assume that $\sum_k a_k < \infty$ and $\sum_\ell b_\ell < \infty$. Then for every bijection $\sigma : \mathbb{N}^2 \rightarrow \mathbb{N}$, the series $\sum_n h_n$ with elements $h_{\sigma(k,\ell)} = f_k g_\ell$ converges absolutely uniformly in G to $(\sum_k f_k)(\sum_\ell g_\ell)$.*

Proof. We have

$$\sum_\ell a_k b_\ell = a_k \sum_\ell b_\ell < \infty, \quad (44)$$

and

$$\sum_k \left(\sum_\ell a_k b_\ell \right) = \sum_k a_k \left(\sum_\ell b_\ell \right) = \left(\sum_k a_k \right) \left(\sum_\ell b_\ell \right) < \infty. \quad (45)$$

Note that these imply

$$\sum_\ell f_k g_\ell = f_k \left(\sum_\ell g_\ell \right), \quad \text{and} \quad \sum_k \left(\sum_\ell f_k g_\ell \right) = \left(\sum_k f_k \right) \left(\sum_\ell g_\ell \right). \quad (46)$$

The proof is established upon employing [Theorem 28\(b\)](#) with $f_{k,\ell} = f_k g_\ell$. \square