8. If $f(x) = \cos^2 x$, what is it's MacLaurin series? What is $\int \cos^2 x$?

Since $\cos^2 x = 1/2 + \frac{\cos 2x}{2}$ and $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

then $\cos^2 x = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}\right]$

as such $\int \cos^2 x = \frac{1}{2}\left[x + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!}\right]$

I think this would be well-suited for the final exam as it requires an understanding between the link of series, sums, and integrals, yet isn't a function that is completely trivial.

Source: A Yahoo! question asking for the series of $\sin^2(2x)$. URL available upon request

#8 Problem statement:

With the following 3 vectors, find the volume of the parallelepiped contained between them:

A(-8, 15, 3)
B(3, -6, -8)
C(16, 12, 3)

Full solution:

Using the formula $|(\vec{u} \times \vec{v}) \cdot \vec{w}|$,

$A \times C = (45-36, 48+24, -96-240) = (9, 72, -336)$
$(A \times C) \cdot B = 27 - 432 + 2688 = 2283$

Final answer: $2283 \; u^3$

Explanation:

I believe we have not practiced the triple product enough and I had forgotten about it. This might surprise a few people, but as long as the formula is remembered, it is fairly straightforward. It is a nice problem to test the dot and the cross product in a single question. Putting it at the beginning of the exam would be a great confidence booster!

8. If the following series converges, compute its sum:

$$\sum_{n=0}^{\infty} \frac{12}{n^2+4n+3}$$

① Factor denominator into $(n+1)(n+3)$

② Use a partial fraction decomposition to get $A=6$, $B=-6$

$$\frac{12}{(n+1)(n+3)} = \frac{A}{(n+1)} + \frac{B}{(n+3)}$$

$A(n+3) + B(n+1) = 12$

(plug in $n=-1$ and $n=-3$)

$-2A = 12$; $A = -6$

$2B = 12$; $B = 6$

③ Rewrite $\frac{6}{(n+1)} - \frac{6}{(n+3)}$ as a series and expand:

$$6\sum_{n=0}^{\infty} \frac{1}{(n+1)} - \frac{1}{(n+3)} = \left(1-\tfrac{1}{3}\right) + \left(\tfrac{1}{2}-\tfrac{1}{4}\right) + \left(\tfrac{1}{3}-\tfrac{1}{5}\right) + \left(\tfrac{1}{4}-\tfrac{1}{6}\right) + \cdots$$

④ Note that terms begin to cancel and will continue to do so.

We can compute the limit of the series to see if it converges, and to what:

$$\lim_{n\to\infty} 6\sum_{n=0}^{\infty} \frac{1}{n+1} - \frac{1}{n+3} = 6 \lim_{n\to\infty}\left(1+\tfrac{1}{2}-\frac{1}{n+3}\right)$$

$$= 6\left(1+\tfrac{1}{2}-\frac{1}{\infty+3}\right) = 6\left(1+\tfrac{1}{2}-0\right) = 9$$

$\boxed{\therefore \text{ the sum converges; and it converges to } 9.}$

- This kind of problem is not uncommon when evaluating knowledge and understanding of series/sequences. I like it because it requires good understanding of partial sums and their limits, and knowing which steps to take requires knowing how these concepts fit together, as well as some clever algebra.

- Took inspiration from webwork #1, Q21.

8. Derive the first four terms of the Maclaurin series of the following.

$f(x) = \tan(x)$

Solution:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$ using Maclaurin series.

Then, compute the long division

$$x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7$$

$$1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \cdots \;\big)\; x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \tfrac{1}{5040}x^7 + \cdots$$

$$-\left(x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 - \tfrac{1}{720}x^7\right) + \cdots$$

$$\tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \tfrac{1}{840}x^7 - \cdots$$

$$\tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \tfrac{1}{72}x^7 - \cdots$$

$$\tfrac{2}{15}x^5 - \tfrac{4}{315}x^7 + \cdots$$

$$\tfrac{2}{15}x^5 - \tfrac{2}{30}x^7 + \cdots$$

$$\tfrac{17}{315}x^7 - \cdots$$

$$\tfrac{17}{315}x^7 - \cdots$$

Thus,

$$\tan(x) = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7.$$

Why? It is a different type of question from the one on midterm. Using the memorized maclaurin series to determine another function's Maclaurin series will require some computational skill, basic identity from Calculus I, and understanding on the Taylor series.

