Solutions to the final exam problems

Math 222 Winter 2015

1. (a) Derive the Maclaurin series for the function $f(x) = \sin^2 x$.
   (b) What is the convergence radius of the resulting series?
   (c) Evaluate the 99-th and 100-th derivatives of $f(x)$ at $x = 0$.

Solution: This is one of the problems suggested by students, with a little augmentation. The idea is to use the identity

$$\sin^2 x = \frac{1-\cos 2x}{2}, \tag{1}$$

which can be deduced from the double angle formula

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x. \tag{2}$$

By substituting $t = 2x$ into the Maclaurin series

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}, \tag{3}$$

we get

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}, \tag{4}$$

and hence

$$\begin{aligned} \sin^2 x &= \frac{1}{2}\Big(1 - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}\Big) = \frac{1}{2}\Big(1 - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}\Big) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n-1}}{(2n)!} x^{2n}. \end{aligned} \tag{5}$$

This completes (a). For (b), we use the ratio test. If $c_n = \frac{(-1)^{n-1}2^{2n-1}}{(2n)!} x^{2n}$, then

$$\frac{|c_{n+1}|}{|c_n|} = x^2 \cdot \frac{2^{2n+1}}{2^{2n-1}} \cdot \frac{(2n)!}{(2n+2)!} = \frac{4x^2}{(2n+1)(2n+2)} \to 0, \qquad \text{as} \quad n \to \infty, \tag{6}$$

for any real number x, meaning that the power series in (5) converges for any real number x. Hence the convergence radius of the power series is equal to ∞.

As for (c), we recall that if the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{7}$$

has a nonzero convergence radius, then

$$f^{(n)}(0) = n! a_n, \qquad \text{for} \quad n = 0, 1, \ldots. \tag{8}$$

Comparing (5) with (7), we see that $a_0 = 0$ and that

$$a_{2k-1} = 0, \qquad \text{and} \qquad a_{2k} = \frac{(-1)^{k-1}2^{2k-1}}{(2k)!}, \qquad \text{for} \quad n = 1, 2, \ldots. \tag{9}$$

This means that

$$f^{(99)}(0) = 0, \qquad \text{and} \qquad f^{(100)}(0) = 100! \cdot \frac{(-1)^{49}2^{99}}{100!} = -2^{99}. \tag{10}$$

Grading: (Note that this is only a rough guideline)

- Point distribution of the parts (a), (b), (c): 5+2.5+2.5.
- Parts (b) and (c) can be marked independently of whether the answer to (a) is correct.
- In (a), incorrect "trigonometric identity" and/or careless mistakes: -1 point.
- In (b) and (c), careless mistakes: -0.5 point.
- Note that careless mistakes that change the course of the solution drastically cannot be treated as "careless mistakes" in the sense of the preceding two items.
- Other details are left to the discretion of the grader.

2. Decide if the following series converge.

   (a) $\displaystyle\sum_{n=1}^{\infty} \big(1 - \cos\frac{1}{n}\big)$.

   Solution: For n large, by the Maclaurin series of cosine, we have

$$1 - \cos\frac{1}{n} \approx 1 - \big(1 - \frac{1}{2n^2}\big) = \frac{1}{2n^2}, \tag{11}$$

   which suggests that the original series $\sum a_n$ converges, and that it should be comparable to the series $\sum n^{-2}$. Then the limit comparison test gives us the limit

$$\lim_{n\to\infty} \frac{a_n}{n^{-2}} = \lim_{n\to\infty} n^2\big(1 - \cos\frac{1}{n}\big). \tag{12}$$

   In order to compute this limit, we replace it by

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}, \tag{13}$$

   where we have applied L'Hôpital's rule twice. The conclusion is that

$$\lim_{n\to\infty} \frac{a_n}{n^{-2}} = \frac{1}{2}, \tag{14}$$

   and hence the series $\sum a_n$ converges.

   Grading: (Note that this is only a rough guideline)

   - Total mark: 5 points.
   - Reaching the correct conclusion by incorrect reasoning: 0 points.

- Reaching the correct conclusion by heuristic reasoning: 3 points.
- Making it rigorous (by using some convergence test): 2 points.
- Careless mistakes: -1 point.
- Note that careless mistakes that change the course of the solution drastically cannot be treated as "careless mistakes" in the sense of the preceding item.
- Incorrect application of the n-th term test: -2 points.
- Other details are left to the discretion of the grader.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.

