MATH 222 PRACTICE PROBLEMS FOR MIDTERM

1. Does the following series converge or diverge?
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$.
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{\sqrt{n}}$.
2. (a) Give a convincing argument that the following inequalities are true.
   $$\int_1^n \log x \,\mathrm{d}x \leq \log 1 + \log 2 + \log 3 + \ldots + \log n \leq \int_1^{n+1} \log x \,\mathrm{d}x,$$
   for any integer $n \geq 1$.
   (b) Show that
   $$n^n e^{1-n} \leq n! \leq (n+1)^{n+1} e^{-n},$$
   for any integer $n \geq 1$.
   (c) Show that
   $$\lim_{n\to\infty} \frac{n!}{n^n} = 0, \qquad \lim_{n\to\infty} \sqrt[n]{n!} = \infty, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{r^n}{n!} = 0,$$
   for any (fixed) real number r.
3. (a) (*Binomial theorem*) Derive the Maclaurin series for $(1+x)^m$, where m is a positive integer. From this, derive a formula for $(a+b)^m$.
   (b) (*Binomial series*) Find the Maclaurin series for $(1+x)^a$, where a is a real number and a is *not* among $0, 1, 2, \ldots$. Determine its radius of convergence.
   (c) Starting with the fact that
   $$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}},$$
   derive the Maclaurin series for $\arcsin x$.
4. Derive the Maclaurin series for the following functions.
   (a) $E(x) = \displaystyle\int_0^x e^{t^2} \mathrm{d}t$.
   (b) $S(x) = \displaystyle\int_0^x \frac{\sin t}{t} \mathrm{d}t$.
5. Decide if the series
   $$\sum_{n=1}^{\infty} \log \cos \frac{1}{n},$$

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converges. (*Hint*: Try to compare it with a p-series by the limit comparison test. The Maclaurin series for $\cos x$ and $\log(1+x)$ may be useful.)

6. Consider the series
$$\sum_{n=0}^{\infty}(-1)^n a_n x^{n^2},$$
where $a_0 = 1$ and $a_{n+1} = 3^n a_n$ for each $n = 0, 1, 2, \ldots$. Determine the values of x for which the series converges.

7. Find the convergence radius of the following power series.
   (a) $\displaystyle\sum_{n=0}^{\infty}(n^3 + a^n)(x-3)^n$, where $a \geq 0$ is a given real number.
   (b) $\displaystyle\sum_{n=0}^{\infty}\log\big(1+\frac{1}{n}\big)(x+1)^n$.

8. Consider the space curve given by
$$X(t) = (1+\cos t, \sin t, 2\sin\tfrac{t}{2}),$$
where t is a real parameter.
   (a) Show that $|X(t)| = 2$ (i.e., the length of $X(t)$ equals 2) for any parameter value t.
   (b) Compute the distance between the point $X(t)$ and the line L that passes through the points $(1,0,0)$ and $(1,0,1)$.
   (c) Compute the unit tangent $T(t)$, the unit normal $N(t)$, the binormal $B(t)$, and the curvature $\kappa(t)$ of the curve, as functions of the parameter t.
   (d) Describe the curve in words, giving enough details so that a person with some knowledge of calculus would be able to sketch the curve from your description.
   (e) Sketch the curve.