8. Using Maclaurin series, evaluate

$$\lim_{x \to 0} \frac{e^x - \sin(x^2)}{e^x - \tan(x^2)}$$

$$\frac{e^x - \sin(x^2)}{e^x - \frac{\sin(x^2)}{\cos(x^2)}} = \frac{e^x - \sin(x^2)}{\frac{e^x\cos(x^2) - \sin(x^2)}{\cos(x^2)}} = \frac{e^x\cos(x^2) - \sin(x^2)\cos(x^2)}{e^x\cos(x^2) - \sin(x^2)}$$

$$= \frac{(1 + x + \ldots)(1 - \frac{x^4}{2!} + \ldots) - (x^2 - \frac{x^6}{3!} + \ldots)(1 - \frac{x^4}{2!} + \ldots)}{(1 + x + \ldots)(1 - \frac{x^4}{2!} + \ldots) - (x^2 - \frac{x^6}{3!} + \ldots)}$$

$$= \frac{1 - \frac{x^4}{2!} + x - \frac{x^5}{2!} + \ldots - x^2 + \frac{x^6}{2!} + \frac{x^6}{3!} - \frac{x^{10}}{3!2!} + \ldots}{1 - \frac{x^4}{2!} + x - \frac{x^5}{2!} + \ldots - x^2 + \frac{x^6}{3!} - \ldots}$$

$$\lim_{x \to 0} \frac{e^x\cos(x^2) - \sin(x^2)\cos(x^2)}{e^x\cos(x^2) - \sin(x^2)} = \frac{1}{1} = 1$$

since the rest of the terms tend to 0 as $x \to 0$.

* modified example from Calculus II – Mardsen and Weinstein.

This question would be well suited for the exam as it requires some outside thinking on the most optimal way to solve the problem (ie. not needing many terms, changing tan to $\frac{\sin}{\cos}$). It also tests the use of (x^2) as opposed to x in the sin/cos Maclaurin series

8. Reparametrize the curve $r(t) = \langle e^{2t}\cos 2t, 2, e^{2t}\sin(2t) \rangle$ in terms of arc length measured from the point $t=0$

Solution: $r'(t) = e^{2t}(-2\sin 2t) + ((\cos 2t)(2e^{2t})$

0

$e^{2t}(2\cos 2t) + \sin(2t) 2e^{2t}$

$= \langle 2e^{2t}\cos 2t - 2e^{2t}\sin 2t, 0, 2e^{2t}\cos 2t + 2e^{2t}\sin 2t \rangle$

$$\|r'(t)\| = \sqrt{4e^{4t}\cos^2(2t) - 4e^{4t}\cos(2t)\sin(2t) + 4e^{4t}\sin^2(2t) + 4e^{4t}\cos^2(2t) + 4e^{4t}\sin(2t)\cos(2t) + 4e^{4t}\sin^2(2t)}$$

$$= \sqrt{8e^{4t}\cos^2(2t) + 8e^{4t}\sin^2(2t)}$$

$$= \sqrt{8e^{4t}(\cos^2(2t) + \sin^2(2t))}$$

$$= \sqrt{8 \cdot e^{4t}} = 2\sqrt{2}e^{2t}$$

$$L = \int_0^t 2\sqrt{2} \cdot e^{2x}\,dx$$

$$= \sqrt{2}e^{2x}\Big]_0^t$$

$$= \sqrt{2}e^{2t} - \sqrt{2}e^0$$

$$= \sqrt{2}e^{2t} - \sqrt{2}$$

$$\sqrt{2} \cdot e^{2t} = S + \sqrt{2}$$

$$2t = \ln\left(\frac{S+1}{\sqrt{2}}\right)$$

$$t = \tfrac{1}{2}\ln\left(\frac{S+1}{\sqrt{2}}\right)$$

(plug into "r")

Basic concept of curves paramatrized in terms of arc length

This can be applied to many other different problems

8. Problem Statement:

Determine whether or not the following three vectors lie on the same plane:

$\vec{a} = \langle 1, -4, 7 \rangle$, $\vec{b} = \langle 3, 3, -3 \rangle$, $\vec{c} = \langle 7, 2, 5 \rangle$

Solution:

If the volume of the parallelepiped defined by thes vectors is 0, then they lie on the same plane. Otherwise, they don't.

Volume $= |\vec{a} \cdot (\vec{b} \times \vec{c})|$

So first we compute the cross product $(\vec{b} \times \vec{c})$

$$(\vec{b} \times \vec{c}) = \begin{vmatrix} i & j & k \\ 3 & 3 & -3 \\ 7 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -3 \\ 2 & 5 \end{vmatrix} i - \begin{vmatrix} 3 & -3 \\ 7 & 5 \end{vmatrix} j + \begin{vmatrix} 3 & 3 \\ 7 & 2 \end{vmatrix} k$$

$$= \langle 21i, -36j, -15k \rangle$$

Now we take the dot product of $\vec{a}$ and this vector:

$$\langle 1, -4, 7 \rangle \cdot \langle 21, -36, -15 \rangle$$
$$= (1 \cdot 21) + (-4 \cdot -36) + (7 \cdot -15)$$
$$= 21 + 144 - 105$$
$$= 60$$

Thus, the vectors are not all on the same plane.

Explanation: This question is well suited for the final exam because it tests the fundamentals of vectors. One must either have a conceptual understanding of vectors in space as well as the geometric meaning of the dot product and cross product in order to understand the volume formula and what that means in terms of co-planar vectors, and further know how to compute the above products, all important in a fundamental understanding of vectors.

Sources: Paul's Notes and an online calculator to check that my answer is correct.

8. Bonus Question

Find the maclaurin series for $c(x) = \sin^2(x)$.

solution

At first glance, this question would seem difficult to most. The student must remember a trig identity in order to simplify the question and to be able to come to a conclusion.

