Solutions to selected midterm problems

1. Consider a circle centred at $(x_1, y_1) \in \mathbb{R}^2$ of radius $r_1 > 0$, and another circle centred at $(x_2, y_2) \in \mathbb{R}^2$ of radius $r_2 > 0$. Let $(x, y) \in \mathbb{R}^2$ be a point of intersection of the two circles.

   (a) Intuitively, if we slightly vary any of the 6 parameters $(x_1, y_1, r_1, x_2, y_2, r_2)$, the intersection point (x, y) will move also slightly. Under what condition can we apply the implicit function theorem to guarantee that (x, y) is a differentiable function of the 6 parameters, as these parameters vary in a 6-dimensional open cube centred at the current value of the parameters?

   (b) What is the geometric meaning of the condition derived in (a)?

   (c) Compute the partial derivatives $\dfrac{\partial x}{\partial x_1}$ and $\dfrac{\partial x}{\partial r_1}$.

Solution: Let us introduce the notations $\alpha = (x_1, y_1, r_1, x_2, y_2, r_2)$ and $\beta = (x, y)$, and let us denote by α^* and β^* the current values of the parameters. We know that the point $\beta^* = (x^*, y^*)$ is a point of intersection of the two circles. On the other hand, a point (x, y) is an intersection point if and only if the following equations are satisfied:

$$
\begin{aligned}
(x - x_1)^2 + (y - y_1)^2 &= r_1^2, \\
(x - x_2)^2 + (y - y_2)^2 &= r_2^2.
\end{aligned}
$$

We write this as $\phi(\alpha, \beta) = 0$, where $\phi : \mathbb{R}^8 \to \mathbb{R}^2$ is defined by

$$
\phi(x_1, y_1, r_1, x_2, y_2, r_2, x, y) = \begin{pmatrix} (x - x_1)^2 + (y - y_1)^2 - r_1^2 \\ (x - x_2)^2 + (y - y_2)^2 - r_2^2 \end{pmatrix}.
$$

Since the components of ϕ are polynomials, ϕ is continuously differentiable (in fact smooth) in $\mathbb{R}^8$. We have $\phi(\alpha^*, \beta^*) = 0$. The only remaining condition to check in order to apply the implicit function theorem and express β as a function of α near (α^*, β^*) is the invertibility of the matrix $D_\beta \phi(\alpha^*, \beta^*)$. Thus we compute

$$
D_\beta \phi(\alpha, \beta) = \begin{pmatrix} 2(x - x_1) & 2(y - y_1) \\ 2(x - x_2) & 2(y - y_2) \end{pmatrix},
$$

and therefore

$$
\det D_\beta \phi(\alpha, \beta) = 4(x - x_1)(y - y_2) - 4(y - y_1)(x - x_2).
$$

Hence if

$$
(x^* - x_1^*)(y^* - y_2^*) \neq (y^* - y_1^*)(x^* - x_2^*), \tag{1}
$$

then there exist $\delta > 0$ and a continuously differentiable function $h : Q_\delta^6(\alpha^*) \to \mathbb{R}^2$ such that $\phi(\alpha, h(\alpha)) = 0$ for all $\alpha \in Q_\delta^6(\alpha^*)$. In other words, the original intersection point (x^*, y^*) moves in a (continuously) differentiable manner as the 6 parameters $(x_1, y_1, r_1, x_2, y_2, r_2)$ take values in the 6-dimensional cube $Q_\delta^6(\alpha^*)$.

As for (b), suppose that (1) does *not* hold, that is, we have

$$(x^* - x_1^*)(y^* - y_2^*) = (y^* - y_1^*)(x^* - x_2^*).$$

Since (x^*, y^*) is at the distance $r_1^* > 0$ from (x_1^*, y_1^*), the quantities $x^* - x_1^*$ and $y^* - y_1^*$ cannot vanish at the same time. Without loss of generality, let $x^* - x_1^* \neq 0$. Then the aforementioned condition is equivalent to

$$y^* - y_2^* = \frac{(y^* - y_1^*)(x^* - x_2^*)}{x^* - x_1^*}.$$

If $y^* - y_2^* = 0$, then $x^* - x_2^* \neq 0$. If $y^* - y_2^* \neq 0$, then $(y^* - y_1^*)(x^* - x_2^*) \neq 0$, and hence $x^* - x_2^* \neq 0$. In any case, we can divide by $x^* - x_2^*$, yielding

$$\frac{y^* - y_2^*}{x^* - x_2^*} = \frac{y^* - y_1^*}{x^* - x_1^*}.$$

This means that the vectors $(x^* - x_1^*, y^* - y_1^*)$ and $(x^* - x_2^*, y^* - y_2^*)$ are on the same line, or equivalently, the two circles are tangential to each other at (x^*, y^*). Therefore, condition (1) says that the two circles are *not* tangent to each other, or that the two circles intersect at more than a single point.

