HINTS TO PRACTICE PROBLEMS FOR MIDTERM

MATH 222 WINTER 2015

1. Does the following series converge or diverge?
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$. *Hint*: Apply the n-th term test, and the fact that $\lim \sqrt[n]{n} = 1$.
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{\sqrt{n}}$. *Hint*: Apply the alternating series test.
2. (a) Give a convincing argument that the following inequalities are true.
$$\int_1^n \log x \,\mathrm{d}x \le \log 1 + \log 2 + \log 3 + \ldots + \log n \le \int_1^{n+1} \log x \,\mathrm{d}x, \tag{1}$$
   for any integer $n \ge 1$. *Hint*: Use the geometric argument as in the proof of the integral test, or what is the same, observe that
$$\int_{k-1}^{k} \log x \,\mathrm{d}x \le \log k \le \int_k^{k+1} \log x \,\mathrm{d}x,$$
   because $\log x$ is increasing, and sum over $k = 2, 3, \ldots, n$.
   (b) Show that
$$n^n e^{1-n} \le n! \le (n+1)^{n+1} e^{-n}, \tag{2}$$
   for any integer $n \ge 1$. *Hint*: Compute the integrals in the estimate (1).
   (c) Show that
$$\lim_{n\to\infty} \frac{n!}{n^n} = 0, \qquad \lim_{n\to\infty} \sqrt[n]{n!} = \infty, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{r^n}{n!} = 0,$$
   for any (fixed) real number r. *Hint*: Use (2) and the squeeze theorem.
3. (a) (*Binomial theorem*) Derive the Maclaurin series for $(1+x)^m$, where m is a positive integer. From this, derive a formula for $(a+b)^m$. *Hint*: Recall that $c_n = f^{(n)}(0)/n!$ for the Maclaurin series coefficients. Write $(a+b)^m = a^m(1+\frac{b}{a})^m$ for $a \ne 0$.
   (b) (*Binomial series*) Find the Maclaurin series for $(1+x)^a$, where a is a real number and a is *not* among $0, 1, 2, \ldots$. Determine its radius of convergence. *Hint*: Use the formula $c_n = f^{(n)}(0)/n!$ for the Maclaurin series coefficients. Once the coefficients are known, apply the ratio test to find the convergence radius.
   (c) Starting with the fact that
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}},$$

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derive the Maclaurin series for $\arcsin x$. *Hint*: With the help of (b), derive the Maclaurin series for $(1-x^2)^{-\frac{1}{2}}$. Then integrate the Maclaurin series term by term.

4. Derive the Maclaurin series for the following functions.
   (a) $E(x) = \displaystyle\int_0^x e^{t^2}\mathrm{d}t$. *Hint*: Based on the Maclaurin series of e^x, derive the Maclaurin series for e^{t^2}. Then integrate the series term by term.
   (b) $S(x) = \displaystyle\int_0^x \frac{\sin t}{t}\mathrm{d}t$. *Hint*: Based on the Maclaurin series of $\sin x$, derive the Maclaurin series for $\frac{\sin t}{t}$. Then integrate the series term by term.
5. Decide if the series
$$\sum_{n=1}^{\infty} \log\cos\frac{1}{n}, \tag{3}$$
converges. *Hint*: For n large, we have
$$\cos\frac{1}{n} \approx 1 - \frac{1}{2n^2},$$
and so
$$\log\cos\frac{1}{n} \approx \log(1-\frac{1}{2n^2}) \approx -\frac{1}{2n^2}.$$
This suggests that the series (3) converges, and that it should be comparable to the series $\sum n^{-2}$. Then the limit comparison test gives us the limit
$$\lim_{n\to\infty} n^2 \log\cos\frac{1}{n}.$$
In order to compute this limit, we replace it by
$$\lim_{x\to 0} \frac{\log\cos x}{x^2},$$
which is amenable to L'Hôpital's rule.
6. Consider the series
$$\sum_{n=0}^{\infty} (-1)^n a_n x^{n^2},$$
where $a_0 = 1$ and $a_{n+1} = 3^n a_n$ for each $n = 0, 1, 2, \ldots$. Determine the values of x for which the series converges. *Hint*: By applying the ratio test to the series, we end up with the limit
$$\lim_{n\to\infty} \frac{|a_{n+1}x^{(n+1)^2}|}{|a_n x^{n^2}|} = \lim_{n\to\infty} 3^n x^{(n+1)^2-n^2} = \lim_{n\to\infty} 3^n x^{2n+1},$$
which can be computed, depending on whether $3x^2 > 1$, $3x^2 = 1$, or $3x^2 < 1$.
7. Find the convergence radius of the following power series.
   (a) $\displaystyle\sum_{n=0}^{\infty} (n^3 + a^n)(x-3)^n$, where $a \geq 0$ is a given real number. *Hint*: Apply the ratio test. The answer will depend on the value of a, in particular on whether or not $a > 1$.

(b) $\displaystyle\sum_{n=0}^{\infty} \log\big(1+\frac{1}{n}\big)(y+1)^n$. *Hint*: The ratio test leads us to the limit

$$\lim_{n\to\infty} \frac{\log(1+\frac{1}{n+1})}{\log(1+\frac{1}{n})},$$

which can be computed by applying L'Hôpital's rule to

$$\lim_{x\to\infty} \frac{\log(1+\frac{1}{x+1})}{\log(1+\frac{1}{x})} = \lim_{x\to\infty} \frac{\log(x+2)-\log(x+1)}{\log(x+1)-\log x}.$$

8. Consider the space curve given by

$$X(t) = (1+\cos t, \sin t, 2\sin\tfrac{t}{2}),$$

where t is a real parameter.

(a) Show that $|X(t)| = 2$ (i.e., the length of $X(t)$ equals 2) for any parameter value t. *Hint*: If $X(t) = (x_1(t), x_2(t), x_3(t))$, what this question is asking is to compute $\sqrt{x_1(t)^2 + x_2(t)^2 + x_3(t)^2}$.

(b) Compute the distance between the point $X(t)$ and the line L that passes through the points $(1,0,0)$ and $(1,0,1)$. *Hint*: If $X(t) = (x_1(t), x_2(t), x_3(t))$, what this question is asking is to compute $\sqrt{(x_1(t)-1)^2 + x_2(t)^2}$.

(c) Compute the unit tangent $T(t)$, the unit normal $N(t)$, the binormal $B(t)$, and the curvature $\kappa(t)$ of the curve, as functions of the parameter t. *Hint*: These are routine computations.

(d) Describe the curve in words, giving enough details so that a person with some knowledge of calculus would be able to sketch the curve from your description. *Hint*: What is the geometric significance of the information obtained in (a) and (b)?

(e) Sketch the curve. *Hint*: Use the information obtained in (a) and (b).