Recall: $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$.

we know that:

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ by substituting in $2x$,

$\Rightarrow \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$

we can now substitute this into the original identity:

$$\sin^2(x) = \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right)$$

$$= \frac{1}{2}\left(1 - \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2^{2n})(x^{2n})}{(2n)!}\right)\right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (2^{2n})(x^{2n})}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n (2^{2n-1})}{(2n)!} x^{2n} \qquad \square$$

This would be well suited for the exam because it is more difficult than the midterm question (due to the trig identity), but easy enough to solve with our calculus 3 knowledge.

Sources:

'Table of Trigonometric Identities' on www.sosmath.com/trig/Trig5/trig5/trig5.html.

⑧ Bonus Q:

Given a curve: $\vec{x}(t) = \langle 4t^3, 2-t^2, t^3 \rangle$, determine where we are on the curve after traveling a distance of 10.

Solution:

$$x'(t) = \langle 12t^2, -2t, 3t^2 \rangle$$

$$\|x'(t)\| = \sqrt{144t^4 + 4t^2 + 9t^4} = \sqrt{t^2(4 + 153t^2)} = t\sqrt{4+153t^2}$$

The arc length:

$$S(t) = \int_0^t t\sqrt{4+153t^2}\,dt = \frac{1}{2\cdot 153\cdot\frac{3}{2}}(4+153t^2)^{3/2}\Big|_0^t = \frac{1}{459}(4+153t^2)^{3/2}\Big|_0^t$$

$$S(t) = \frac{(4+153t^2)^{3/2}}{459} - \frac{4^{3/2}}{459} = \frac{(4+153t^2)^{3/2} - 8}{459}$$

Solve for t:

$$459s + 8 = (4+153t^2)^{3/2}$$

$$(459s+8)^{2/3} = 4 + 153t^2$$

$$\sqrt{\frac{(459s+8)^{2/3} - 4}{153}} = t$$

$x(t(s))$ when $s = 10$ → $r(t(10))$

$$t(10) = \sqrt{\frac{((459\cdot 10)+8)^{2/3} - 4}{153}} = 1.33$$

$$x(t(10)) = x(1.33) = \langle 12\cdot(1.33)^2, -2(1.33), 3(1.33)^2 \rangle$$

$$= \langle 21.23, -2.66, 5.31 \rangle$$

The numbers didn't end up working in terms of being able to compute without a calculator, but the idea behind the question is to combine theory with application of the concept.

⑧ 1) Problem Statement :

Find the limit if it exists : $\lim_{n\to\infty} n^3\left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)$

2) Full Solution :

Here we use Taylor series for $\sin(x)$ to expand $\sin(\frac{1}{n})$

$$\sin\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{3!n^3} + \frac{1}{5!n^5} - \frac{1}{7!n^7} + \cdots$$

$$\frac{1}{n} - \sin\left(\frac{1}{n}\right) = \frac{1}{3!n^3} - \frac{1}{5!n^5} + \frac{1}{7!n^7} - \cdots$$

when $n \to \infty$, $\frac{1}{n}$ will be close to 0, hence $\frac{1}{n} - \sin(\frac{1}{n}) \approx \frac{1}{3!n^3} = \frac{1}{6n^3}$

$$\lim_{n\to\infty} n^3\left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right) \approx \lim_{n\to\infty} n^3 \cdot \frac{1}{6n^3} = \boxed{\frac{1}{6}}$$ final answer

3) why this question is the bomb :

In all exercises I've done, there's no taylor series in computing limits. If students realizes using taylor series to solve the limit problem, it shows he/she understands the fundamental usefulness of taylor expansion, which is to simplify things out and to transform one equation into another in order to facilitate the computation. It's also not a "calculation-intensive" question, it evaluates understanding of a core concept and is straight to the point.

Bonus question

Find the sum of the series $\sum_{n=0}^{\infty} (-1)^n \frac{(x+5)^{2n+1}}{(2n+9)!}$

1. This series reminds us of the Maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

2. We can first make the Maclaurin series for $\sin x$ look more like the given series by replacing x by $x+5$:

$$\sin(x+5) = \sum_{n=0}^{\infty} (-1)^n \frac{(x+5)^{2n+1}}{(2n+1)!} \quad (1)$$
$$= (x+5) - \frac{(x+5)^3}{3!} + \frac{(x+5)^5}{5!} - \frac{(x+5)^7}{7!} + \dots$$

3. We see that the exponent of the numerator matches the denominator we apply the factorial to (ie $2n+1$). Similarly, we have to match the exponent from the numerator to the number we apply the factorial to in the denominator: $2n+9$.
In order to do this, let's multiply and divide (1) by $(x+5)^8$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x+5)^{2n+1}}{(2n+9)!} = \frac{1}{(x+5)^8} \sum_{n=0}^{\infty} (-1)^n \frac{(x+5)^{2n+9}}{(2n+9)!}$$
$$= (x+5)^{-8} \Big[\underbrace{\frac{(x+5)^9}{9!} - \frac{(x+5)^{11}}{11!} + \frac{(x+5)^{13}}{13!} \dots}_{\text{It is the same series as for } \sin(x+5) \text{ but without the first 4 terms}} \Big]$$

$$= (x+5)^{-8} \Big[\sum_{n=0}^{\infty} (-1)^n \frac{(x+5)^{2n+1}}{2n+1} - \sum_{n=0}^{4} (-1)^n \frac{(x+5)^{2n+1}}{2n+1} \Big]$$
$$= (x+5)^{-8} \Big[\sin(x+5) - (x+5) + \frac{(x+5)^3}{3!} - \frac{(x+5)^5}{5!} + \frac{(x+5)^7}{7!} \dots \Big]$$

8) Problem for Final Exam:

Q: Find the Taylor Series for $f(x) = \frac{1}{x^2}$ about $x = 1$.