To compute the derivatives of x with respect to the parameters α, we use the formula

$$D_\alpha \beta = Dh = -(D_\beta \phi)^{-1} D_\alpha \phi.$$

We have

$$(D_\beta \phi)^{-1} = \frac{1}{2\Delta} \begin{pmatrix} y - y_2 & y_1 - y \\ x_2 - x & x - x_1 \end{pmatrix},$$

where $\Delta = (x - x_1)(y - y_2) - (y - y_1)(x - x_2) \neq 0$. Furthermore, we have

$$\frac{\partial \phi}{\partial x_1} = \begin{pmatrix} -2(x - x_1) \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \frac{\partial \phi}{\partial r_1} = \begin{pmatrix} 2r_1 \\ 0 \end{pmatrix},$$

which yield

$$\frac{\partial \alpha}{\partial x_1} = -(D_\beta \phi)^{-1} \frac{\partial \phi}{\partial x_1} = -\frac{1}{2\Delta} \begin{pmatrix} y - y_2 & y_1 - y \\ x_2 - x & x - x_1 \end{pmatrix} \begin{pmatrix} -2(x - x_1) \\ 0 \end{pmatrix} = \frac{x - x_1}{\Delta} \begin{pmatrix} y - y_2 \\ x_2 - x \end{pmatrix},$$

and

$$\frac{\partial \alpha}{\partial r_1} = -(D_\beta \phi)^{-1} \frac{\partial \phi}{\partial r_1} = -\frac{1}{2\Delta} \begin{pmatrix} y - y_2 & y_1 - y \\ x_2 - x & x - x_1 \end{pmatrix} \begin{pmatrix} 2r_1 \\ 0 \end{pmatrix} = -\frac{r_1}{\Delta} \begin{pmatrix} y - y_2 \\ x_2 - x \end{pmatrix}.$$

Since x is simply the first component of α, we can read off

$$\frac{\partial x}{\partial x_1} = \frac{(x - x_1)(y - y_2)}{(x - x_1)(y - y_2) - (y - y_1)(x - x_2)},$$

and

$$\frac{\partial x}{\partial r_1} = \frac{(y_2 - y)r_1}{(x - x_1)(y - y_2) - (y - y_1)(x - x_2)}.$$

2. Consider a circle centred at $(X, Y) \in \mathbb{R}^2$ of radius $R > 0$, and a line passing through the origin that makes an angle α in anticlockwise direction with the positive x-axis. Let $(x, y) \in \mathbb{R}^2$ be a point of intersection of the circle and the line.

   (a) Intuitively, if we slightly vary any of the 4 parameters (X, Y, R, α), the intersection point (x, y) will move also slightly. Under what condition can we apply the implicit function theorem to guarantee that (x, y) is a differentiable function of the 4 parameters, as these parameters vary in a 4-dimensional open cube centred at the current value of the parameters?

   (b) What is the geometric meaning of the condition derived in (a)?

   (c) Compute the partial derivatives $\dfrac{\partial x}{\partial X}$ and $\dfrac{\partial x}{\partial \alpha}$.

Solution: Let us introduce the notations $\xi = (X, Y, R, \alpha)$ and $\eta = (x, y)$, and let us denote by ξ^* and η^* the current values of the parameters. We know that the point $\eta^* = (x^*, y^*)$ is a point of intersection of the circle and the line. On the other hand, a point (x, y) is an intersection point if and only if the following equations are satisfied:

$$
\begin{gathered}
(x - X)^2 + (y - Y)^2 = R^2, \\
x \sin\alpha - y \cos\alpha = 0.
\end{gathered}
$$

We write this as $\phi(\xi, \eta) = 0$, where $\phi : \mathbb{R}^6 \to \mathbb{R}^2$ is defined by

$$
\phi(X, Y, R, \alpha, x, y) = \begin{pmatrix} (x - X)^2 + (y - Y)^2 - R^2 \\ x \sin\alpha - y \cos\alpha \end{pmatrix}.
$$