Solution: The function $f(x) = \frac{1}{x(\log x)^2}$ is positive and decreasing for $x \in (1, \infty)$, so by the integral test, the convergence of the given series is the same as the finiteness of the integral

$$I = \int_2^{\infty} \frac{\mathrm{d}x}{x(\log x)^2}. \tag{15}$$

This integral can be computed with the help of the substitution $x = e^t$. Since $(e^t)' = e^t$ and $\log e^t = t$, we have

$$I = \int_{\log 2}^{\infty} \frac{e^t \mathrm{d}t}{e^t \cdot t^2} = \int_{\log 2}^{\infty} \frac{\mathrm{d}t}{t^2} = -\frac{1}{t}\Big|_{\log 2}^{\infty} = \frac{1}{\log 2}, \tag{16}$$

and hence the given series converges.

Note: In the exam version of this problem, the summation erroneously starts at $n = 1$, which would make the first term of the series meaningless since $\log 1 = 0$. I hope that you recognized this and focused on the meaningful part of the series. However, points will not be deducted if you worked with the problem as it is, and concluded, by a correct line or reasoning, either that the series diverges, or that the expression does not make sense.

Grading: (Note that this is only a rough guideline)

- See the notes in the preceding paragraph.
- Total mark: 5 points.
- Reaching the correct conclusion by incorrect reasoning: 0 points.
- Reaching the correct conclusion by heuristic reasoning: 3 points.
- Making it rigorous (by using some convergence test): 2 points.
- Careless mistakes: -1 point.
- Incorrect application of the n-th term test: -2 points.
- Note that careless mistakes that change the course of the solution drastically cannot be treated as "careless mistakes" in the sense of the preceding two items.
- Other details are left to the discretion of the grader.

3. Compute the unit tangent $T(t)$, the unit normal $N(t)$, the binormal $B(t)$, the curvature $\kappa(t)$, and the torsion $\tau(t)$, for the helix $X(t) = (\cos 3t, \sin 3t, 4t)$, where $-\infty < t < \infty$. Then evaluate each of the aforementioned quantities at the parameter value $t = \frac{\pi}{3}$.

Solution: This is one of the problems suggested by students, with the computation of torsion added to it. We have

$$X'(t) = (-3\sin 3t, 3\cos 3t, 4), \qquad |X'(t)| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 16} = 5, \tag{17}$$

and thus

$$T(t) = \frac{X'(t)}{|X'(t)|} = \big(-\frac{3}{5}\sin 3t, \frac{3}{5}\cos 3t, \frac{4}{5}\big). \tag{18}$$

Similarly, we have

$$T'(t) = \big(-\frac{9}{5}\cos 3t, -\frac{9}{5}\sin 3t, 0\big), \qquad |T'(t)| = \frac{9}{5}, \tag{19}$$

and thus

$$N(t) = \frac{T'(t)}{|T'(t)|} = (-\cos 3t, -\sin 3t, 0). \tag{20}$$

At this point, the curvature can be computed as

$$\kappa(t) = \frac{|T'(t)|}{|X'(t)|} = \frac{9}{5}\cdot\frac{1}{5} = \frac{9}{25}. \tag{21}$$

Furthermore, for the binormal vector, we have

$$B(t) = T(t)\times N(t) = \big(\frac{4}{5}\sin 3t, -\frac{4}{5}\cos 3t, \frac{3}{5}\big). \tag{22}$$

The torsion can be computed by the formula

$$\tau = \frac{(X'\times X'')\cdot X'''}{|X'\times X''|^2}, \tag{23}$$

but an easier way is to use the third equation of the Frenet-Serret formula, which reads

$$\frac{B'(t)}{|X'(t)|} = -\tau(t)N(t). \tag{24}$$

Note that the left hand side represents the derivative with respect to an arclength parameter (which is a consequence of the chain rule). From (22), we get

$$B'(t) = \big(\frac{12}{5}\cos 3t, \frac{12}{5}\sin 3t, 0\big), \tag{25}$$

and by comparing this with (20) and (24), and taking into account that $|X'(t)| = 5$, we infer

$$\tau(t) = \frac{12}{25}. \tag{26}$$

Finally, the evaluation of $T(t)$, $N(t)$, and $B(t)$ at $t = \frac{\pi}{3}$ gives

$$T(\tfrac{\pi}{3}) = (0, -\tfrac{3}{5}, \tfrac{4}{5}), \qquad N(\tfrac{\pi}{3}) = (1, 0, 0), \qquad \text{and} \qquad B(\tfrac{\pi}{3}) = (0, \tfrac{4}{5}, \tfrac{3}{5}). \tag{27}$$