Sol: To start, we need to get a formula for $f^{(n)}(1)$, so take a couple derivatives and evaluate at $x = 1 \Rightarrow$

$f^{(0)}(x) = 1/x^2$ $\qquad$ $f^{(0)}(1) = 1$

$f^{(1)}(x) = -2/x^3$ $\qquad$ $f^{(1)}(1) = 2/(1)^3 = 2$

$f^{(2)}(x) = (2)(3)/x^4$ $\qquad$ $f^{(2)}(1) = (2)(3)/1^4 = 6$

$f^{(3)}(x) = -2(3)(4)/x^5$ $\qquad$ $f^{(3)}(1) = 2(3)(4)/(1)^5 = 24$

$\vdots$ $\qquad$ $\vdots$

$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{(x^{n+2})}$ $\qquad$ $f^{(n)}(1) = \frac{(-1)^n (n+1)!}{(1)^{n+2}} = (n+1)!$

Then the Taylor Series for the function $\Rightarrow$

$$\frac{1}{x^2} = \sum_{n=0}^{\infty} \frac{f^{n}(1)}{n!} \cdot (x-1)^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} \cdot (x-1)^n = \boxed{\sum_{n=0}^{\infty} (n+1)(x-1)^n}$$

8. Bonus: Consider the helix $x(t) = (\cos(-3t), \sin(-3t), 4t)$ compute at $t = \frac{\pi}{3}$ of the unit tangent $T(t)$, unit normal $N(t)$, unit binormal $B(t)$ and curvature $k(t)$.

$$x(t) = (\cos(-3t), \sin(-3t), 4t)$$
$$x'(t) = (3\sin(-3t), -3\cos(-3t), 4)$$
$$|x'(t)| = \sqrt{9+14} = 5$$

$$T = \frac{x'(t)}{|x'(t)|} = \left(\frac{3}{5}\sin(-3t), -\frac{3}{5}\cos(-3t), \frac{4}{5}\right)$$

$$\text{at } T = \frac{\pi}{3} = \left(\frac{3}{5}\sin(-\pi), -3\cos(-\pi), \frac{4}{5}\right) = \left(0, 3, \frac{4}{5}\right)$$

$$T' = \left(-\frac{9}{5}\cos(-3t), -\frac{9}{5}\sin(-3t), 0\right)$$
$$|T'| = \frac{9}{5}$$

$$N = \frac{T'}{|T'|} = (-\cos(-3t), -\sin(-3t), 0)$$
$$\text{at } T = \frac{\pi}{3} = (-\cos(-\pi), -\sin(-\pi), 0) = (1, 0, 0)$$

$$B = \frac{T \times N}{|T \times N|} \qquad T \times N = \left(\frac{4}{5}\sin(-3t), \frac{4}{5}\cos(-3t), -\frac{3}{5}\right)$$
$$|T \times N| = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$$

$$B = \left(\frac{4}{5}\sin(-3t), \frac{4}{5}\cos(-3t), -\frac{3}{5}\right)$$
$$\text{at } T = \frac{\pi}{3} = \left(\frac{4}{5}\sin(-\pi), \frac{4}{5}\cos(-\pi), -\frac{3}{5}\right) = \left(0, -\frac{4}{5}, -\frac{3}{5}\right)$$

$$\kappa = \frac{|T'|}{|x'|} = \frac{9}{5} \cdot \frac{1}{5} = \frac{9}{25}$$

This problem is similar to webwork assignment 3 Q.31.
This problem tests on virtually all the aspects of space curves and requires understanding the relationships between the helix, its unit tangent, unit normal, unit binormal, and curvature. It is also not too computation extensive provided you know what you are doing and consists of many sections where point marks can be given.