Since the components of ϕ are smooth functions, ϕ is continuously differentiable (in fact smooth) in $\mathbb{R}^6$. We have $\phi(\xi^*, \eta^*) = 0$. The only remaining condition to check in order to apply the implicit function theorem and express η as a function of ξ near (ξ^*, η^*) is the invertibility of the matrix $D_\eta \phi(\xi^*, \eta^*)$. Thus we compute

$$
D_\eta \phi(\xi, \eta) = \begin{pmatrix} 2(x - X) & 2(y - Y) \\ \sin\alpha & -\cos\alpha \end{pmatrix},
$$

and therefore

$$
\det D_\eta \phi(\xi, \eta) = -2(x - X)\cos\alpha - 2(y - Y)\sin\alpha.
$$

Hence if

$$
(x - X)\cos\alpha + (y - Y)\sin\alpha \neq 0, \tag{2}
$$

then there exist $\delta > 0$ and a continuously differentiable function $h : Q^4_\delta(\alpha^*) \to \mathbb{R}^2$ such that $\phi(\xi, h(\xi)) = 0$ for all $\xi \in Q^4_\delta(\xi^*)$. In other words, the original intersection point (x^*, y^*) moves in a (continuously) differentiable manner as the 4 parameters (X, Y, R, α) take values in the 4-dimensional cube $Q^4_\delta(\xi^*)$.

As for (b), consider the negation of (2), and notice that the condition

$$
(x^* - X^*)\cos\alpha^* + (y^* - Y^*)\sin\alpha^* = 0,
$$

means that the vectors $(x^* - X^*, y^* - Y^*)$ and $(\cos\alpha, \sin\alpha)$ are perpendicular to each other, or equivalently, the circle and the line are tangential to each other at (x^*, y^*). Therefore, condition (2) says that the line *cannot* be tangent to the circle, or that the line cuts the circle at more than a single point.

To compute the derivatives of x with respect to the parameters ξ, we use the formula

$$D_\xi \eta = Dh = -(D_\eta \phi)^{-1} D_\xi \phi.$$

We have

$$(D_\eta \phi)^{-1} = \frac{1}{2\Delta} \begin{pmatrix} \cos\alpha & 2(y - Y) \\ \sin\alpha & -2(x - X) \end{pmatrix},$$

where $\Delta = (x - X)\cos\alpha + (y - Y)\sin\alpha \neq 0$. We also have

$$\frac{\partial \phi}{\partial X} = \begin{pmatrix} -2(x - X) \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \frac{\partial \phi}{\partial \alpha} = \begin{pmatrix} 0 \\ x\cos\alpha + y\sin\alpha \end{pmatrix},$$

which yield

$$\frac{\partial \eta}{\partial X} = -(D_\eta \phi)^{-1} \frac{\partial \phi}{\partial X} = -\frac{1}{2\Delta} \begin{pmatrix} \cos\alpha & 2(y - Y) \\ \sin\alpha & -2(x - X) \end{pmatrix} \begin{pmatrix} -2(x - X) \\ 0 \end{pmatrix} = \frac{x - X}{\Delta} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix},$$

and

$$\begin{aligned} \frac{\partial \eta}{\partial \alpha} &= -(D_\eta \phi)^{-1} \frac{\partial \phi}{\partial \alpha} = -\frac{1}{2\Delta} \begin{pmatrix} \cos\alpha & 2(y - Y) \\ \sin\alpha & -2(x - X) \end{pmatrix} \begin{pmatrix} 0 \\ x\cos\alpha + y\sin\alpha \end{pmatrix} \\ &= \frac{x\cos\alpha + y\sin\alpha}{\Delta} \begin{pmatrix} Y - y \\ x - X \end{pmatrix}. \end{aligned}$$

Since x is simply the first component of η, we can read off

$$\frac{\partial x}{\partial X} = \frac{(X - x)\cos\alpha}{(x - X)\cos\alpha + (y - Y)\sin\alpha},$$

and

$$\frac{\partial x}{\partial \alpha} = \frac{(Y - y)(x\cos\alpha + y\sin\alpha)}{(x - X)\cos\alpha + (y - Y)\sin\alpha}.$$