Grading: (Note that this is only a rough guideline)

- Total mark: 10 points.
- Careless mistakes: -1.5 point.
- Incorrect application of the chain, product, or quotient rule: -3 points.
- Note that careless mistakes that change the course of the solution drastically cannot be treated as "careless mistakes" in the sense of the preceding two items.
- Other details are left to the discretion of the grader.

4. Find, with full justifications, the maximum and minimum values of

$$f(x, y) = \frac{x}{1 + x^2 + y^2}. \tag{28}$$

Solution: This is basically a problem from the last written assignment. The critical points of f must satisfy

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= \frac{1 + x^2 + y^2 - x \cdot 2x}{(1 + x^2 + y^2)^2} = \frac{1 - x^2 + y^2}{(1 + x^2 + y^2)^2} = 0, \\ \frac{\partial}{\partial y} f(x, y) &= \frac{0 - x \cdot 2y}{(1 + x^2 + y^2)^2} = -\frac{2xy}{(1 + x^2 + y^2)^2} = 0, \end{aligned} \tag{29}$$

which imply the equations

$$\begin{aligned} x^2 - y^2 &= 1, \\ xy &= 0. \end{aligned} \tag{30}$$

The only solutions of this system are

$$(x_1, y_1) = (1, 0), \qquad \text{and} \qquad (x_2, y_2) = (-1, 0). \tag{31}$$

For the values of f at the critical points, we have

$$f(x_1, y_1) = \frac{1}{2}, \qquad \text{and} \qquad f(x_2, y_2) = -\frac{1}{2}. \tag{32}$$

The question is: Are they the maximum and minimum values of f? To answer this question, we first try to show that $f(x, y)$ is close to 0 when the point (x, y) is far away from the origin. Given (x, y), let $r = \sqrt{x^2 + y^2}$ be the distance from (x, y) to the origin $(0, 0)$. Then we have $|x| \leq r$ and $|y| \leq r$, and so

$$|f(x, y)| = \frac{|x|}{1 + x^2 + y^2} \leq \frac{r}{1 + r^2}. \tag{33}$$

Moreover, if (x, y) is *outside* the open disk $D_R = \{(x, y) : x^2 + y^2 < R^2\}$ of radius $R > 0$, that is, if $r \geq R$, then we have

$$|f(x, y)| \leq \frac{r}{1 + r^2} \leq \frac{r}{r^2} = \frac{1}{r} \leq \frac{1}{R}. \tag{34}$$

In particular, fixing $R = 3$, we get

$$|f(x, y)| \leq \frac{1}{3} \qquad \text{for} \quad x^2 + y^2 \geq R^2. \tag{35}$$

Since both points (x_1, y_2) are (x_2, y_2) are in the disk D_R, we conclude that any possible maximizers and minimizers must be contained in the disk D_R. Now we apply the Weierstrass existence theorem in the *closed* disk $\bar{D}_R = \{(x, y) : x^2 + y^2 \leq R^2\}$, to infer that there exist a maximizer and a minimizer of f over the closed disk $\bar{D}_R$. By (35), neither a maximizer nor a minimizer can be on the boundary of $\bar{D}_R$, so they must be in the open disk D_R. This means that any maximizer must be a critical point of f in D_R, and comparing the values (32), we infer that there is only one maximizer of f over $\bar{D}_R$, and the maximizer is the point (x_1, y_1). Similarly, there is only one minimizer of f over $\bar{D}_R$, and the minimizer is the point (x_2, y_2). Note that at this point all we know is that (x_1, y_1) is the maximizer of f over $\bar{D}_R$ and that (x_2, y_2) is the minimizer of f over $\bar{D}_R$. However, invoking (35) once again, we conclude that (x_1, y_1) is indeed the maximizer of f over $\mathbb{R}^2$ and that (x_2, y_2) is indeed the minimizer of f over $\mathbb{R}^2$. The final answer is that the maximum value of f in $\mathbb{R}^2$ is $\frac{1}{2}$, and the minimum value is $-\frac{1}{2}$.