8. Problem: a) Apply integral Test to $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

b) Does the value of the infinite series converge or diverge to the answer in a)? Explain

Solution a) $\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}\,dx$

$= \lim_{b\to\infty} \left[\arctan\right]_1^b$

$= \lim_{b\to\infty} (\arctan b - \arctan 1]$

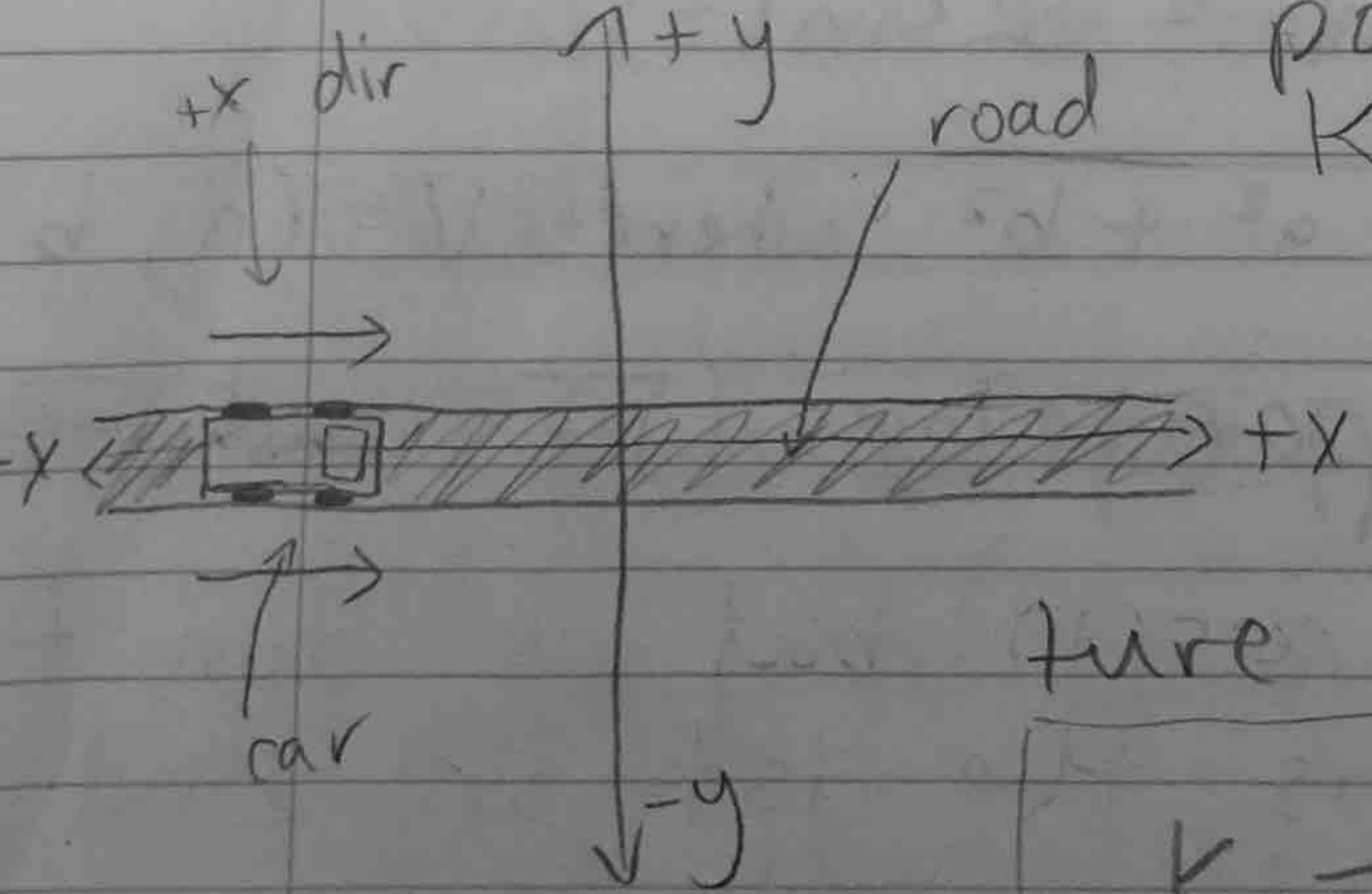
$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}.$

b) No. the integral test tells you nothing about the value of the series. It just tells you whether the series converges or diverges.

Explanation: This question achieves two tasks. It first tests whether students remember how to do basic integration. The second task it achieves by testing whether students are aware that the values from the integral do not represent the values from the infinite series. I belive this would be a good question to be placed in the final.

Question:

A car is driving down a straight road that, when looked upon from above, lies upon the x-axis of an x-y plane (see figure below). The car is travelling along this road in the "positive-x" direction. When the car reaches the point (0, 0), some unknown event occurs and the car goes off the road in the path of a curve with the curvature formula:



$$\boxed{K = \frac{|2\sin(t)|}{(4+\cos^2(t))^{3/2}}}$$

over the interval $t = [0, 2\pi]$, where t represents time in seconds.

unit tangent vector of the curve

Given that $T(0) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$...

Give the parametric equation $X(t)$ that represents the path of the car over $t = [0 \to 2\pi]$ and draw a rough sketch of this curve. (include the car's straight line motion before the point $(0, 0)$).

Give a plausible explanation for what might have happened to have made the car travel along this curve so suddenly.

Answer: Knowing that $K = \dfrac{|X'(t) \times X''(t)|}{|X'(t)|^3}$

because $K = \dfrac{|2\sin(t)|}{\left(\sqrt{4+\cos^2(t)}\right)^3}$, we can gather

that $|X'(t)| = \sqrt{4+\cos^2(t)}$ and

$|X'(t) \times X''(t)| = |2\sin(t)|$

~~bc~~ because $|V| = \sqrt{a^2+b^2}$ where $V = (a, b)$

$X'(t)$ must have parameters ($\sqrt{4}$ and $\sqrt{\cos t^2}$) or (2 and $\cos(t)$) but we don't know which is the 1st or 2nd parameter yet.

The order can be figured out by computing the unit tangent vector $T(t) = \frac{X'(t)}{|X'(t)|}$ for both equations

$$\begin{cases} X'(t) = (2, \cos(t)) & \text{and} \\ X'(t) = (\cos(t), 2) \end{cases}$$

and comparing them at $T(0)$ to the given $T(0) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ function.

Doing so shows that $X'(t) = (2, \cos(t))$ is the correct equation.

($|X'(t) \times X''(t)|$ can also be computed and checked if it equals $|2\sin(t)|$ but this step is uneccesary.)

$X(t)$ can be fomd by integrating $X'(t)$...

$$\int X'(t) = (2t + A, \sin(t) + B)$$

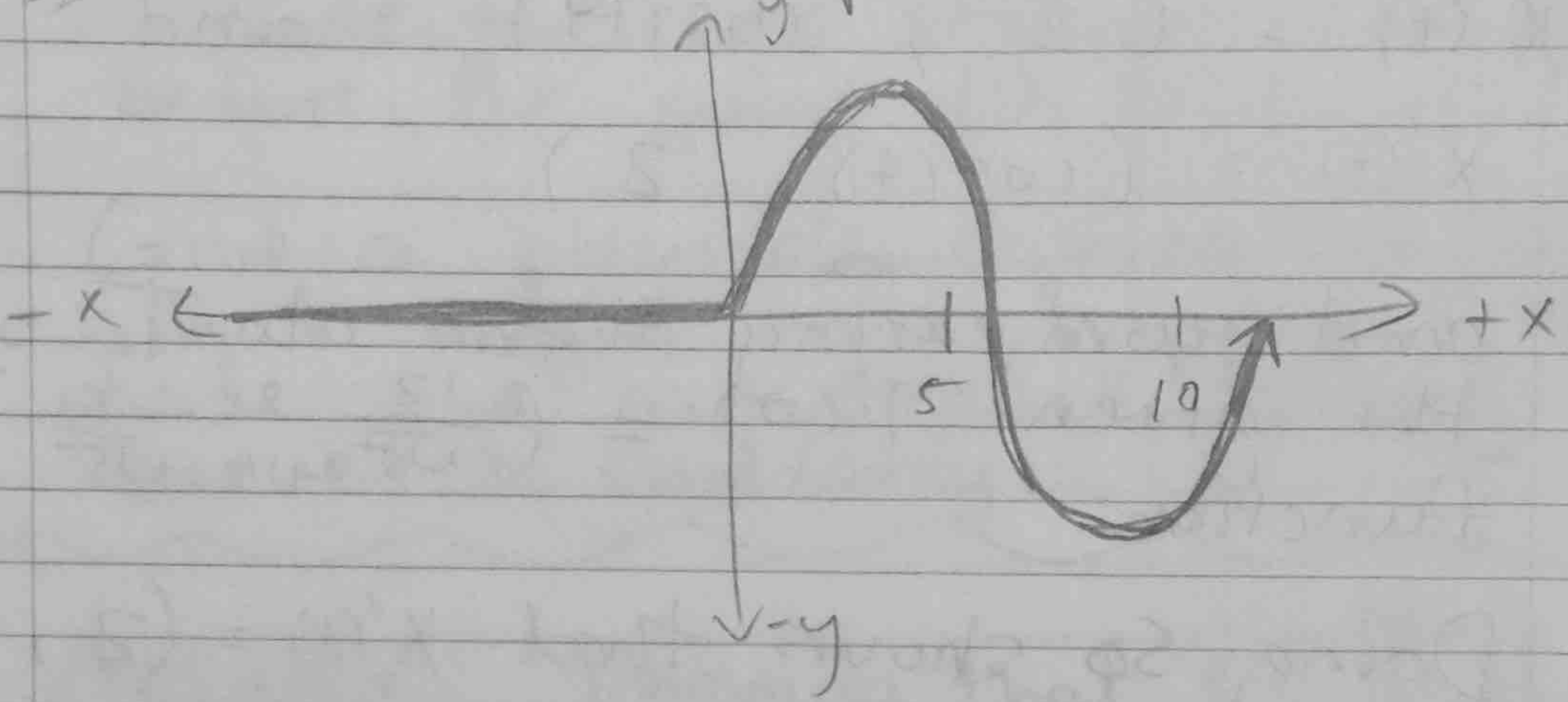
where A and B are some constants but because the car started at point $(0, 0)$ at $t = 0$.

$[0 = 2(0) + A]$ and $[0 = (0) + B]$

so $A = B = 0$ and

$X(t) = (2t, \sin(t))$

$X(t)$ Sketched over $t = [0, 2\pi]$ with initial straight path:



What caused this path is up to the marker to decide what is plausible

examples:

- The driver swerved to avoid hitting a dog

- The driver fell asleep for a few seconds and veered off course.

etc.