Grading: (Note that this is only a rough guideline)

- Total mark: 10 points.
- Correct equations for the critical points : 2 points.
- Finding the critical points : 2 points.
- Evaluation of f at the critical points: 1 point.
- Arguing heuristically that $|f|$ must be very small far from the origin: 2 points.
- Rigorous justification that $|f|$ must be small far from the origin: 3 points.
- The argument in the last paragraph of the solution: 2 points.
- Careless mistakes: -1.5 point.
- Incorrect application of the chain, product, or quotient rule: -3 points.
- Note that careless mistakes that change the course of the solution drastically cannot be treated as “careless mistakes” in the sense of the preceding two items.
- Other details are left to the discretion of the grader.

5. Compute the following integrals.

   (a) $\displaystyle\int_0^1 \int_y^1 x e^{-x^3} \mathrm{d}x\, \mathrm{d}y.$

   Solution: Since the inner integral over x looks like one of those integrals that cannot be computed, let us try switching the order of integrations. Firstly, the domain of integration is $D = \{0 \le y \le 1, y \le x \le 1\}$, in the “type II” fashion. We can write it in the “type I” fashion as $D = \{0 \le x \le 1, 0 \le y \le x\}$, and hence

   $$I = \int_0^1 \int_y^1 x e^{-x^3} \mathrm{d}x\, \mathrm{d}y = \int_0^1 \int_0^x x e^{-x^3} \mathrm{d}y\, \mathrm{d}x = \int_0^1 x^2 e^{-x^3} \mathrm{d}x \qquad (36)$$

   Upon using the substitution $x = \sqrt[3]{t}$, we get

   $$I = \int_0^1 x^2 e^{-x^3} \mathrm{d}x = \frac{1}{3} \int_0^1 e^{-t} \mathrm{d}t = -\frac{e^{-t}}{3}\Big|_0^1 = \frac{e-1}{3e}, \qquad (37)$$

   which is the final answer.

   Grading: (Note that this is only a rough guideline)

   - Total mark: 5 points.
   - Correct interchange of integrals: 2 points.

- Computation of the inner integral: 2 points.
- Careless mistakes: −1 point.
- Note that careless mistakes that change the course of the solution drastically cannot be treated as "careless mistakes" in the sense of the preceding item.
- Other details are left to the discretion of the grader.

(b) $\iint_D x^2y^2 \,\mathrm{d}A$, where $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is the unit disk.

Solution: In polar coordinates, the integral takes the form

$$\begin{aligned} I = \iint_D x^2y^2 \,\mathrm{d}A &= \int_0^1 \int_0^{2\pi} (r\sin\theta)^2(r\cos\theta)^2 \cdot r \cdot \mathrm{d}\theta \,\mathrm{d}r \\ &= \int_0^1 \int_0^{2\pi} r^5 \sin^2\theta \cos^2\theta \,\mathrm{d}\theta \,\mathrm{d}r. \end{aligned} \tag{38}$$

The inner integral can be computed as

$$\begin{aligned} \int_0^{2\pi} \sin^2\theta \cos^2\theta \,\mathrm{d}\theta &= \frac{1}{4}\int_0^{2\pi} \sin^2(2\theta) \,\mathrm{d}\theta = \frac{1}{8}\int_0^{2\pi} \big(1 - \cos 4\theta\big) \,\mathrm{d}\theta \\ &= \frac{1}{8}\Big(\theta - \frac{\sin 4\theta}{4}\Big)\Big|_0^{2\pi} = \frac{\pi}{4}, \end{aligned} \tag{39}$$

and hence the original integral is

$$I = \int_0^1 \int_0^{2\pi} r^5 \sin^2\theta \cos^2\theta \,\mathrm{d}\theta \,\mathrm{d}r = \frac{\pi}{4}\int_0^1 r^5 \,\mathrm{d}r = \frac{\pi}{4} \cdot \frac{r^6}{6}\Big|_0^1 = \frac{\pi}{24}. \tag{40}$$

Grading: (Note that this is only a rough guideline)

- Total mark: 5 points.
- Correct set up of the integral (either in polar or Cartesian coordinates): 2 points.
- Computation of the inner integral: 2 points.
- Careless mistakes: −1 point.
- Note that careless mistakes that change the course of the solution drastically cannot be treated as "careless mistakes" in the sense of the preceding item.
- Other details are left to the discretion of the grader.