$\boxed{\#8}$ BONUS: $\int_0^1 t^2\,dt = \frac{t^3}{3}\Big|_0^1 = \boxed{1/3}$ JUST KIDDING...

FIND THE INTERVAL OF CONVERGENCE FOR $\sum_{n=1}^{\infty} \frac{5^n x^n (n+3)}{(n+5)}$

$a_{n+1} = \frac{5 \cdot 5^n x^n x (n+3+1)}{(n+5+1)} \longrightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{5 \cdot \cancel{5^n} \cancel{x^n} x(n+4)}{(n+6)} \cdot \frac{(n+5)}{(n+3)\cancel{x^n}\cancel{5^n}}\right|$

$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{5x(n+4)(n+5)}{(n+6)(n+3)}\right| \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 5|x| \lim_{n\to\infty}\left|\frac{n^2+9n+20}{n^2+9n+18}\right| = \underline{5|x|}$

FOR a_n TO CONVERGE $\rightarrow$ $-1 < 5x < 1$ $\therefore$ $-1/5 < x < 1/5$

TEST BOUNDS:

FOR $x = 1/5 \rightarrow \sum_{n=1}^{\infty} \frac{(n+3)}{(n+5)}$ $\rightarrow$ DIVERGES $\because \lim_{n\to\infty} \frac{(n+3)}{(n+5)} = 1$ (n^{th} TERM TEST)

FOR $x = 1/5 \rightarrow \sum_{n=1}^{\infty} (-1)^n (n+3)/(n+5)$ $\rightarrow$ DIVERGES BY n^{th} TERM TEST

$\therefore$ INTERVAL OF CONVERGENCE: $(-1/5, 1/5)$

8)

Find a value of x so that the statement below is true

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\sum_{n=0}^{\infty} x^n = \frac{2^{\frac{3}{2}}}{4-\pi}$$

To solve this question, all one needs to know is $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ and $\sum_0^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin(x)$

We will rewrite the statement as

$$\frac{\sin(x)}{1-x} = \frac{2^{\frac{3}{2}}}{4-\pi}$$

Now, we will rewrite the right hand side of the equation: $\frac{\sin(x)}{1-x} = \frac{2^{\frac{3}{2}}}{4(1-\frac{\pi}{4})} = \frac{2^{\frac{3}{2}}}{2^2(1-\pi)} = \frac{2^{-\frac{1}{2}}}{1-\frac{\pi}{4}}$

From there, we see that $\sin(x) = \frac{1}{\sqrt{2}}$ and $1 - x = 1 - \frac{\pi}{4} \rightarrow x = \frac{\pi}{4}$

Problem 8. *(Bonus problem)*

1. *Using the definition of Taylor series, find the first three non-zero terms of the Maclaurin series (Taylor series about 0) of* $f(x) = \ln(1 + \cos x)$ [1]

2. *Let the first term from part 1 be* $x_1(t)$*, the second term be* $x_2(t)$ *and the third term be* $x_3(t)$ *for some space curve* $X(t)$*. Find the curvature* $\kappa(t)$ *of* $X(t)$

Solution. 1.

$$f(x) = \ln(1 + \cos x)\ ,\ f(0) = \ln(2) \tag{78}$$

$$f'(x) = -\tan\frac{x}{2}\ ,\ f'(0) = 0 \tag{79}$$

$$f''(x) = -\frac{1}{1 + \cos x}\ ,\ f''(0) = -\frac{1}{2} \tag{80}$$

$$f^{(3)}(x) = -\frac{\sin x}{(1 + \cos x)^2}\ ,\ f^{(3)}(0) = 0 \tag{81}$$

$$f^{(4)}(x) = -\frac{2\cos x + \sin^2 x \cos x + 2}{(1 + \cos x)^4}\ ,\ f^{(4)}(0) = -\frac{1}{4} \tag{82}$$

$$f(x) = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f^{(3)}(0)}{3!}(x)^3 + \frac{f^{(4)}(0)}{4!}(x)^4 + \ldots \tag{83}$$

$$f(x) = \ln(2) - \frac{x^2}{4} - \frac{x^4}{96} - \ldots \tag{84}$$

[1] Inspiration for this question comes from the IB Math HL Specimen paper 3 for first examinations in 2014, calculus option

2.

$$X(t) = (\ln(2), \frac{-t^2}{4}, \frac{-t^4}{96}) \tag{85}$$

$$X'(t) = (0, \frac{-t}{2}, \frac{-t^3}{32}), \quad |X'| = (\frac{t}{2})\sqrt{1 + \frac{t^4}{64}} \tag{86}$$

$$X''(t) = (0, \frac{-1}{2}, \frac{-3t^2}{32}) \tag{87}$$

$$X' \times X'' = (\frac{t^3}{32}, 0, 0) \Rightarrow |X' \times X''| = \frac{t^3}{32} \tag{88}$$

$$\kappa(t) = \frac{|X' \times X''|}{|X'|^3} \tag{89}$$

$$\therefore \kappa(t) = \frac{\frac{t^3}{32}}{\frac{t^3}{8}\sqrt{(1 + \frac{t^4}{64})^3}} = \frac{1}{4\sqrt{(1 + \frac{t^4}{64})^3}} \tag{90}$$

This question serves to bring together several key concepts we have learned across at least a couple different topics. It involves memory and calculation of important formulas (both the formula for Taylor series and for curvature).

8. Determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \log\left(\tan\left(\sin\left(\cos\left(\frac{(x+1)^n}{(n+1)!}\right)\right)\right)\right)$$

At first glance, this problem might appear to many students to be cruel and unusual punishment; the nested trigonometric and logarithm functions are deliberately designed to cause panic. However, if the student recalls their n-th term test for divergence, this question becomes straight-forward.

$$\lim_{n\to\infty} \log\left(\tan\left(\sin\left(\cos\left(\frac{(x+1)^n}{(n+1)!}\right)\right)\right)\right) = \log\left(\tan\left(\sin\left(\cos\left(0\right)\right)\right)\right) = \log\left(\tan\left(\sin(1)\right)\right) \neq 0$$

Thus, the series diverges. This problem requires the knowledge of one of the more easily forgotten tests and of the fact that:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

It also determines whether a student can avoid being distracted by trivial components, such as the nested functions and the slight modification of the above expression. The student would have to look beyond the "scary-looking" layers, and realize it is quite simple. Since its solution is relatively short, it should not be worth too many marks. However, it would be an excellent bonus question or part of a list of other series which require testing.

I have devised this question myself, but it was inspired by 3(b) of this assignment; its expression enclosed by the logarithm added more challenge, so I decided to enclose even more functions. The assignment question, however, is convergent; I deliberately chose an expression with a known limit instead, and used nested functions trivially, knowing it would remain divergent, since $\cos(0) = 1$.

Problem 8. Find the center of the osculating circle to $X(t) = (2e^t, e^{2t})$ as a function of t.

Solution

$$X(t) = (2e^t, e^{2t})$$
$$X'(t) = (2e^t, 2e^{2t})$$
$$X''(t) = (2e^t, 4e^{2t})$$

The curvature κ is given by:

$$\begin{aligned}
\kappa &= \frac{|X' \times X''|}{|X'|^3} \\
&= \frac{|(2e^t)(4e^{2t}) - (2e^{2t})(2e^t)|}{\sqrt{(2e^t)^2 + (2e^{2t})^2}^3} \\
&= \frac{|8e^{3t} - 4e^{3t}|}{\sqrt{4e^{2t} + 4e^{4t}}^3} \\
&= \frac{|4e^{3t}|}{8e^{3t}\sqrt{1 + e^{2t}}^3} \\
&= \frac{1}{2\sqrt{1 + e^{2t}}^3}
\end{aligned}$$

The unit tangent to the function is:

$$
\begin{aligned}
T &= \frac{X'}{|X'|} \\
&= \frac{(2e^t, 2e^{2t})}{\sqrt{4e^{2t} + 4e^{4t}}} \\
&= \frac{(2e^t, 2e^{2t})}{2e^t\sqrt{1+e^{2t}}} \\
&= \left(\frac{1}{\sqrt{1+e^{2t}}}, \frac{e^t}{\sqrt{1+e^{2t}}}\right)
\end{aligned}
$$

$X' \times X''$ is positive for all t, so the curve is always turning leftwards and so the normal is directed $90°$ counterclockwise from the tangent vector. Therefore:

$$
N = \left(-\frac{e^t}{\sqrt{1+e^{2t}}}, \frac{1}{\sqrt{1+e^{2t}}}\right)
$$

The radius of curvature of the circle, r is:

$$
\begin{aligned}
r &= \frac{1}{\kappa} \\
&= \left(\frac{1}{2\sqrt{1+e^{2t}}^3}\right)^{-1} \\
&= 2\sqrt{1+e^{2t}}^3
\end{aligned}
$$

The center of curvature for a given t is placed at r units away from $X(t)$ in the direction of N. Therefore, it is located at:

$$
\begin{aligned}
C &= rN + X(t) \\
&= (2\sqrt{1+e^{2t}}^3)\left(-\frac{e^t}{\sqrt{1+e^{2t}}}, \frac{1}{\sqrt{1+e^{2t}}}\right) + (2e^t, e^{2t}) \\
&= \left(-\frac{2e^t\sqrt{1+e^{2t}}^3}{\sqrt{1+e^{2t}}} + 2e^t, \frac{2\sqrt{1+e^{2t}}^3}{\sqrt{1+e^{2t}}} + e^{2t}\right) \\
&= \Big(-2e^t(1+e^t) + 2e^t, 2(1+e^{2t}) + e^{2t}\Big) \\
&= \Big(-2e^t - 2e^{3t} + 2e^t, 2 + 2e^{2t} + e^{2t}\Big) \\
&= \Big(-2e^{3t}, 2 + 3e^{2t}\Big)
\end{aligned}
$$

Justification

This question is reasonably non-trivial while not seemingly unduly difficult at first glance or throughout the calculations. Furthermore, no question concerning an osculating circle or its center has been covered on previous assignments or in class, save for more trivial examples. However, all the concepts involved have nonetheless been thoroughly covered.